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The theory of weakly nonlinear waves in a homogeneous, conservative medium can often be developed in terms of a Hamiltonian which retains only cubic nonlinearities. This approach has been stressed by Ken Watson in studies of the interactions among internal waves[1,2] and between surface and internal waves[3] in the ocean.

The process of relaxation of a single wave mode in a bath of ambient modes can be described by a model of this type: The test wave Hamiltonian. This model was proposed by Watson[3] for the study of the interaction between a single internal wave and a spectrum of surface waves--and thus as a mechanism for the transfer of energy from the ocean surface to its interior.

In this model the wave actions are represented by $\{J_T; J_i, J_i'\ i = 1,2,\ldots M\}$ where J_T represents the test

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wave and the M pairs (J_i, J_i') are ambient waves which form M interacting triads with J_T. The subscripts refer to wavenumbers and interactions are allowed only when

$$\underset{\sim}{k}_T = \underset{\sim}{k}_i \pm \underset{\sim}{k}_i' \quad . \tag{1}$$

We refer to the two possible triads in Eq. (1) as sum and difference interactions.

The test wave Hamiltonian may be written

$$\mathcal{H} = \omega_T J_T + \sum_{i=1}^{M} (\omega_i J_i + \omega_i' J_i') - \sum_{n=1}^{M} \varepsilon_n^{\pm} \sqrt{J_T J_n J_n'} \; \cos(\theta_n \pm \theta_n' - \theta_T) . \tag{2}$$

Here the θ's represent wave phases, and the ε's are coupling coefficients which are generally functions of the three wavenumbers. To linear order $(\varepsilon \to 0)$ the phases evolve as the linear frequency: $\dot{\theta} = \omega + O(\varepsilon)$, where ω of course depends upon $\underset{\sim}{k}$.

Numerical integration of the equations of motion for the Hamiltonian (2) indicates that it is completely integrable.[4] The evidence is threefold. First, two orbits initially close together separate only linearly in time (Lyapunov exponents are zero). In a nonintegrable system the separation is typically exponential. Second, the Poincaré surface of section for the two triad (M=2) system (which is, in this case, two-dimensional) apparently consists of smooth level curves indicating the existence of an additional integral. Finally, a quantitative test of the smoothness of the curves (the residue method of J. Greene) shows that to double precision accuracy they are indeed smooth. A discussion of these issues is given in greater detail in Ref. 4 and in my dissertation[5] (see, however, Ref. 8).

The numerical evidence presented above seems insensitive to the parameters of the Hamiltonian. I am therefore lead to the conjecture that Eq. (2) is integrable for arbitrary ω's and ε's . I should note, however, that Ref. 4 deals only with the difference interactions in Eq. (2). As we will see below, the Hamiltonian with only sum interactions also appears integrable; however, when there is a mixture of sum and difference interactions the behavior is unknown.

Integrability in the sense of Liouville[6] means the existence of $N=2M+1$ integrals--that is, one integral for each degree of freedom. I use the term integrals for a set of functions on phase space which are functionally independent and in involution (all Poisson brackets zero). Constants of motion are merely time independent ($\{F,\mathcal{H}\} = 0$) functions. Liouville's theorem on integrability shows that once the N integrals are known the equations of motion can be integrated by quadrature. Essentially, the integrals can be used as canonical momenta, and the Hamiltonian expressed in terms of these momenta is independent of the conjugate variables.

For the test wave Hamiltonian, we can immediately find M+2 of the integrals. The first is the Hamiltonian itself and the others are

$$I_n = J_n - sJ_n' \,, \qquad n = 1,2,\ldots M,$$

$$I_T = J_T + \sum_{n=1}^{M} J_n \,. \tag{3}$$

Here, $s = +1(-1)$ for sum (difference) triads. These integrals are related to symmetries of $\mathcal{H}$: Each of the I_n is obtained by noting that the transformation

