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Dynamo Effect and Current Drive due to Magnetic Fluctuations in Sheared Magnetic Field

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Abstract

It is shown that when the mean electromotive force in a plasma is evaluated in terms of the two-point correlation of driven magnetic field fluctuations (as would be appropriate for problems related to current drive by waves), there is no β effect; i.e., the terms associated with gradients of the mean field vanish. The relevance of this result to the problem of current drive is discussed.

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I. Dynamo Effect due to Magnetic Fluctuations in Sheared Field

The possibility of generating the mean electromotive force (EMF) in a conducting fluid by random velocity fields is well recognized in the context of geo-dynamo and astrophysical problems.[1,2] The idea is to specify some average spectral property of the field (e.g. the two-point correlation) and then use the quasilinear induction equation to study the evolution of the mean magnetic field in terms of these properties. This approach seems to be appropriate for geo-dynamo problems where it has been argued that various processes in the earth's mantle and core can set up such a random velocity field. If the scale length L of the mean magnetic field $\mathbf{B}$ is much larger than the scalelength ℓ of the velocity field, then the mean EMF $\mathcal{E}$ can be expanded in a rapidly convergent series[3] as

$$\mathcal{E}_i = \alpha_{ij}\mathbf{B}_0 + \beta_{ijk}\frac{\partial \mathbf{B}_j}{\partial x_k} + \gamma_{ijk\ell}\frac{\partial^2 \mathbf{B}_j}{\partial x_k \partial x_\ell} + \cdots \tag{1}$$

where α_{ij}, β_{ijk}, etc., are pseudo-tensors which depend upon the spectral properties of the v-field. In Eq. (1) the first term is dominant; however, the second term is also important as it is of the same order as the diffusion term in the induction equation. The subsequent terms can be neglected if L is sufficiently larger than ℓ. It has been shown that if the velocity field lacks reflection symmetry, as is the case when it carries finite helicity, then $\alpha_{ij} \neq 0$ and the first term in Eq. (1) is finite. This is usually called the "α-effect".[1,2] However, if the v-field is reflectionally symmetric (and also isotropic and homogeneous), then $\alpha_{ij} = 0$ and the dominant term is the one associated with β_{ijk}. This is called the β-effect.[3] As can be easily seen, the β-effect modifies the resistivity of the fluid. It has been found that usually in all cases where β can be calculated explicitly, it increases the fluid resistivity.

Recently, the problem of non-resonant current drive in fusion plasmas has rekindled the interest in calculating the mean EMF due to externally injected waves.[4,5,6] However, as these waves are injected by antennas, thereby "stirring" the plasma electromagnetically (as

opposed to "mechanical stirring" in geo-dynamo problems), it is more natural to take the spectral properties of the magnetic fluctuations $\widetilde{b}$ as given. One then might argue that, if the b-field carries a finite helicity, then an analogous α-effect may be generated which can then drive currents in the plasma. This is precisely the principle behind driving currents in the plasma by polarized waves. Now, since fusion plasmas are sheared (because they carry a finite current), it is of interest to investigate whether there is an analogous β-effect. This question is of importance because in a realistic situation, one would expect an α-effect from the asymmetric part of the b-field and a β-effect from the symmetric part. The β-effect may reduce the efficiency of the current drive that one may infer by considering the α-effect alone. In this letter we have investigated this problem and found that the β-effect is identically zero. That is, when the mean EMF is evaluated in terms of two-point correlations of the random magnetic field fluctuations with the use of the linearized equation of motion, the terms proportional to the gradients of magnetic field mutually cancel.

To show briefly (details will be given elsewhere) this we consider an incompressible conducting fluid governed by the following equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} + \eta \nabla^2 \mathbf{B} \tag{2}$$

$$\rho \frac{\partial \mathbf{v}}{\partial r} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = (B \cdot \nabla)B + \nu \nabla^2 \mathbf{v} - \nabla P \tag{3}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0 \tag{4}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{B}$ is the magnetic field, η and ν are the resistivity and viscosity, respectively, and P is the total pressure. Next we separate the variables into a mean and a fluctuating part as

$$\mathbf{v} = \overline{v} + v \ , \ \mathbf{B} = \overline{B} + b \ , \ P = \overline{P} + p \tag{5}$$

with

$$\overline{v} = \langle v \rangle = \langle b \rangle = \langle p \rangle = 0 \ . \tag{6}$$

