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Relaxed States with Plasma Flow

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Abstract

In the theory of relaxation, a plasma reaches a state of minimum energy subject to constant magnetic helicity. In this state the plasma velocity is zero. Several authors have attempted to extend the theory, by introducing a number of different helicity invariants, so as to obtain relaxed states with plasma flow. It is shown here that these generalized invariants are special cases of two basic self-helicities, one for electrons and one for ions. The validity of the generalized invariants is discussed and contrasted with that of the original magnetic helicity.

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I. Introduction

The theory of relaxation of turbulent plasma has been very successful in predicting and interpreting the results of plasma experiments such as Reverse Field Pinch, Spheromak and Multipinch.[1,2] According to this theory the plasma relaxes to a configuration of minimum energy subject to the constraint of constant total magnetic helicity

$$K_0 = \int \mathbf{A} \cdot \mathbf{B} \, d\tau \, . \tag{1}$$

In the resulting equilibria, the fluid velocity is zero and the magnetic field satisfies

$$\nabla \times \mathbf{B} = \lambda \mathbf{B} \tag{2}$$

where λ is a constant.

The success of this theory has led to several attempts to extend it to describe equilibria with plasma flow.[3,4,5] These extensions involve several different constraints. The object of this note is to show how these different constraints can be unified and to comment on their validity in the light of the original theory.

II. Invariants

If any vector field $\mathbf{P}$ satisfies

$$\frac{\partial \mathbf{P}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{P}) = \nabla \chi \tag{3}$$

and $\mathbf{v}$ is subject to arbitrary variations (e.g. representing a turbulent velocity), then the variations of $\mathbf{P}$ are also arbitrary, except for the constraints that

$$\oint_C \frac{\partial \mathbf{P}}{\partial t} \cdot d\ell = 0 \tag{4}$$

where C is any closed field line of $\nabla \times \mathbf{P}$. An equivalent expression of these constraints is that the helicity

$$K_i = \int_{C_i} \mathbf{P} \cdot \nabla \times \mathbf{P} \, d\tau \tag{5}$$

in any closed flux tube C_i is constant.

The essential feature of relaxation theory is that in the presence of small dissipative effects, such as resistivity, field lines cannot be identified (lines break and rejoin) so that only the *total* helicity defined by

$$K_0 \int_{V_0} \mathbf{P} \cdot \nabla \times \mathbf{P} \, d\tau \tag{6}$$

(where V_0 is the volume of the system) remains a good invariant. (Here and elsewhere we assume appropriate boundary conditions, e.g. that $\mathbf{v} = 0$.)

For a plasma of cold ions and electrons with negligible interaction we have, for both species,

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} \right) - \mathbf{v} \cdot (\nabla \times \mathbf{v}) = \frac{-e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{7}$$

where

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} , \qquad \mathbf{B} = \nabla \times \mathbf{A} . \tag{8}$$

These equations can be re-arranged to give

$$\frac{\partial \mathbf{P}_e}{\partial t} + \mathbf{v}_e \times (\nabla \times \mathbf{P}_e) = \nabla \lambda_e \tag{9a}$$

$$\frac{\partial \mathbf{P}_i}{\partial t} + \mathbf{v}_i \times (\nabla \times \mathbf{P}_i) = \nabla \lambda_i \tag{9b}$$

where

$$\mathbf{P}_i = \left(\frac{M \mathbf{v}_i}{e} + \mathbf{A} \right) \quad \text{and} \quad \mathbf{P}_e = \left(\frac{m \mathbf{v}_e}{e} - \mathbf{A} \right) \tag{10}$$

with M and m the ion and electron mass. Consequently, for a cold plasma, there are two self-helicities which are invariant;

$$\int_{V_0} \mathbf{P}_e \cdot \nabla \times \mathbf{P}_e \quad \text{and} \quad \int_{V_0} \mathbf{P}_i \cdot \nabla \times \mathbf{P}_i \tag{11}$$

(in general there will be an invariant self-helicity for each independent species.)