$$\theta_n \rightarrow \theta_n + \psi \,,$$

$$\theta_n' \rightarrow \theta_n' - s\psi \,,$$

leaves $\mathcal{H}$ invariant. The remaining integral results from the symmetry

$$\theta_T \rightarrow \theta_T + \psi \,,$$

$$\theta_n \rightarrow \theta_n + \psi \,, \text{ for } n = 1,2,\ldots M$$

To make further progress it is convenient to introduce the action-amplitude variables (see Watson's discussion in Ref. 1)

$$a_T = \sqrt{J_T}\, \exp^{-i\theta_T} \quad a_n = \sqrt{J_n}\, \exp^{-i\theta_n}, \; b_n = \sqrt{J_n'}\, \exp^{-i\theta_n'} \,. \tag{4}$$

These variables are not quite canonical, obeying the Poisson bracket relations

$$\{a_n^*, a_m\} = i\delta_{m,n} \qquad \{b_n^*, b_m\} = i\delta_{n,m} \qquad \{a_T^*, a_T\} = i \tag{5}$$

with all other brackets zero. If the Hamiltonian, Eq. (2), is written in terms of Eq. (4)

$$\mathcal{H} = \omega_T a_T^* a_T + \sum_{n=1}^{M} \left(\omega_n a_n^* a_n + \omega_n' b_n^* b_n\right) - \frac{1}{2}\sum_{n=1}^{M} \left\{ \begin{matrix} \varepsilon_n^+ a_T a_n^* b_n^* \\ \varepsilon_n^- a_T a_n^* b_n \end{matrix} + \text{complex conjugate} \right\} \tag{6}$$

then the equations of motion become

$$\dot{a}_n = \{a_n, \mathcal{H}\} = \{a_n, a_m^*\} \frac{\partial \mathcal{H}}{\partial a_m^*} = -i \frac{\partial \mathcal{H}}{\partial a_n^*}$$

or, more explicitly

$$\dot{a}_T = -i\omega_T a_T + \frac{i}{2}\sum_{n=1}^{M} \left\{ \begin{matrix} \varepsilon_n^{+*} a_n b_n \\ \varepsilon_n^{-*} a_n b_n^* \end{matrix} \right\}$$

$$\dot{a}_n = -i\omega_n a_n + \frac{i}{2} \left\{ \begin{matrix} \varepsilon_n^+ a_T b_n^* \\ \varepsilon_n^- a_T b_n \end{matrix} \right\}$$

$$\dot{b}_n = -i\omega_n' b_n + \frac{i}{2} \left\{ \begin{matrix} \varepsilon_n^+ a_T a_n^* \\ \varepsilon_n^{-*} a_T^* a_n \end{matrix} \right\} \tag{7}$$

Here we have allowed complex valued coupling coefficients, ε_n.

The integrals for Eq. (6) can be obtained for a special set of parameter values by generalizing a result of Hald.[7] Hald found constants of motion in addition to those in Eq. (3) for a system which is equivalent to our $M=2$ case if all the frequencies are zero and if the coupling coefficients are equal. For Hald's system these constants are quadratic in the amplitudes (a_n, b_n).

When the frequencies are nonzero, it turns out that the magnitudes of Hald's constants are time independent.[5] Special values of the parameters are required for this result:

$$\varepsilon_n^2 = \varepsilon^2 \quad ,$$

$$\Delta_n^{\pm} = \omega_n \pm \omega_n' - \omega_0 = \Delta \,. \tag{8}$$

Here ε is any complex constant and Δ is a real constant representing the resonance mismatch.

For the Hamiltonian, Eq. (6), with Eq. (8), the new constants are

$$|I_{ij}|^2 = \left\{ \begin{array}{l} \left|\varepsilon_i^+ a_i a_j^* - \varepsilon_j^+ b_i^* b_j\right|^2 \\ \left|\varepsilon_i^- a_i a_j^* + \varepsilon_j^- b_i b_j^*\right|^2 \end{array} \right\} \, , \; i,j = 1,2,\ldots M \,. \tag{9}$$

Here we require the triads labeled i and j to be either <u>both</u> sum or <u>both</u> difference interactions. From now on we consider only the cases where the triads are either <u>all sum</u> or <u>all difference</u> triads. The mixed case requires additional constants for integrability. Since $|I_{ij}|^2 = |I_{ji}|^2$ and $|I_{ii}|^2 = I_i^2$ there are $\frac{1}{2}M(M-1)$ new constants in Eq. (9).