Using Eq. (4) in Eqs. (1) and (2) we have three equations for $\overline{B}, v$ and b:

$$\frac{\partial \overline{B}}{\partial r} = \nabla \times \mathcal{E} + \eta \nabla^2 \overline{B} \tag{7}$$

$$\frac{\partial b}{\partial t} = (\overline{B} \cdot \nabla) v - (v \cdot \nabla) \overline{B} + \nabla \times G_1 + \eta \nabla^2 b \tag{8}$$

$$\frac{\partial v}{\partial t} + G_2 = -\nabla p + (\overline{B} \cdot \nabla) b + (b \cdot \nabla) \overline{B} + G_3 + \nu \nabla^2 v \tag{9}$$

where

$$\mathcal{E} = \langle v \times b \rangle \ , \tag{10}$$

$$G_1 = (v \times b) - \langle v \times b \rangle \tag{11}$$

$$G_2 = (v \cdot \nabla) v - \langle (v \cdot \nabla) v \rangle \tag{12}$$

$$G_3 = (b \cdot \nabla) b - \langle (b \cdot \nabla) b \rangle \ . \tag{13}$$

Note the appearance of the mean EMF $\mathcal{E}$ as a source term in the equation for $\overline{B}$. Within the approximation of first-order smoothing[3]; i.e., $v_0 t_0 / \ell \ll 1$ (where v_0 is the root mean square value of v and t_0 is some characteristic time of the field), the terms G_1, G_2, and G_3 can be neglected. To introduce the inhomogeneity of the mean magnetic field, we take[7]

$$\overline{B}_i(x_i) = x_k \frac{\partial \overline{B}_i}{\partial x_k} \tag{14}$$

where $\frac{\partial \overline{B}_i}{\partial x_k}$ is uniform. This permits taking the Fourier transform of the fluctuations in the presence of inhomogeneity according to

$$\int_0^\infty \int_{-\infty}^\infty x_i \frac{\partial b_j}{\partial x_k} e^{-i(k \cdot x - \omega t)} = -\widehat{b}_j \, \delta_{ik} - k_k \frac{\partial \widehat{b}_j}{\partial k_i} \tag{15}$$

where $\widehat{b}_j$ is the Fourier transform of b_j, and ω and $\mathbf{k}$ are real. Fourier transforming the variables in Eqs. (8) and (9) according to Eq. (15), we have

$$(-i\omega + \eta k^2) \widehat{b}_i = -k_j \frac{\partial \widehat{v}_i}{\partial k_k} \frac{\partial \overline{B}_j}{\partial x_k} - \widehat{v}_j \frac{\partial \overline{B}_i}{\partial x_j} \tag{16}$$

$$\rho(-i\omega + \nu k^2)\widehat{v}_i = +ik_i\widehat{p} - k_j \frac{\partial \widehat{b}_i}{\partial k_k}\frac{\partial \overline{B}_j}{\partial x_k} + \widehat{b}_j \frac{\partial \overline{B}_i}{\partial x_j} . \tag{17}$$

Now we show the usual β-effect[7]; i.e., when the two-point correlation of the velocity field v is given. Since the β-effect depends upon the reflectionally symmetric part of the spectrum, the two-point correlation function is given by

$$\langle v_i(x,t)v_j(x',t')\rangle = \varphi^v_{ij}(\mathbf{k},\omega)\delta(k-k')\delta(\omega-\omega') \tag{18}$$

where

$$\varphi^v_{ij} = \frac{E^v(k,\omega)}{4\pi k^2}(k^2\delta_{ij} - k_ik_j) . \tag{19}$$

We can now calculate $\mathcal{E}$ in terms of $E^v(k,\omega)$ according to

$$\mathcal{E}_i = \mathrm{Re}\left\langle \epsilon_{ijk}\, v_j(x,t)b^*_k(x',t')\right\rangle . \tag{20}$$

Clearly, since $\langle v_iv_j\rangle$ is again, we can only use the induction equation to eliminate b terms of v in Eq. (20) (i.e., we cannot use the equation of motion) to give

$$\mathcal{E}_i = \mathrm{Re}\ \epsilon_{ijk}\iint \frac{1}{(-i\omega+\eta k^2)}\left[\frac{\partial}{\partial k_q}(k_n\varphi^v_{jk}) - \varphi^v_{jq}\,\delta_{kn}\right]\frac{\partial \overline{B}_n}{\partial x_q}\, d\mathbf{k}d\omega \tag{21}$$

where Eq. (18) has been used to perform the integrations over k' and ω'. Only the second term in Eq. (21) makes a nonzero contribution. Using Eq. (19) for φ^v_{jq} we have

$$\mathcal{E}_i = -\epsilon_{ijk}\frac{\eta}{4\pi}\int_0^\infty\int_{-\infty}^{+\infty}\frac{E^v(k,\omega)}{(\omega^2+\eta^2k^4)}\left[\delta_{jq} - \frac{k_jk_q}{k^2}\right] d\mathbf{k}d\omega\,\frac{\partial \overline{B}_k}{\partial x_q} . \tag{22}$$

Using $k_i = k\cos\theta$ and $d\mathbf{k} = k^2dk\sin\theta d\theta d\varphi$, the integral in Eq. (22) can be simplified as

$$\mathcal{E}_i = -\frac{2}{3}\eta\int_0^\infty\int_{-\infty}^{+\infty}\frac{k^2E^v(k,\omega)}{(\omega^2+\eta^2k^4)}\,dkd\omega\,(\nabla\times\mathbf{B})_i . \tag{23}$$

This coincides with Moffat's expression for the β-effect.[7] Since $E^v(k,\omega)\geq 0$, therefore $\mathcal{E}_i < 0$, and hence the β-effect enhances the resistivity.

