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Fractal Orbits and Passive Transport in Scaling Turbulence

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Abstract

The passive advection in time-dependent 2D and 3D flows is considered. The velocity field $\mathbf{v}(\mathbf{r}, t)$ is assumed to have power spectra of both scales and frequencies in respectively wide inertial ranges. *The quasi-linear limit* of passive transport is introduced and studied analytically using the method of *the virtual separation of scales.* The fractal dimension of particle trajectories and the propagation rates of an impurity are calculated.

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The scaling nature of fluid and plasma turbulence results in a self-similar behavior of Lagrangian trajectories and interfaces. Being a concise characteristic of self-similarity, the concept of fractal dimension [1] is widely used in the description of fluid turbulence [2, 3, 4, 5, 6] and begins entering the plasma physics [7]. In the present paper we shall focus on fractal description of passive tracers in incompressible time-dependent random flows. A variety of important problems can be cast into the form of a passive advection,

$$d\mathbf{r}/dt = \mathbf{v}(\mathbf{r}, t), \tag{1}$$

where $\mathbf{v}(\mathbf{r}, t) = \nabla \times \boldsymbol{\psi}(\mathbf{r}, t)$ is a prescribed velocity field featuring one or another kind of randomness, $\boldsymbol{\psi}(\mathbf{r}, t)$ is the vector potential, and $\mathbf{r}$ denotes the position of fluid (or passive pollutant) particle. In the case of plasma transport of particular interest is the two-dimensional Hamiltonian turbulence with $\boldsymbol{\psi}(\mathbf{r}, t) = \psi(x, y, t)\hat{\mathbf{z}}$. Here the particle motion is governed by the Hamiltonian (stream function) $\psi(x, y, t)$. The strong confining magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ makes the cross-field plasma transport effectively two-dimensional, even for a 3D plasma turbulence. In general case of electromagnetic perturbations, ψ is a combination of the electric potential and the parallel component of the magnetic vector potential [9]. No special remarks are necessary to confirm the appropriateness of the 2D approximation for the description of the chaotic transport in geophysical fluid dynamics.

We study time-dependent flows with the vector potential $\boldsymbol{\psi}(x, y, t)$ being a random function with a power Kolmogorov-like spectrum $|\boldsymbol{\psi}(\mathbf{k}, \omega)|^2 \propto k^{-\gamma}$ and the model dispersion relation $\omega(\mathbf{k}) \propto k^{\delta}$. We use the previously introduced method of the virtual separation of scales [10] to calculate the rate of the propagation of a passive impurity, as well as the fractal dimension of the fluid particle orbits. In general, the fractal dimension can serve as a measure of the Lagrangian chaos and it is shown to take different values for different regimes of chaotic advection. In two dimensions, the problem considered here is similar to the one studied in Ref. [6] (where the authors put $\delta = 1$). Our results agree with the scalings found in [6] numerically, however, our analytical arguments are quite different from those of Ref. [6]. We also discuss the interconnection of our results with those obtained using a continuum nonlocal advection-diffusion theory [8].

It is convenient to rewrite the Fourier spectrum of ψ in terms of the λ-components,

$$\psi(\mathbf{r},t)=\sum_{\lambda}\psi_{\lambda}(\mathbf{r},t),\quad \psi_{\lambda}(\mathbf{r},t)=\int_{1/2<|\mathbf{k}|\lambda<1}\psi(\mathbf{k},t)e^{i\mathbf{kr}}d\mathbf{k}. \tag{2}$$

Here $\psi_{\lambda}(\mathbf{r},t)$ is the λ-component of the vector potential and the sum in Eq. (2) is taken over the geometrical progression of scales

$$\lambda_i=\lambda_0,\mu\lambda_0,\mu^2\lambda_0,...,\lambda_m, \tag{3}$$

where the multiplier $\mu=2$ and $[\lambda_0,\lambda_m]$ is the inertial range ($\lambda_0\ll\lambda_m$), beyond which ψ_λ is assumed to be negligible. The λ-component $\psi_{\lambda}(\mathbf{r},t)$ possesses a single characteristic spatial scale λ. If the complex phases of $\psi(\mathbf{k},t)$ are random, as one expects for turbulent flows [4], then the Fourier spectrum $|\psi(\mathbf{k},t)|^2\propto k^{-\gamma}$ yields the λ-spectrum

$$\psi_\lambda\equiv[\psi_{\lambda}(\mathbf{r},t)]_{\mathrm{rms}}=\psi_0(\lambda/\lambda_0)^H,\quad H=(\gamma-2)/2. \tag{4}$$

One can analogously write the λ-spectrum of the velocity: $v_\lambda=v_0(\lambda/\lambda_0)^{H-1}$.