These constants are not integrals, however, since they are not in involution. It is easy to see that the Poisson brackets $\{|I_{ij}|^2, I_0\}$ and $\{|I_{ij}|^2, I_n\}$ are zero. The bracket of two of the new constants is

$$\left\{|I_{ij}|^2, |I_{k\ell}|^2\right\} = i\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_\ell\Big\{\varepsilon_i\delta_{i,\ell}\left(I_{ij}I_{jk}I_{ki} - I_{ik}I_{kj}I_{ji}\right)$$

$$+ \varepsilon_j\delta_{j,k}\left(I_{ji}I_{i\ell}I_{\ell j} - I_{j\ell}I_{\ell i}I_{ij}\right)$$

$$+ \varepsilon_k\delta_{k,i}\left(I_{kj}I_{j\ell}I_{\ell k} - I_{k\ell}I_{\ell j}I_{jk}\right)$$

$$+ \varepsilon_\ell\delta_{\ell,j}\left(I_{\ell i}I_{ik}I_{k\ell} - I_{\ell k}I_{ki}I_{i\ell}\right)\Big\} \qquad (10)$$

It follows from Eq. (10) that if none of the four indices, i, j, k, ℓ are equal, the bracket is zero. Furthermore, it is easy to see that

$$\left\{\varepsilon_i|I_{ij}|^2 + \varepsilon_k|I_{kj}|^2, |I_{ik}|^2\right\} = 0 . \qquad (11)$$

The integrals can be constructed from linear combinations like that in Eq. (11) and we obtain

$$C_n = \sum_{j=1}^{n} \varepsilon_j|I_{jn+1}|^2 , \quad n = 1,2,\ldots,M-1 . \qquad (12)$$

These integrals are involutive

$$\{C_n, C_m\} = 0 .$$

Furthermore, it is easy to see that these integrals are all independent. Each of the I_n depends on a new variable J_n' making them independent from I_T and I_m, and similarly each of the C_n is a function of a new phase θ_{n+1}.

A complete set of 2M+1 integrals for the test wave Hamiltonian, Eq. (6), given the parameters of Eq. (8), and either all-sum or all-difference interactions is given by

$$\mathcal{H},\ I_T,\ I_1,\ I_2,\ \ldots,\ I_M,\ C_1,\ C_2,\ \ldots,\ C_{M-1}$$

As the simplest example, when $M = 2$, the new integral is

$$\frac{1}{\varepsilon_1} C_1 = |I_{12}|^2 = J_1 J_2 + J_1' J_2' - 2s\varepsilon_1^s \varepsilon_2^s \sqrt{J_1 J_2 J_1' J_2'}$$

$$\times \cos\left[\theta_1 - \theta_2 + s(\theta_1' - \theta_2')\right]$$

While these integrals prove integrability for the equal coupling coefficient, equal resonance mismatch model, numerical evidence indicates integrability more generally. As an example, I present a surface of section when $\Delta_1 \neq \Delta_2$, $\varepsilon_1^2 \neq \varepsilon_2^2$ with difference interactions (Fig. 1). There is no visible evidence of stochasticity. It remains a challenge to discover the integrals for this case, if indeed they exist.[8]

Finally we note that additional interactions will typically destroy the integrability of the test wave system. For example, when a triad involving J_1, J_1' and J_2 is added, the orbits become obviously stochastic.[9]

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Figure Captions

1. Surface of section for the two triad (M=2) difference Hamiltonian. The four known integrals have the values $H = -0.1$, $I_T = 2.01$, $I_1 = 1.1$, $I_2 = 1.6$. The coupling parameters are $\varepsilon_1 = -0.37$, $\varepsilon_2 = -1.0$ and $\Delta_1 = 0.2$, $\Delta_2 = 0.13$. The variables P, Q are defined by $\sqrt{2}\,a = P + iQ$. The surface shown is given by $Q_2 = 0$.

