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Part II. 2D Transport of Passive Scalar

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Statistical Geometry of Multiscale Isolines. Part II. 2D Transport of Passive Scalar

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Abstract

The propagation of a passive tracer in a vorticous incompressible fluid with a small background molecular diffusivity D_0 is usually asymptotically diffusional with an effective diffusivity D_{eff}, which is much greater than D_0. The convective-diffusive transport in a two-dimensional steady flow $\mathbf{v} = \nabla\psi(x,y) \times \hat{\mathbf{z}}$ admits an approach, based on the statistical geometry of stream-lines (= isolines of the stream-function $\psi(x,y)$). This kind of analysis was completed previously [1,2] for a zero-mean, random flow with a single characteristic space scale λ_0 using the percolation theory. The result $D_{\text{eff}} \approx \psi_0^{10/13} D_0^{3/13}$, for $\psi_0 = \langle|\psi|\rangle \gg D_0$, was expressed in terms of 2D percolation exponents. In the present paper, this approach is extended to a random multiscale flow with spectral components $\psi_\lambda \propto \lambda^H$ in a wide scaling range $\lambda_0 < \lambda < \lambda_m$. Using the stream-lines analysis based on the method of the virtual separation of scales, which is

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introduced in a companion paper [3], the expressions for the effective diffusivity D_{eff} and the corresponding mixing length for various flow spectra are obtained.

I. Introduction

The convective-diffusive transport problem of a passive scalar, described by the equation

$$\partial n/\partial t + \mathbf{v}\nabla n = D_0 \nabla^2 n \ , \tag{1}$$

has numerous applications in hydrodynamics, plasma, and solid state physics [1,2,4-10]. Here n denotes the density of an impurity, $\mathbf{v}(\mathbf{r},t) = \nabla \times \boldsymbol{\psi}(\mathbf{r},t)$ is an incompressible velocity field, and D_0 represents the background (molecular) diffusion coefficient. The frame of reference is chosen so that on the average the fluid rests: $\langle \mathbf{v}(\mathbf{r},t)\rangle = 0$. According to the Taylor postulate [4], for a turbulent flow, the asymptotic behavior of the impurity propagation is assumed to be diffusional. This means that one can average Eq. (1) over space and time to obtain

$$\partial \langle n \rangle / \partial t = D_{\text{eff}} \, \nabla^2 \langle n \rangle \ , \tag{2}$$

with $\langle n \rangle$ being the space-time averaged admixture density. In the presence of the background diffusion D_0, a random or turbulent nature of the velocity field is not necessary for the effective diffusion D_{eff} to be established. There exists a proof [11] that a sufficient condition for an asymptotically diffusional walk is the boundedness of the flow vector potential: $|\boldsymbol{\psi}(\mathbf{r},t)| \le \psi_{\text{max}}$. In this case one has [5,11] $D_0 \le D_{\text{eff}} \le D_0 + \psi^2_{\text{max}}/D_0$. The condition of the boundedness of $\boldsymbol{\psi}$ does not seem too stringent. For example, Refs. [9,10] reported a superdiffusional propagation of an admixture, for 2D flows with $\langle \mathbf{v}(x,y)\rangle = 0$ but with unbounded stream-functions $\psi(x,y)$ (in a 2D-geometry, $\boldsymbol{\psi}(\mathbf{r}) = \psi(x,y)\hat{\mathbf{z}}$).

It should be emphasized that, if valid, Eq. (2) describes the passive scalar behavior only asymptotically, when $\langle n \rangle$ varies sufficiently slowly both in space and time. Thus, we introduce the diffusive mixing, or correlation length ξ_m and mixing time τ_m: the averaging in (2) is to be made over such a spatial scale a that $\xi_m \ll a \ll R$, R being the characteristic scale for $\langle n \rangle$. A similar restriction is imposed on the time-average.

Along with the general space-time averaging of Eq. (1) with the assumed result (2), the problem of the passive transport may also be stated in two particular ways. The first, *spatial statement* describes a boundary value problem for the admixture density n. In two dimensions, one may consider the time-average admixture flux J_{12} between two opposite (longer) sides of a rectangle $a \times A$ ($a \ll A$), when the constant values of the density, n_1 and n_2, are maintained on these sides (see Fig. 1). Due to the linearity of Eq. (1), the flux J_{12} is directly proportional to the density difference and the distance A, so that the ratio

$$D(a) = aJ_{12}/A(n_1 - n_2) \tag{3}$$

depends only on the size a. To emphasize this dependence, in what follows, we will refer to the transport between the opposite sides of the box as *the box diffusion*, with a similar terminology for the quantity (3). If there exists such a size, or mixing length, ξ_m that for $a \gg \xi_m$ the box diffusion coefficient $D(a)$ tends to a constant, then the constant is the effective diffusion D_{eff}. The spatial problem was studies in Refs. [7,8] where the mixing length was simply equal to the period of the flow pattern.

The second, *temporal statement* of the problem considers the evolution of the initial condition $n(\mathbf{r}, 0) = \delta(\mathbf{r} - \mathbf{r}_0)$ due to equation (1). Here, the average square displacement is calculated to define the quantity

$$D(t) = (1/4t) \int (\mathbf{r} - \mathbf{r}_0)^2 n(\mathbf{r}, t) d^2\mathbf{r} \ . \tag{4}$$

If there exists such a time τ_m that for $t \gg \tau_m$ $D(t)$ saturates to a constant, then the constant is the effective diffusion D_{eff} and τ_m is the mixing time. The temporal problem can be equivalently formulated in terms of particle orbits described by

$$d\mathbf{r}/dt = \mathbf{v}(\mathbf{r}, t) + \mathbf{v}_D(t) \ , \tag{5}$$

where the term $\mathbf{v}_D(t)$ corresponds to non-correlated random kicks, which produce the background diffusion D_0. The correlation function of $\mathbf{v}_D(t)$ is given by

$$\langle \mathbf{v}_D(t)\mathbf{v}_D(t)\rangle = 4D_0\delta(t-t') \ . \tag{6}$$

The representation of Eq. (5) emphasizes the intrinsic nonlinearity of the problem involved. For a random velocity field, one may speak about "a random nonlinearity", since the function $\mathbf{v}(\mathbf{r})$ is a random nonlinear function. As generic tools of linear analysis prove helpless for the calculation of the effective transport, the basic equation (1) may be regarded as "false linear".