Now let us consider the case when the fluid is "stirred" electromagnetically. In this case we regard the properties of the magnetic field fluctuations as given. In particular we take the two-point correlation of the magnetic fluctuations to be given according to

$$\left\langle b_i(x,k) b_j^*(x',t') \right\rangle = \varphi_{ij}^b \delta(k+k')\delta(\omega+\omega') \tag{24}$$

where

$$\varphi_{ij}^b = \frac{E^b(k,\omega)}{4\pi k^4}\left(k^2\delta_{ij} - k_i k_y\right) . \tag{25}$$

The mean EMF is still given by Eq. (18). However, since $\langle b_i b_j \rangle$ is given, we now cannot use the induction equation. Rather, we must use the equation of motion to eliminate $\widehat{v}$ in terms of $\widehat{b}$ to express $\mathcal{E}_i$ in terms of $\langle b_i b_j \rangle$. Thus we have

$$\mathcal{E}_i = \mathrm{Re}\, \frac{\epsilon_{ijk}}{\rho} \int_0^\infty \int_{-\infty}^{+\infty} \frac{1}{(-i\omega + \eta k^2)} \left[+ i k_j \widehat{p}\widehat{b}_k - \frac{\partial}{\partial k_q}\left(k_n \varphi_{jk}^b\right) + \varphi_{jq}^b \,\delta_{kn} \right] d\mathbf{k} d\omega \, \frac{\partial \overline{B}_n}{\partial x_q} . \tag{26}$$

Note the extra contribution to $\mathcal{E}_i$ when fluctuations in the total pressure beat with the magnetic fluctuations $\widehat{b}_k$. As stated earlier, the second term does not contribute to $\mathcal{E}_i$. The contribution from the third term is similar to the contribution from the second term in Eq. (21), but with η replaced by ν and with the opposite sign. Hence, the contribution from the third term is

$$\text{Third term} \; = \; +\frac{2}{3}\frac{\nu}{\rho} \int_0^\infty \int_0^\infty \frac{k^2 E^b(k,\omega)}{(\omega^2 + \nu^2 k^4)} \, dk d\omega (\nabla \times \overline{B})_i \; . \tag{27}$$

Since $E^b \geq 0$, it appears that in this case the β-effect decreases the plasma resistivity. However, we still have to evaluate the term due to the total pressure fluctuations; i.e., the first term. Using $k_i \widehat{v}_i = 0$, from Eq. (17) we have

$$\widehat{p} = i2k_j \frac{\partial \overline{B}_j}{\partial x_k} \widehat{b}_k \tag{28}$$

where we have used

$$0 = \frac{\partial k_i b_i}{\partial k_j} = k_i \frac{\partial b_i}{\partial k_j} + b_j \; . \tag{29}$$

Using Eq. (28) in Eq. (26), we can write the contribution from the first-term as

$$\text{First term} \;=\; -2\,\frac{\epsilon_{ijk}}{4\pi\rho}\int_0^\infty\!\!\int_{-\infty}^\infty \frac{k_j k_n}{k^2}\,\frac{E^b(k,\omega)}{k^2}\left[\delta_{\rho k}-\frac{k_\ell k_k}{k^2}\right]+d\mathbf{k}d\omega\,\frac{\partial\overline{B}_n}{\partial x_\ell}\;. \tag{30}$$

The contribution from the $k_\ell k_k/k^2$ term in Eq. (30) is zero because, along with $k_j k_n$, it comprises the ith component of $(\mathbf{k}\times\mathbf{k})$. The remaining integral is finite only when $j = n$ and $\ell = k$. Again using $k_i = k\cos\theta$ and $d\mathbf{k} = k^2 dk\sin\theta d\theta d\varphi$ we can show that

$$\text{Third term} \;=\; -\frac{2}{3}\,\frac{\nu}{\rho}\int_0^\infty\!\!\int_0^\infty \frac{k^2 E^b(k,\omega)}{(\omega^2+\nu^2k^4)}\,dkd\omega\,(\nabla\times\overline{B})_i\;. \tag{31}$$

This cancels with the third term, giving

$$\mathcal{E}_i = 0\;. \tag{32}$$

The physical reason for this cancellation is clear. Externally driven magnetic fluctuations cause fluctuations of the total pressure. This gives rise to an additional EMF which is in the opposite direction to the EMF due to the β-effect. When the velocity field is perturbed, there are no such contributions.

It should be noted that these results are quite general and independent of any particular geometry. These results are relevant to current drive since they imply that there is no increase of the plasma resistivity and hence no degradation of current drive efficiency due to externally driven magnetic fluctuations .

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