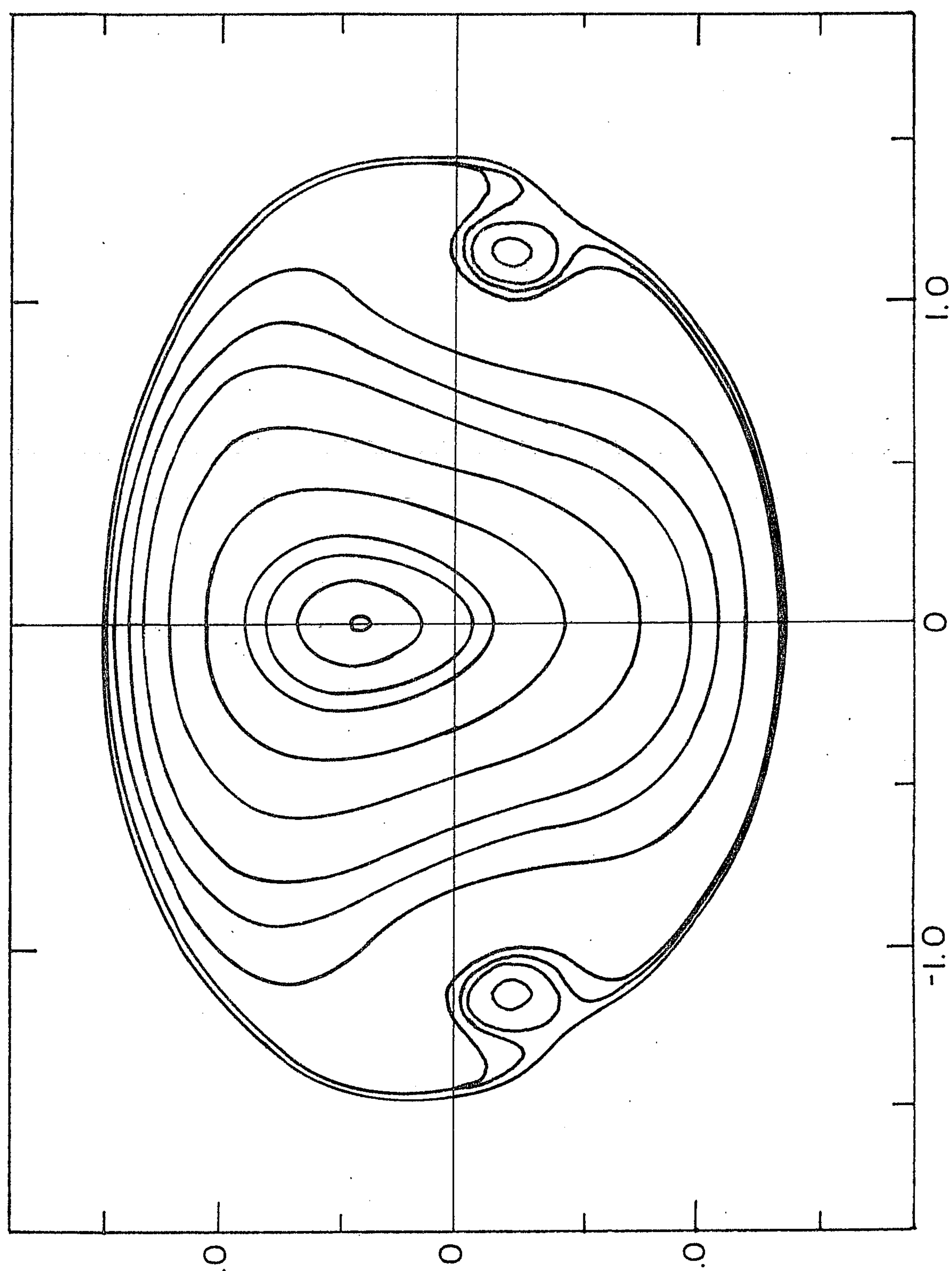

Fig. 1