General ideas of turbulent cascade imply the existence of a single characteristic frequency ω_λ of the time dependence of $\psi_{\lambda}(\mathbf{r},t)$. Suppose the power scaling of the form $\omega_\lambda=\omega_0(\lambda/\lambda_0)^{-\delta}$. Unlike the dispersion relation $\omega(\mathbf{k})$, the characteristic frequency ω_λ is more universal since it naturally accounts for the nonlinear resonance broadening.

Let us first consider the two-dimensional case $\psi(\mathbf{r},t)=\psi(x,y,t)\hat{\mathbf{z}}$. In the hypothetical case of a stationary (integrable) Hamiltonian (or approximately for $\omega_\lambda\ll v_\lambda/\lambda$), the (x,y)-trajectories of fluid particles coincide with the contours of $\psi(x,y)$. Any contour of a random function is almost surely (i.e. with the probability one) closed [11], hence for $\omega_0=0$ there is no turbulent diffusion. If, however, one deals with a contour of the size $a\gg\lambda_0$ then on the scales $\lambda \in [\lambda_0, a]$ the trajectory behaves very sinuously, and is indeed a fractal curve with the Hausdorff dimension [10]

$$d_h = (10 - 3H)/7, \quad -3/4 < H < 1. \tag{5}$$

It is pointed out that the fractal dimension (5) cannot be calculated using the Mandelbrot formula for a cross-section [1] $d_{hf} = d' - 1$, where $d' = 3 - H$ is the fractal dimension of the 3D graph of $\psi(x, y)$. (Note that at $0 < H < 1$ the square of the λ-component of $\psi(x, y)$ is essentially the same as the delta-variance [1], hence in the scaling range $[\lambda_0, \lambda_m]$ $\psi(x, y)$ is a fractional Brownian plane-to-line function.) The horizontal cross-section $\psi(x, y) = h$ is a multi-connected fractal set with the self-similar range $[\lambda_0, a_h]$, where a_h is the maximum size of the isolines [10]. A separate contour of $\psi(x, y)$ (the particle orbit) is a subset of this set and hence has a smaller fractal dimension. One can see that at $H < 1$ $d_h < d_{hf} = 2 - H$. The most general approach to studying the statistics of isosets of a random function is the continuum percolation theory [12, 11, 10].

Consider now the essentially nonstationary limit $\omega_\lambda \geq v_\lambda/\lambda$. Suppose for simplicity $\omega_0 \geq v_0/\lambda_0$ and $\delta \leq 2 - H$, so that on every scale λ the flow is changed not slower than particles pass the distance λ. Analogously to the plasma transport notations [7], the limit $\omega_\lambda \gg v_\lambda/\lambda$ can be referred to as *the quasi-linear limit*, since, to a first approximation, one can neglect the dependence of the RHS of Eq. (1) on $\mathbf{r}$, and the equation of motion becomes trivially linear (but not as trivially tractable). In the quasi-linear limit, the particle trajectories have nothing to do with the level lines of ψ. In particular, the 2D-orbits exhibit multiple self-intersections ($x(t)$ and $y(t)$ behave independently) and, as shown below, have larger fractal dimension than (5). The following analysis is valid for both two- and three-dimensional turbulence.

The technique of the virtual separation of scales [10] means considering the flow of the form (2)–(3), however with the parameter $\mu \gg 1$ in place of $\mu = 2$. The final results are obtained as the marginal applicability ($\mu = 2$) of the artificial approximation of the strongly separated scales λ_i. So let us take $\mu \gg 1$ and follow the particle motion with progressive time. Let H be less than one. Then on the time scales $0 < t < \omega_{\lambda_0}^{-1}$ the displacement is a linear function of time dominated by the short-scale velocity: $r(t) \approx v_0 t$. At $\omega_{\lambda_0}^{-1} \ll t \ll \omega_{\lambda_1}^{-1}$, the λ_0-component of the velocity undergoes fast oscillations, hence leading to the diffusive motion $r_d(t) \approx (D_{\lambda_0} t)^{1/2}$, $D_{\lambda_0} = v_0^2/\omega_0$. This diffusion is accompanied by the linear convective transfer $r_c(t) \approx v_{\lambda_1} t$. The ratio of these two displacements by the time $t = \omega_{\lambda_1}^{-1}$ is

$$r_d(t)/r_c(t) \approx (D_{\lambda_0}/D_{\lambda_1})^{1/2}, \quad D_\lambda \equiv v_\lambda^2/\omega_\lambda.$$

If $D_\lambda \propto \lambda^{\delta+2H-2}$ is decreased with increasing λ, viz. $\delta < 2(1-H)$, then the diffusion D_{λ_0} predominates the convection v_{λ_1}. This argument can be repeated for longer scales. The conclusion (which is also valid in the limit $\mu \to 2$) is that in this case the particle motion is dominated by the shortest scale flow component; the manner of transport is diffusive with the coefficient of the order of D_{λ_0}. Thus the fractal dimension of particle orbits $d = 2$ meaning an ultimate degree of Lagrangian chaos.