When plasma pressure is not negligible, Eq. (7) contains an extra term $\nabla p/\rho$. This can be incorporated into Eqs. (9), leaving the invariants unaffected, if $p = p(\rho)$. This assumption covers i) uniform incompressible plasma (ρ = constant) — as considered by Turner,[5] ii) isentropic plasma ($p\rho^{\gamma}$ = constant) as considered by Sudan,[3] iii) uniform temperature plasma ($p = \rho T$) as considered by Finn and Anderson.[4]

In the limit of small electron mass the electron invariant becomes the usual magnetic helicity

$$I_0 = \int \mathbf{A} \cdot \mathbf{B}\, d\tau \tag{12}$$

and the ion invariant becomes

$$I_1 = \int \left(\frac{M\mathbf{v}}{e} + \mathbf{A}\right) \cdot \nabla \times \left(\frac{M\mathbf{v}}{e} + \mathbf{A}\right) \tag{13}$$

where $\mathbf{v}$ is now the plasma velocity. Alternatively, we may subtract (12) from (13) to yield an invariant

$$I_2 = \int \mathbf{v} \cdot \left(\mathbf{B} + \frac{M}{2e} \nabla \times \mathbf{v}\right) d\tau \ . \tag{14}$$

A further approximation, neglecting the ion mass, is equivalent to neglecting the Hall term in the extended Ohms law

$$\mathbf{E} + (\mathbf{v} \times \mathbf{B}) = (\mathbf{j} \times \mathbf{B})/n\, e \tag{15}$$

and yields the so-called 'cross helicity' invariant

$$I_3 = \int (\mathbf{v} \cdot \mathbf{B}) d\tau \ . \tag{16}$$

However the term 'cross-helicity' now seems inappropriate since the basic invariants (11) are both self-helicities. We will refer to (I_1, I_2, I_3) as generalized helicities and retain the term magnetic helicity for the original quantity (I_0).

III. Relaxation

Before the above invariants can be used to find relaxed states, one must consider the role of density and pressure. We will continue to neglect pressure, again effectively considering a cold plasma, and we treat the density in the same manner as the helicity. That is; in an ideal plasma the mass on each closed flux tube would be constant, but the loss of identity among different flux tubes inherent in relaxation leaves only the total mass as an invariant — just as it left only the total helicity invariant. Thus we are led to consider relaxation to a minimum of the energy

$$E = \int \left(\rho \frac{v^2}{2} + \frac{B^2}{2} \right) d\tau \tag{17}$$

subject to the constancy of

$$M = \int \rho \, d\tau \tag{18a}$$

$$I_0 = \int \mathbf{A} \cdot \mathbf{B} \, d\tau \tag{18b}$$

$$I_1 = \int \left(\frac{M\mathbf{v}}{e} + \mathbf{A} \right) \cdot \nabla \times \left(\frac{M\mathbf{v}}{e} + \mathbf{A} \right) d\tau \tag{18c}$$

under variations $\delta\mathbf{A}, \delta\mathbf{v}$ and $\delta\rho$.

The resulting equations are

$$\nabla \times \mathbf{B} - \lambda \mathbf{B} + \frac{\mu M}{e} \nabla \times \mathbf{v} = 0 \tag{19a}$$

$$\frac{e}{M} \rho \mathbf{v} - \mu \mathbf{B} + \frac{\mu M}{e} \nabla \times \mathbf{v} = 0 \tag{19b}$$

$$v^2 = \text{constant} \tag{19c}$$

where λ and μ are Lagrange multipliers.

Equations (19) are precisely the zero pressure limit of those used by Sudan.[3] (However, his constant density solution does not exist at zero pressure.)

If we assume that the Larmor radius is small compared to the equilibrium scale length, then $\nabla \times \mathbf{v}$ may be neglected in (19b) (but not in (19a) as λ/μ is not yet determined). Then

($\mu' = \mu M/e$)

$$\nabla \times \mathbf{B} - \lambda \mathbf{B} + \mu' \nabla \times \mathbf{v} = 0 \tag{20a}$$

$$\rho \mathbf{v} - \mu' \mathbf{B} = 0 \tag{20b}$$

$$v^2 = \text{ constant }. \tag{20c}$$

Equations (20) are the zero pressure limit of those used by Finn and Antonsen;[4] they represent small Larmor radius equilibria in which kinetic and magnetic energies are comparable ($nMv^2 \sim B^2$). Of course, if the kinetic energy is negligible the magnetic field reverts to the usual form (2).

Finally, we note that since Turner[5] did not use a mass conservation constraint, but instead assumed constant density, he had only two equations for equilibrium, corresponding to (20a) and (20b).