The temporal statement in the form of Eq. (5) is usually used in computer simulations [7,1] where the square particle displacement is averaged over the ensemble of random kicks $\mathbf{v}_D(t)$ or, equivalently, the ensemble of particles. In Ref. [11] a mixed analytical approach was taken: The authors considered the temporal problem with the initial condition $n(\mathbf{r},0) = \delta(\mathbf{r}-\mathbf{r}_0)$ and spatially averaged the quantity (4) over $\mathbf{r}_0$.

In the absence of a flow, $\mathbf{v}\equiv 0$, the spatial and the temporal problems are known to be equivalent, with $D_{\text{eff}} = D_0$ and the mixing parameters $\xi_m = 0$, $\tau_m = 0$, which is simply due to the absence of any characteristic length and time. When a convection is present, and both $D(a)$ and $D(t)$ do saturate, it is widely adopted that the two saturated values are the same:

$$D_{\text{eff}} = \lim_{a\to\infty} D(a) = \lim_{t\to\infty} D(t) \ . \tag{7}$$

The authors of this paper, however, are not aware about a rigorous proof of this hypothesis.

The assumption of the passive transport meaning the $\mathbf{v}$-field in (1) to be independent of the distribution of n, is actually not always valid. Particularly, for anomalous heat conduction in a high-temperature plasma, the convection itself may be caused by temperature or density gradient driven instabilities. Nevertheless, besides its fundamental value, the approximation of a passive impurity is an unescapable step in the understanding of the active scalar problem.

Another remark concerns the evolution equation of a vector

$$\partial \mathbf{B}/\partial t - \nabla \times (\mathrm{v} \times \mathbf{B}) = \eta \nabla^2 \mathbf{B} \tag{8}$$

modeling, for example, the magnetic dynamo in a conducting medium [12–14] with $\mathbf{B}$ denoting the magnetic field, or the evolution of the vorticity $\mathbf{B} = \nabla \times \mathrm{v}$ in a liquid [6] with η being the magnetic or kinematic viscosity, respectively. While the former case admits the approximation of a passive vector unless the magnetic energy $B^2/8\pi$ is comparable with kinetic one, the latter is principally of the active type (this reflects the nonlinearity being inherent for fluid dynamics). Not intending to elaborate this issue in detail, let us only notice that the techniques developed in the present paper could be also applied to the 2D passive vector problem.

As showed Zeldovich [5] (see also Ref. [1]), an arbitrary convection of an incompressible fluid ($\nabla \cdot \mathrm{v} = 0$) increases the admixture transport, hence always $D_{\mathrm{eff}} > D_0$. The most interesting is the case of a strong convection $\psi_{\mathrm{max}} \gg D_0$. Then, obviously, $D_{\mathrm{eff}} \gg D_0$; this is the scope of our further interest. Most likely, the assessment $D_{\mathrm{eff}} \approx \psi_{\mathrm{max}}$ would be valid for a generic 3D flow. It could be argued by the possibility of a stochastic behavior of incompressible streamlines in three dimensions, even for steady flows [15,16].

In the case of a two-dimensinal time-independent flow, $\mathrm{v} = \nabla \psi(x, y) \times \hat{\mathbf{z}}$, the stream-lines coincide with the isolines of $\psi(x, y)$. Since almost any such line is closed [2,3], a diffusive character of the contaminant propagation will arise only for the nonzero molecular diffusion D_0. So the asymptotic $D_{\mathrm{eff}} \approx \psi_{\mathrm{max}}^{1-\zeta} D_0^{\zeta}$, $\psi_{\mathrm{max}} \gg D_0$, $0 < \zeta < 1$ has been suggested [17]. One of the first results in this scope, with $\zeta = 1/2$, was been obtained independently in papers [7] and [8]. The authors examined the regular cellular system of vortices

$$\psi(x, y) = \psi_0 \sin(k_x x) \sin(k_y y) \tag{9}$$

and found

$$D_{\rm eff} = A\sqrt{D_0\,\psi_0}\;, D_0 \ll \psi_0\;, \qquad (10)$$

with the numerical coefficient A of the order of unity.

In Refs. [1,2] it was pointed out that the flow (9) is structurally unstable: a vanishing perturbation will destroy the regular system of elementary convective cells and gather them into several conglomerates, so that the isolines of $\psi(x,y)$ of an arbitrary length will be present. This circumstance explains the importance of studying in [1,2] the structurally stable generic stream-function $\psi(x,y)$, which is assumed to be random and to have a single space scale λ ("the λ-flow", for its definition and properties see Part I [3]). The analysis of fractal and percolation properties of monoscale isolines in papers [1,2] led to the result $\zeta = 3/13$ expressed through percolation exponents. The effective diffusivity in the λ-flow was estimated as

$$D_{\rm eff} \approx \left(D_0\psi^{\nu+2}\right)^{1/(\nu+3)} = D_0^{3/13}\psi^{10/13}\;, \quad D_0 \ll \psi, \qquad (11)$$

where $\nu = 4/3$ is the correlation length exponent of the 2D percolation problem. The effective diffusion (11) sets up on space and time scales above the mixing length and mixing time

$$\xi_m = \lambda(\psi/D_0)^{\nu/(\nu+3)}\;, \quad \tau_m = \lambda^2/D_0\;, \qquad (12)$$

respectively.

The objective of the present work is to extend our earlier approach [1,2] and to calculate the effective diffusivity in a multiscale flow. The latter is assumed to have a power spectrum inside a wide range of wavelengths $\lambda_0 < \lambda < \lambda_m,\ \lambda_0 \ll \lambda_m$:

$$\psi_\lambda = \psi_0(\lambda/\lambda_0)^H\;, \qquad (13)$$

where ψ_λ denotes the root-mean-square amplitude of the stream-function λ-component

$$\psi_\lambda(\mathbf{r}) = \int_{1/2<|\mathbf{k}|\lambda<1} \psi(\mathbf{r})e^{i\mathbf{kr}}d^2\mathbf{k}\;.$$

The spectrum (13) corresponds to the Fourier spectrum $|\psi(\mathbf{k})| \propto k^{-(1+H)}$.