In the opposite case $\delta > 2(1-H)$, for $\omega_{\lambda_0}^{-1} \ll t \ll \omega_{\lambda_1}^{-1}$ the convection due to v_{λ_1} is more important than the diffusion D_{λ_0}, hence on this time interval one can neglect the contribution of $\mathbf{v}_{\lambda_0}(\mathbf{r}, t)$ to the particle motion. Generally, for $\omega_{\lambda_{i-1}}^{-1} \ll t \ll \omega_{\lambda_i}^{-1}$ the displacement $r(t)$ behaves as though the λ-components of the flow with $\lambda \leq \lambda_{i-1}$ were "switched off" (see Fig. 1). As a result, in the limit of a smooth spectrum ($\mu = 2$), which corresponds to "the rounding of the corners" in Fig. 1, we obtain the power dependence $r(t) \approx v_{\lambda(t)} t$, $\omega_{\lambda(t)} t \equiv 1$, or

$$r(t) \approx \frac{v_0}{\omega_0}(\omega_0 t)^\xi, \quad \xi = \frac{\delta + H - 1}{\delta}, \quad \frac{v_0}{\omega_0} < r < \frac{v_{\lambda_m}}{\omega_{\lambda_m}}. \tag{6}$$

Under the assumed conditions, $2 - 2H \leq \delta < 2 - H$, we have $1/2 \leq \xi < 1$, so that the Lagrangian motion is superdiffusive. In the considered regime, the Cartesian coordinates of the fluid particle $r_\alpha(t)$ behave similarly to the displacement (6) and can be shown to be independent fractional (persistent) Brownian functions. Hence the fractal dimension of the trajectory $\{\mathbf{r}(t)\}$ equals [1]

$$d = 1/\xi = \delta/(\delta + H - 1), \quad 2(1-H) \le \delta < 2 - H, \quad 0 < H < 1. \tag{7}$$

It is easy to see that the fractal dimension (7) is greater than the dimension (5) for the stationary 2D flow thus assuming a "more chaotic" motion. Note also that $d \le 2$ and the equality is achieved for $\delta = 2(1-H)$, which corresponds to the standard Brownian motion. For the case $\delta = 1$, $H = (\gamma - 2)/2$, Eq. (7) yields $d = 2/(\gamma - 2)$, in accord with the results of Ref. [6] obtained from the numerical calculation of the Fourier spectra of $x(t)$, $y(t)$ for different γ $(3 < \gamma < 4)$.

One can also be interested in the manner of motion beyond the scaling range (6). Note that for $\delta < 2 - H$ the average particle displacement $r(\omega_{\lambda_m}^{-1}) \approx r_m \equiv v_{\lambda_m}/\omega_{\lambda_m}$ is still much less than λ_m, the upper turbulent scale, which is typically defined by fluid boundaries. Bearing in mind the above argument of separated scales, the answer is simple: at $t \gg \omega_{\lambda_m}^{-1}$ one has a usual Brownian motion with the diffusion coefficient D_{λ_m}. Thus in the second scaling range $[r_m, \lambda_m]$ (and also for $\lambda > \lambda_m$ if there are no boundaries), the Lagrangian orbits have the fractal dimension $d = 2$.

Our next remark concerns the extension of the above results to the stationary 3D flow. Koch and Brady [8] reported the displacement scaling, $r(t) \propto t^{2/(2+\gamma')}$, for the propagation of a passive tracer in a random steady flow with the velocity covariance $U(r) = \langle \mathbf{v}(\mathbf{r})\mathbf{v}(\mathbf{0})\rangle \propto r^{-\gamma'}$. This result is intriguingly similar to Eq. (6) if we would formally put $\delta = 2 - H$ and notice that at $1/2 < H < 1$ the correlation function $U(r)$ has a power scaling with $\gamma' = 2(1-H)$. The coincidence of the exponents, however different are the time-dependent and the stationary problem, is not accidental. Indeed, unlike the 2D situation, the steadiness of the flow in three dimensions does not generally imply the existence of any new integral of motion. So one can reasonably assume that in a generic case the stochastic walk of time-independent stream lines ("Lagrangian turbulence") is not subject to any topological restrictions. In particular, this means that the fractal dimension of a stream line in a single-scale flow, say $\mathbf{v}_\lambda(\mathbf{r})$, shall equal two. From the viewpoint of a tracer exploring the stream line, this implies the diffusion coefficient $D_\lambda \approx \lambda v_\lambda$, which is mathematically identical to the introduced above if we formally put $\omega_0 = v_0/\lambda_0$ and $\delta = 2 - H$. Hence with this ansatz, the previous (time-dependent) arguments apply to steady 3D flows as well. In two dimensions, the low-frequency limit, $\omega_\lambda \ll v_\lambda/\lambda$, is much more complicated due to the approximate conservation of $\psi(x, y, t)$ [11, 13].