IV. Discussion

We have seen that the different invariants invoked to obtain relaxed states with flow[3,4,5] are special cases of the two basic self-helicity invariants

$$\int \mathbf{P}_i \cdot \nabla \times \mathbf{P}_i \quad \text{and} \quad \int \mathbf{P}_e \cdot \nabla \times \mathbf{P}_e \tag{21}$$

We now consider the validity of these invariants compared to that of the original magnetic helicity. In this context it is important to recall that relaxation theory is not a mathematical variational principle (such as Hamilton's principle) involving virtual displacements. It refers to real displacements in a turbulent system. Then there are two complementary aspects to be considered.

i) Given that the ideal system has an infinity of invariants, why are the two global helicities singled out in a slightly dissipative system?

ii) Given that dissipation is present, why can the global helicities be regarded as constant?

Point i) has already been mentioned; the distinction between global helicities and other invariants is that the latter are significant only when the field lines with which they are associated can be identified. (They cannot, for example, be identified solely from knowledge of the field $\mathbf{P}$). The global helicities require no such identification and can be computed from the field $\mathbf{P}$ alone. Consequently they remain valid even when the field lines can no longer be identified.

Point ii) requires that the global helicities should decay more slowly than the energy. One aspect of this concerns the role of turbulent fluctuations at short wavelength. In a resistive plasma the rate of change of the original magnetic helicity I_0 is

$$\frac{dI_0}{dt} \sim i \int \eta \mathbf{k} \cdot \mathbf{B}_i \times \mathbf{B}_k^* \tag{22}$$

where $\mathbf{B}_k$ are the fourier components of the fluctuating field, whereas the rate of change of energy is

$$\frac{dE}{dt} \sim \int \eta k^2 |\mathbf{B}_k|^2 \ . \tag{23}$$

Consequently energy dissipation remains finite at scale lengths such that $\eta k^2 \sim \mathcal{O}(1)$ whereas helicity dissipation is only $\mathcal{O}(\eta^{1/2})$ at this scale. (The small scale fluctuations discussed here may be related to field line reconnection through the condition $\mathbf{B} \cdot \nabla\lambda = 0$ for a stationary plasma (i.e. λ must be uniform along the field). When two field lines having different values of λ connect, the subsequent adjustment is brought about by short wavelength, Alfvén-wave, motion. Alternatively, small scale fluctuations may arise as thin current sheets, either regular, as in the model of sawtooth reconnection,[6] or random, as in spatially intermittent turbulence. The fact that the argument is relevant to any small scale magnetic fluctuations is presumably responsible for the robustness of the concept of relaxation.)

Now let us consider the situation for the generalized invariants I_1, I_2, I_3. The argument that it is impossible to identify flux tubes in the presence of dissipation, so that only the global helicities are important, is still valid. However the argument for the constancy of the global helicities relative to energy is not. Firstly, the generalized helicities are not conserved even in a perfectly conducting viscous plasma. Secondly, the effect of small scale fluctuations on the generalized helicities is quite different to their effect on the magnetic helicity. For example, the rate of change of the cross-helicity I_3 is

$$\frac{dI_3}{dt} \sim \int (\eta + \mu) k^2 \left(\mathbf{v}_k \cdot \mathbf{B}_k^* + \mathbf{v}_k^* \mathbf{B}_k\right) \tag{24}$$

while the rate of change of energy is

$$\frac{dE}{dt} \sim \int \left(\eta k^2 |\mathbf{B}_k|^2 + \mu k^2 \rho |\mathbf{v}_k|^2\right) \ . \tag{25}$$

Consequently, in so far as the turbulence involves fluctuations in both velocity and magnetic field, the decay of cross-helicity and energy both remain finite at scales $\eta k^2 \sim \mathcal{O}(1)$.

V. Conslusions

The various generalized helicities, used to replace the magnetic helicity in relaxation theory, are special cases of two self-helicities — one related to electrons and one to ions. The assumption that the plasma tends to a state of minimum energy subject to invariance of these generalized helicities, leads to states with plasma flow which have been derived earlier.[3,4,5]

However, while there are strong arguments for the unique invariance of the *magnetic* helicity in a turbulent resistive plasma, these arguments do not appear to apply to the invariance of the *generalized* helicities. Consequently, while we can expect, and experiment confirms, that turbulent plasmas frequently relax to the stationary states derived using the magnetic helicity, it is only in special circumstances (perhaps involving injection of high-energy beams) that we could expect to observe relaxation to the states with plasma flow that are derived using the generalized helicities.

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