To study the transport properties of the monoscale flow, we employ the technique of the virtual scale separation introduced in [3]. A central point in our analysis is the introducing of "the partial diffusion" D_λ (Eqs. (24)–(26)) that describes a cross-stream passive transport on an intermediate scale λ. The effective diffusion D_{eff} and the mixing length ξ_m are then calculated as the intersection of the box diffusion $D(a)$ and the partial diffusion D_λ: $D_{\text{eff}} = D(\xi_m) = D_{\xi_m}$.

The paper is organized in the following manner. In Sec. II, we discuss the box diffusion in the absence of the background diffusivity D_0 and show that $D_{\text{eff}} \to 0$ for $D_0 \to 0$. In Sec. III, an auxiliary problem is concerned about the passive transport in a λ-flow perturbed by a weak homogeneous stream. In Sec. IV, we discuss the spatial problem of diffusion in a multiscale flow. Here, the result of Sec. III is used to calculate the partial diffusion D_λ and hence the effective diffusion D_{eff} and the mixing length ξ_m. In Sec. V we summarize the results and follow the transition between various regimes of the effective diffusion.

II. Box Diffusion

Let us neglect the background diffusion D_0 and study the transport properties of the multiscale flow with the spectrum (13). According to the definition of the box diffusion (3), consider a square $a \times a$, on whose opposite sides AB and CD the specified boundary values n_1 and n_2 of the admixture density are maintained (Fig. 1). The flux J_{12} between AB and CD is then produced by stream-lines with the size of the order of a. Since a multiscale flow may be represented as a superposition of monoscale flows, and the flux is a linear functional of the velocity field[1], we may argue in terms of flux contributions of various monoscale components. Suppose $\lambda_0 < a < \lambda_m$. Then the flux is primarily due to the stream-function component ψ_λ with $\lambda \approx a$, which gives rise to stream-lines like (a) and (b) shown in Fig. 1.

[1]In the absence of a background diffusion.

On the other hand, shorter scales ($\lambda \ll a$) produce stream-lines like as (c) and (d) and clearly do not contribute to the flux J_{12}. As for longer scales, $\lambda \gg a$, there may be at all no one such stream-line intersecting the box. Thus we estimate the box diffusion coefficient as

$$D(a) \approx J_{12}/(n_1 - n_2) \approx \psi_a \ , \quad \lambda_0 < a < \lambda_m. \tag{14}$$

Considering the case $a > \lambda_m$, let us recall the geometry of large convection cells in a monoscale λ-flow [2,3]. Such a cell has the width

$$h(a) \approx \lambda(a/\lambda)^{-1/\nu} \ , \quad a \gg \lambda,$$

hence the resulting λ-scale contribution to the box diffusion is

$$D^{(\lambda)}(a) \approx v_\lambda h(a) \approx \psi_\lambda(\lambda/a)^{1/\nu} \ ,$$

where $\psi_\lambda \propto \lambda^H$. In the interesting case of a nontrivial interaction of scales, $-1/\nu < H < 1$ [3], the biggest contribution to $D(a)$ is done by the longest scale, $\lambda = \lambda_m$, hence

$$D(a) \approx \psi_{\lambda_m} \left(\frac{a}{\lambda_m}\right)^{-1/\nu} \ , \quad a > \lambda_m \ . \tag{15}$$

So in the purely convective transport ($D_0 = 0$), the box diffusion (14)–(15) tends to zero on large scales. For the effective diffusion (7) to be different from zero, we must take into account the finite background diffusion D_0. Then the mixing length ξ_m is defined as the box size such as particles in the convection time between AB and CD diffuse across the stream-lines to a distance of the order of the width of the flow channels connecting the sides AB and CD. To study this cross-stream diffusion, we shall implement the program of separated scales.

III. Transport in Random Monoscale Flow with a Weak Homogeneous Component

Analogously to the calculation of the fractal dimension of a coastline [3], we begin with the consideration of a flow, being the superposition of a λ-flow and a very weak uniform stream:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{u} \ , \ \mathbf{v}_0 = \nabla\psi_0(x, y) \times \hat{\mathbf{z}} \ , \tag{16}$$

$$\mathbf{u} = u\hat{\mathbf{x}} \ , \ u \ll v_0 \ .$$

Here $\psi_0(x, y)$ is a λ_0-flow, and $\mathbf{u}$ is the small homogeneous component.

In Part I we showed that the percolating flux caused by the homogeneous component of the flow is concentrated in channels with the characteristic width

$$\delta_\varepsilon = \lambda_0 \, \varepsilon^{1/(\nu+1)} \tag{17}$$

averagely oriented in the $\hat{\mathbf{x}}$-direction and wandering in the $\hat{\mathbf{y}}$-direction over the distance

$$\Delta_\varepsilon \approx \lambda_0 \, \varepsilon^{-\nu/(\nu+1)} \ , \tag{18}$$

where $\varepsilon = u/v_0$ is the small parameter of the problem.