The conclusions can be drawn as follows.

1. For a power spectrum of turbulence with $\delta \geq 2(1 - H)$, $0 < H < 1$, the propagation rate of a passive scalar is superdiffusive on intermediate scales, with the fractal dimension of particle orbits (7) lying in the range $1 < d < 2$. There is a second scaling range, which is associated with the long-wavelength cut-off λ_m of the turbulence spectrum, where standard Brownian turbulent diffusion ($D \approx D_{\lambda_m}$) is set-up. For $\delta < 2(1 - H)$,there is no superdiffusive scaling range and on all the scales $\lambda > \lambda_0$ a diffusive transport takes place with $D \approx D_{\lambda_0}$.

2. In 2D geometry, the details of relevant scaling laws depend crucially on the ratio $\lambda\omega_\lambda/v_\lambda$, with the quasi-linear regime $\lambda\omega_\lambda/v_\lambda \geq 1$ discussed above and the opposite limit requiring a percolation approach. In three dimensions, the quasi-linear approximation appears to be universally applicable to the investigation of passive transport, meaning the formal ansatz $\omega_\lambda \rightarrow \max(\omega_\lambda, v_\lambda/\lambda)$.

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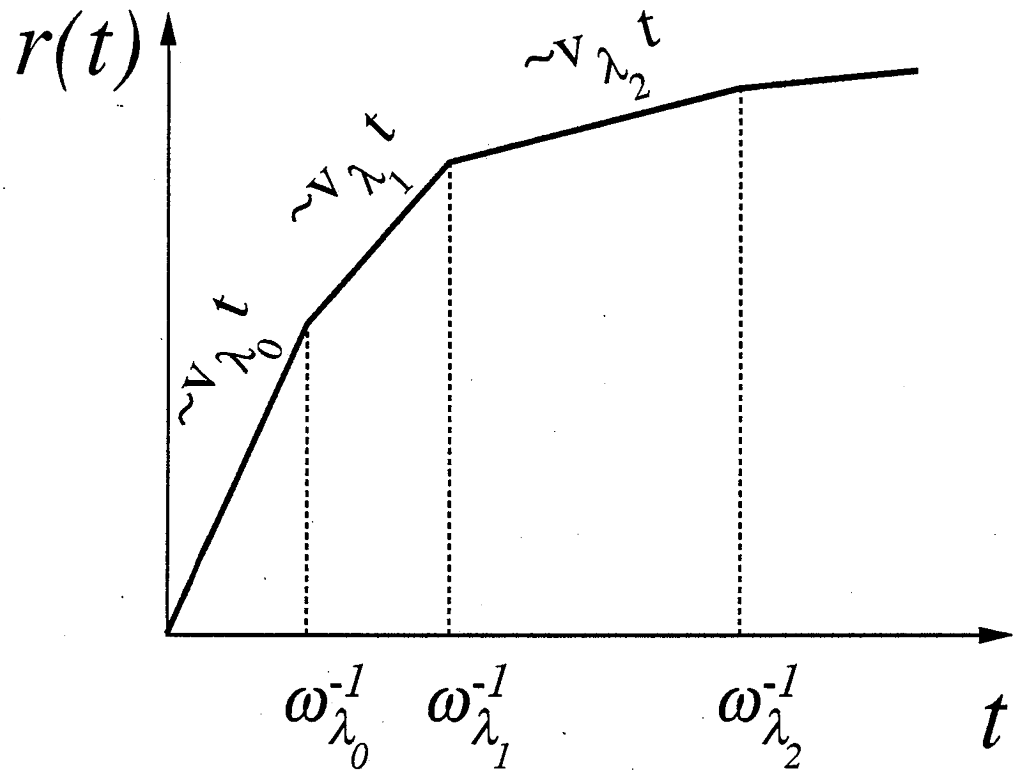

Figure 1. The dependence of fluid particle displacement on time.