In the considered frame of reference, the transport in the $\hat{\mathbf{x}}$-direction is clearly not diffusional, since a mean velocity is present. Consider the diffusion in the $\hat{\mathbf{y}}$-direction. Unless one is interested in transport on scales greater than Δ_ε, the flow pattern looks qualitatively the same as in the absence of the mean flux. Consequently, for Δ_ε exceeding the mixing length ξ_m of the λ_0-flow, the effective diffusion remains the same as in the absence of the mean flow. Using Eqs. (12) and (18) we infer that for $\Delta_\varepsilon > \xi_m$, or $\varepsilon < \varepsilon_*$, where

$$\varepsilon_* = (D_0/\psi_0)^{(\nu+1)/(\nu+3)} \ll 1 \ , \tag{19}$$

the mean flux is too weak to have an essential influence on the transport along $\hat{\mathbf{y}}$, and the expression (11) for the effective diffusivity remains valid. In the opposite case, $\varepsilon > \varepsilon_*$, the distance between the channels Δ_ε becomes shorter than the mixing length ξ_m of Eq. (12). As a reasult, the stream-lines with the diameter a greater than Δ_ε, including the most effective for the transport ($a \approx \xi_m$), will be destroyed by the average stream $\mathbf{u}$. Here, one can conclude that the effective diffusivity D_{yy} across the mean flux (in the $\hat{\mathbf{y}}$-direction) will be diminished. The most "active" particles, giving the greatest contribution to the transport, lie in the channels. In this regime, the value of D_{yy} may be estimated imagining these particles to jump over y on the distance Δ_ε in random direction every trapping time $\tau_\varepsilon \approx \delta_\varepsilon^2/D_0$,

$$D_{yy} \approx \frac{\delta_\varepsilon\, L(\Delta_\varepsilon)}{\Delta_\varepsilon^2} \frac{\Delta_\varepsilon^2}{\tau_\varepsilon} ,$$

with the first factor on the right-hand side corresponding to the fraction of the "active" particles, and δ_ε, Δ_ε determined by the formulas (17) and (18), respectively. The length of the channel $L(a)$ is expressed through the displacement a with the help of the stream-line fractal dimension [2,3] $d_{h0} = 1 + 1/\nu$ as $L(a) \approx \lambda_0(a/\lambda_0)^{d_{h0}}$. So we obtain the average cross-stream diffusivity

$$D_{yy} \approx D_0 \varepsilon^{-(\nu+2)/(\nu+1)} , \; \varepsilon_* < \varepsilon < 1 . \tag{20}$$

In this regime, the mixing length for the effective diffusion (20) is equal to Δ_ε.

The result (20) may also be derived in another way using the analogy between diffusion and conduction. The average diffusivity D_{yy} may be calculated as the conductivity $(R_1 + R_2)^{-1}$ of two "diffusive resistances" connected in series, one of them being the channels (R_1) and the other the flow between the channels (R_2). We can evidently restrict ourselves to the mixing length area $\Delta_\varepsilon \times \Delta_\varepsilon$ where only one channel is present with the transverse diffusive resistance

$$R_1(\varepsilon) = \frac{1}{D_0} \frac{\delta_\varepsilon}{L(\Delta_\varepsilon)} = \frac{1}{D_0} \varepsilon^{\frac{\nu+2}{\nu+1}} .$$

Notice that, for $\varepsilon = \varepsilon_*$, $R_1(\varepsilon) = R_2(\varepsilon) = D_{\text{eff}}^{-1}$, where D_{eff} is given by Eq. (11). For $\varepsilon > \varepsilon_*$, $R_1(\varepsilon)$ is increased. On the other hand, the transport coefficient $R_2^{-1}(\varepsilon)$ is, by definition, the box diffusion and can be calculated using Eq. (15) with $a = \Delta_\varepsilon$ and $\lambda_m = \lambda_0$. Hence, for $\varepsilon > \varepsilon_*$, $R_2(\varepsilon)$ is decreased, so that the transverse diffusive resistance $R_1(\varepsilon)$ of the channels becomes a bottleneck for the transport. Thus we come again to the result (20).

IV. Diffusion of Passive Scalar in 2D Multiscale Flow

In order to obtain the expression for the diffusivity in the multiscale flow, it is suitable to make use of the method of separated scales introduced in Part I. In this way we replace the continuous spectrum flow,

$$\mathbf{v}(x,y) = \mathbf{v}_{\lambda_0}(x,y) + \mathbf{v}_{\lambda_1}(x,y) + \ldots + \mathbf{v}_{\lambda_m}(x,y) \ , \tag{21}$$

$$v_\lambda \equiv [\mathbf{v}_\lambda(x,y)]_{\mathrm{rms}} = v_0(\lambda/\lambda_0)^{H-1} \ ,$$

with $\lambda_{i+1}/\lambda_i \equiv \mu = 2$, by another flow given by the same Eq. (21), but with $\mu \gg 1$. The "correct" value $\mu = 2$ will be considered as marginal for the "incorrect" approximation of separated scales. Actually, the continuous spectrum limit simply means the omitting of the subscripts near λ and is implied not only in the end of our calculations but also throughout further arguments.

For $H > 1$, the longest scale of the velocity field dominates, and the effective diffusion is determined by Eq. (11) with $\psi = \psi_{\lambda_m}$. For $H < 1$, the velocity field $\mathbf{v}_{\lambda_0}(x,y)$ may be regarded as being slightly and quasi-homogeneously perturbed by the rest of the terms on RHS of Eq. (21). The result of the interaction of scales λ_0 and λ_1 is the opening of some stream-lines of $\mathbf{v}_{\lambda_0}(x,y)$ and their collection in channels following the lines of $\mathbf{v}_{\lambda_1}(x,y)$ and wondering around those lines in the stripes with the width (18):

$$\Delta_{0,1} = \lambda_0(v_{\lambda_1}/v_{\lambda_0})^{-\nu/(\nu+1)} = \lambda_0(\lambda_1/\lambda_0)^{(1-H)\nu/(\nu+1)} \ . \tag{22}$$

In what follows, we will refer to these channels as "the (0,1)-channels", with an evident extension for longer scales. Suppose $D_0 \ll \psi_{\lambda_0}, \psi_{\lambda_m}$. If $\Delta_{0,1} \ll \lambda_1$ (i.e., $H > -1/\nu$), we may average the transverse diffusion over the space scale $\Delta_{0,1}$, as we did in the previous section, to obtain a result similar to (20). Having averaged, we find ourselves in the situation with the anisotropic (i.e., basically perpendicular to the convection $\mathbf{v}_{\lambda_1}(x,y)$) diffusivity

$$D_{\lambda_1} = D_0(v_{\lambda_1}/v_{\lambda_0})^{-(\nu+2)/(\nu+1)}\ , \tag{23}$$

and the flow given by Eq. (21) without the first term $\mathbf{v}_{\lambda_0}$, whose contribution is taken into account via the renormalization of the "background" diffusivity (23). To be more accurate, the last equation derived from Eq. (20) is valid provided that the velocity perturbation is not too small: $\varepsilon = v_{\lambda_1}/v_{\lambda_0} > \varepsilon_*$, where ε_* is determined by Eq. (19). In any event, this is the case for the implied continuous spectrum limit ($\varepsilon = 2^{H-1}$), which will be addressed as a final result.

On the second step, we average the passive transport in the same manner over the space scale $\Delta_{1,2}$. This results in[2]

$$D_{\lambda_2} = D_{\lambda_1}(v_{\lambda_2}/v_{\lambda_1})^{-(\nu+2)/(\nu+1)} = D_0(v_{\lambda_2}/v_{\lambda_0})^{-(\nu+2)/(\nu+1)}\ .$$

Generally, we obtain *the partial diffusion*, which describes the particle motion across the stream-lines of $\psi_\lambda(x,y)$ and incorporates the collective effect of the shorter scales of the flow:

$$D_\lambda = D_0\left(\frac{v_\lambda}{v_0}\right)^{-\frac{\nu+2}{\nu+1}} = D_0\left(\frac{\lambda}{\lambda_0}\right)^{\frac{(1-H)(\nu+2)}{\nu+1}}\ . \tag{24}$$

Equation (24) is written for the limit of continuous spectrum ($\mu = 2$).

Formula (24) is valid until D_λ reaches the value of the stream-function amplitude ψ_λ. If this happens with decreasing ψ_λ at $\lambda = \lambda_* \in [\lambda_0, \lambda_m]$ ($H < 0$; see Fig. 2(a)), then the partial diffusion is no longer changed for $\lambda > \lambda_*$, since small components of the stream-functon produce only insignificant (positive) corrections to the diffusion coefficient [5]. The

[2]Here the circumstance, that the longitudinal (with respect to the convection $\mathbf{v}_{\lambda_1}$) diffusivity is different from, viz., much less than, D_{λ_1}, is not important, since along the stream-lines the impurity distribution will be quickly smoothed due to the strong convection, regardless of the longitudinal diffusion. The trapping time in the (1,2)-channels is determined solely by the transverse diffusivity D_{λ_1}.

saturated value of D_λ is clearly the effective diffusion D_{eff} and the intersection point λ_* is the mixing length ξ_m.

If D_λ intersects ψ_λ at $H > 0$ (Fig. 2(b)), then we note that the implied inequality $\varepsilon > \varepsilon_*$ breaks and, instead of Eq. (20), we must calculate $D_{\lambda_{i+1}}$ according to Eq. (11) substituting $D_0 \to D_{\lambda_i}$, $\psi \to \psi_{\lambda_i}$. This leads to the result

$$D_\lambda = \psi_\lambda \ , \quad H > 0 \ , \quad \lambda_0 < \lambda_* < \lambda < \lambda_m \ . \tag{25}$$

Evidently, upon reaching the upper flow scale λ_m, we obtain the effective diffusion $D_{\text{eff}} \approx \psi_m$ and the mixing length $\xi_m = \lambda_m$.

In the case $D_{\lambda_m} < \psi_{\lambda_m}$ (Fig. 2(c,d)), we come to the familiar monoscale problem with the λ_m-flow and the "background" diffusivity D_{λ_m}. Here, the result may be obtained with the help of Eqs. (11), (12). It is pointed out that this limit may be equivalently reduced to the intersection of D_λ and ψ_λ if we formally extend the spectrum of the flow to the scales $\lambda > \lambda_m: \ \psi_\lambda = \psi_{\lambda_m}(\lambda/\lambda_m)^{-1/\nu}$. As discussed earlier [3], this continuation corresponding to the spectral exponent $\widetilde{H} = -1/\nu$ does not change the flow pattern and the resulting transport produced by smaller scales ($\lambda \leq \lambda_m$). Notice that this extended stream-function spectrum yields exactly the box diffusion (14)–(15). So we can also extend the partial diffusion D_λ to longer scales. From Eq. (24) with $H = -1/\nu$ we infer

$$D_\lambda = D_{\lambda_m} \left(\frac{\lambda}{\lambda_m}\right)^{\frac{\nu+2}{\nu}} \ , \quad \lambda_m < \lambda < \lambda_* \ . \tag{26}$$

Now we can formulate a general recipe of the calculation of the effective diffusion and the mixing length in a multiscale 2D flow: The mixing length ξ_m is the scale λ, at which the partial diffusion D_λ given by Eqs. (24)–(26) intersects and trends to exceed the box diffusion $D(\lambda)$ (14)–(15). For $\lambda > \xi_m$, both D_λ and $D(\lambda)$ saturate at their common value at $\lambda = \xi_m$ yielding the effective diffusion D_{eff}. Figure 2 shows different regimes of the variation of D_λ, $D(\lambda)$, and, correspondingly, $D_{\rm eff}$. A simple calculation leads to the following results:

$$\begin{cases} D_{\rm eff} \approx \psi_0 \left(\frac{\psi_0}{D_0}\right)^{\frac{H(\nu+1)}{\nu+2-H(2\nu+3)}} \ , \quad \xi_m = \lambda_* = \lambda_0 \left(\frac{\psi_0}{D_0}\right)^{\frac{\nu+1}{\nu+2-H(2\nu+3)}} \ , \\ \psi_{\lambda_m} < D_{\rm eff} < \psi_{\lambda_0} \ , \ -1/\nu < H < 0 \ ; \end{cases} \tag{27}$$

$$\begin{cases} D_{\rm eff} \approx \psi_{\lambda_m} \ , \quad \xi_m = \lambda_m \ , \\ D_{0*} \equiv \psi_0 \left(\frac{\lambda_0}{\lambda_m}\right)^{\frac{\nu+2-H(2\nu+3)}{\nu+1}} < D_0 < \psi_{\lambda_m} \ , \quad 0 < H < \frac{\nu+2}{2\nu+3} \ ; \end{cases} \tag{28}$$

$$\begin{cases} D_{\rm eff} \approx \psi_{\lambda_0} \left(\frac{D_0}{\psi_{\lambda_0}}\right)^{\frac{1}{\nu+3}} \left(\frac{\lambda_m}{\lambda_0}\right)^{\frac{(H\nu+1)(\nu+2)}{(\nu+1)(\nu+3)}} \ , \quad \xi_m = \lambda_m \left(\frac{\psi_{\lambda_0}}{D_0}\right)^{\frac{\nu(\nu+2-H)}{(\nu+1)(\nu+3)}} \ , \\ D_{\rm eff} < \psi_{\lambda_m} \ , \ -1/\nu < H < 0 \ , \ \text{or } D_{\rm eff} < \psi_0 \ , \ 0 < H < 1 \ . \end{cases} \tag{29}$$

The results (27), (28), and (29) correspond to the regimes (a), (b), and (c,d) of Fig. 2, respectively.

For the completeness of the picture we shall study the case $0 < H < 1$, $\psi_{\lambda_0} < D_0 < \psi_{\lambda_m}$. Here, we can neglect the small-scale components of the flow, i.e., those with $\lambda < \lambda_D$, where $\psi_{\lambda_D} \equiv D_0$, since their contribution to the transport is less than the background diffusion D_0. Replacing in Eq. (29) λ_0 by λ_D we obtain

$$D_{\rm eff} \approx D_0 \left(\frac{\lambda_m}{\lambda_D}\right)^{\frac{(H\nu+1)(\nu+2)}{(\nu+1)(\nu+3)}} \ ,$$

or, equivalently,

$$\begin{cases} D_{\rm eff} \approx \psi_{\lambda_m} \left(\frac{D_0}{\psi_{\lambda_m}}\right)^{\frac{H(2\nu+3)-(\nu+2)}{H(\nu+1)(\nu+3)}} \ , \quad \xi_m = \lambda_m \ , \\ \psi_0 < D_0 < \psi_{\lambda_m} \ , \quad (\nu+2)/(2\nu+3) < H < 1 \ . \end{cases} \tag{30}$$

V. Summary and Discussion

Let us rewrite all the expressions for the effective diffusion once more, with the percolation exponent ν being substituted by its numerical value 4/3. For convenience, here we use the same equation numbers as in Secs. I and IV.

$$\begin{aligned} &\text{If} \quad H < -3/4 \quad \text{and} \quad D_0 < \psi_0 \ , \\ &\text{then} \quad D_{\text{eff}} \approx \psi_0 (D_0/\psi_0)^{3/13} \ ; \end{aligned} \tag{11}$$

$$\begin{aligned} &\text{If} \quad -3/4 < H < 0 \quad \text{and} \quad D_{0*} \equiv \psi_0(\lambda_0/\lambda_m)^{(10-17H)/7} < D_0 < \psi_0 \ , \\ &\text{then} \quad D_{\text{eff}} \approx \psi_0 (D_0/\psi_0)^{-7H/(10-17H)} \ ; \end{aligned} \tag{27}$$

$$\begin{aligned} &\text{If} \quad 0 < H < 10/17 \quad \text{and} \quad D_{0*} < D_0 < \psi_{\lambda_m} \ , \\ &\text{then} \quad D_{\text{eff}} \approx \psi_{\lambda_m} \ ; \end{aligned} \tag{28}$$

$$\begin{aligned} &\text{If} \quad -3/4 < H < 10/17 \quad \text{and} \quad D_0 < D_{0*} \ , \\ &\text{or} \quad 10/17 < H < 1 \quad \text{and} \quad D_0 < \psi_0 \ , \\ &\text{then} \quad D_{\text{eff}} \approx \psi_0 (D_0/\psi_0)^{3/13} (\lambda_m/\lambda_0)^{(40H+30)/91} \ ; \end{aligned} \tag{29}$$

$$\begin{aligned} &\text{If} \quad 10/17 < H < 1 \quad \text{and} \quad \psi_0 < D_0 < \psi_{\lambda_m} \ , \\ &\text{then} \quad D_{\text{eff}} \approx \psi_{\lambda_m} (D_0/\psi_{\lambda_m})^{(51H-30)/91H} \ ; \end{aligned} \tag{30}$$

$$\begin{aligned} &\text{If} \quad H > 1 \quad \text{and} \quad D_0 < \psi_{\lambda_m} \ , \\ &\text{then} \quad D_{\text{eff}} \approx \psi_{\lambda_m} (D_0/\psi_{\lambda_m})^{3/13} \ . \end{aligned} \tag{11}$$

In Figure 3, we plot the dependence $D_{\text{eff}}(D_0)$ for various regimes of the effective diffusion. We see that, for the power spectrum characterized by the exponent H, the whole space of H may be divided into five intervals with distinct behaviors of the effective diffusion. Two of these regimes, namely, $H < -3/4$ and $H > 1$, simply correspond to the monoscale λ_0- and λ_m- flow, respectively, and are given by the monoscale expression (11). Note that for very small background diffusivity D_0, the expression (29) is universal: it holds for the entire interval of the nontrivial interaction of scales, $-3/4 < H < 1$. So the class of universality of the monoscale effective diffusion scaling, $D_{\text{eff}} \propto D_0^{3/13}$, turns out to be quite wide for sufficiently small D_0.

Our consideration supports the conclusion of Ref. [11] that the stream-function ψ of the flow determines the effective diffusion. Given a spectrum of ψ, and the relevant range of wavelengths, one can specify the mixing length ξ_m that distinguishes between the diffusive and nondiffusive regimes of the propagation of passive scalar. For ξ_m to be finite, the stream function should be bounded.

A natural development of this theory will be the consideration of time-dependent turbulent flows [19, 20].

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Figure Captions

[1] Box diffusion in purely convective system. The flux to the right is proportional to the left-side density n_1, while the flux to the left is proportional to n_2. Therefore, the net flux J_{12} is proportional to the difference in the boundary densities.

[2] The double-logarithmic scale-dependence of the stream-function component ψ_λ (thin line), the box diffusion $D(\lambda)$, and the partial diffusion D_λ (thick lines),

(a): $-1/\nu < H < 0, \quad D_{0*} < D_0 < \psi_0$;

(b): $0 < H < (\nu+2)/(2\nu+3), \quad D_{0*} < D_0 < \psi_0$;

(c): $(\nu+2)/(2\nu+3) < H < 1, \quad D_0 < \psi_0$;

(d): $-1/\nu < H < 0, \quad D_0 < D_{0*}$.

[3] The schematic of the dependence $D_{\text{eff}}(D_0)$ in different intervals of the spectral exponent H,

(a): $-3/4 < H < 0$;

(b): $0 < H < 10/17$;

(c): $10/17 < H < 1$.

In the brackets the equation numbers for D_{eff} are indicated.

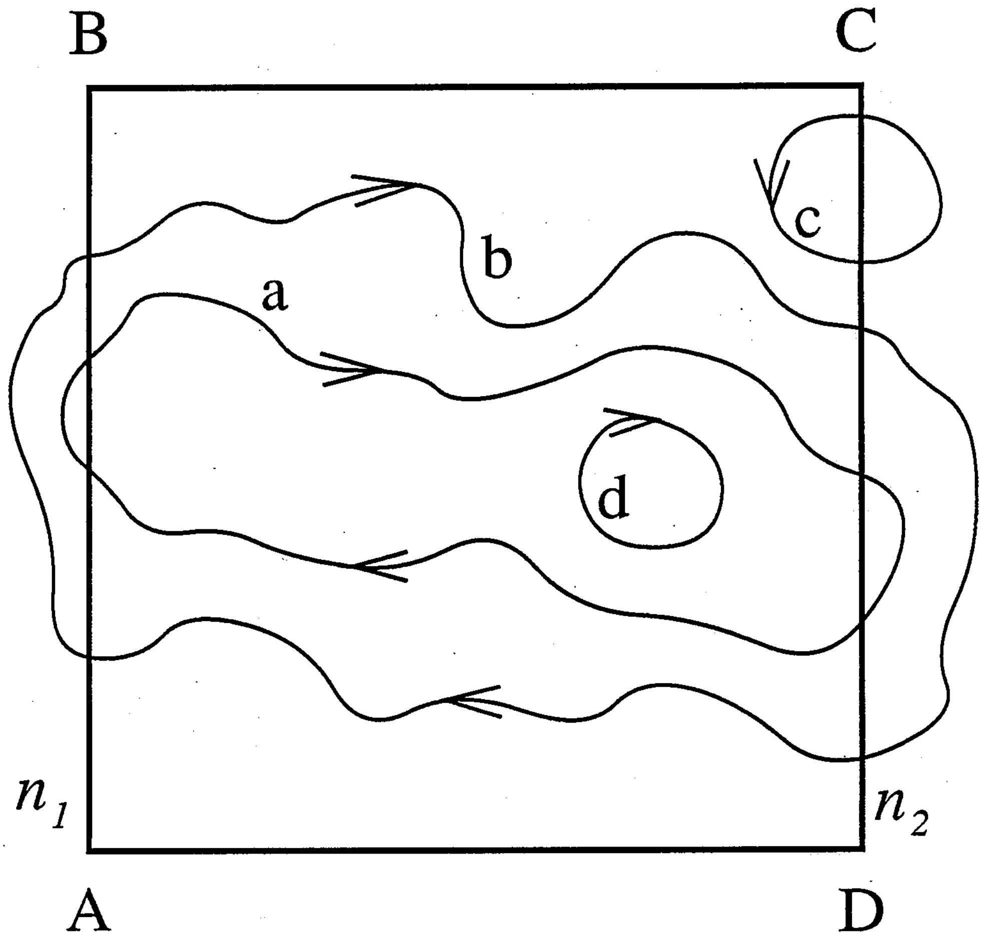
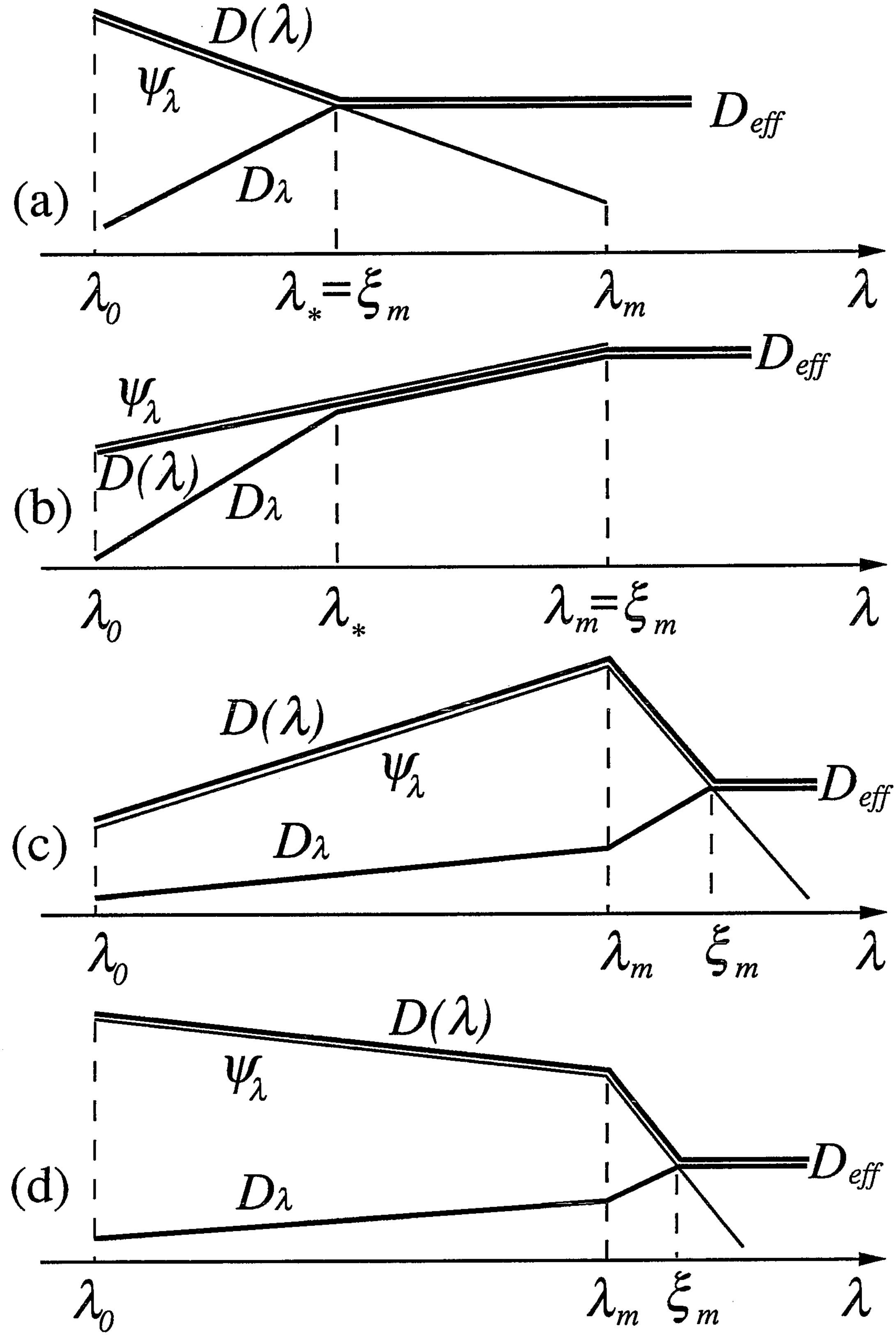


Figure 1.



D(λ)
ψλ
Deff
(a)
Dλ
λ0
λ*=ξm
λm
λ
ψλ
Deff
D(λ)
(b)
Dλ
λ0
λ*
λm=ξm
λ
D(λ)
ψλ
Deff
(c)
Dλ
λ0
λm
ξm
λ
D(λ)
ψλ
(d)
Deff
Dλ
λ0
λm
ξm
λ

Figure 2.

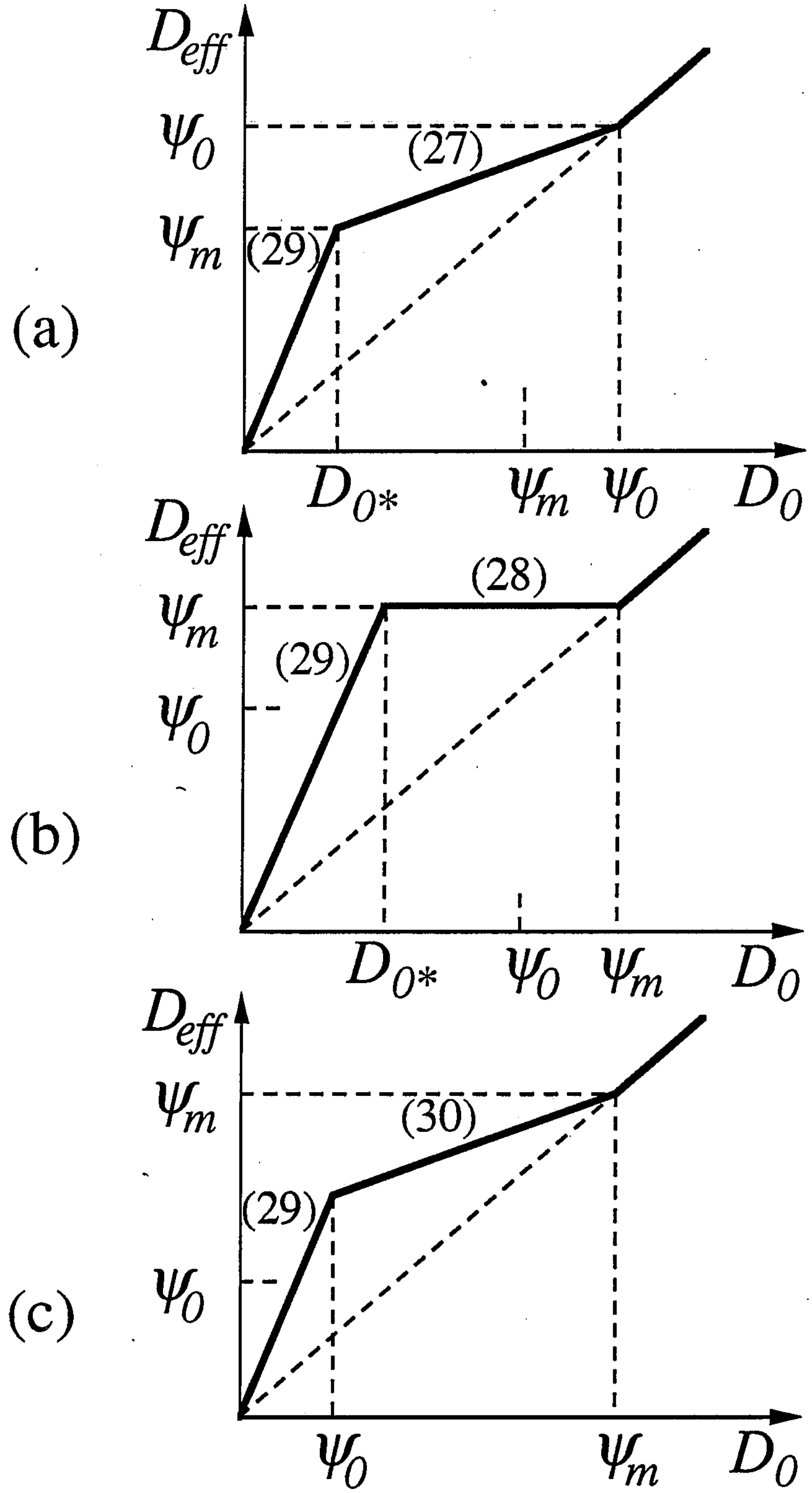

(a)
D_{eff}
ψ_0
ψ_m
(27)
(29)
D_{0*}
ψ_m
ψ_0
D_0
(b)
D_{eff}
ψ_m
ψ_0
(28)
(29)
D_{0*}
ψ_0
ψ_m
D_0
(c)
D_{eff}
ψ_m
ψ_0
(30)
(29)
ψ_0
ψ_m
D_0

Figure 3.