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Effective Plasma Heat Conductivity in "Braided" Magnetic Field. Part II. Percolation Limit

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Errata "Effective Plasma Heat Conductivity in "Braided" Magnetic Field. Part II. Percolation Limit".
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1. [p. 5, 3rd line from the bottom] "$r_\perp^2$" should read "$r_\perp$".

2. [p. 18, Eq. (30b)] Instead of

$$(b_0 z_d/\delta)^{(\nu+1)}\delta^2/t_d, \quad \delta/b_0 < z_d < z_m, \quad \text{(MD)} \qquad (30b)$$

the equation should read

$$(b_0 z_d/\delta)^{\frac{\nu}{\nu+1}}\delta^2/t_d, \quad \delta/b_0 < z_d < z_m, \quad \text{(IR)} \qquad (30b)$$

3. [p. 19, the line under Eq. (43)] "$\nu_e t_d > 1$" should read "$\nu_e t_d < 1$".

4. [p. 29, the paragraph following Eq. (C7)] The sentence

"These field are by no means statistically equivalent."

should be omitted.

5. [p. 31] The reference

Krommes, J.A. (1978) Prog. Theor. Phys. Suppl. **64**, 137.

should be inserted between Kadomtsev & Pogutse (1978) and Krommes, Oberman & Kleva (1983).

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Abstract

This paper is devoted to the problem of the anomalous transport across a magnetic field that includes a small stochastic component $\delta\mathbf{B}$. The perturbation is assumed to be so strongly stretched along the background magnetic field $\mathbf{B}_0$ that the parameter R is large: $R \equiv b_0 L_0/\delta \gg 1$ (here $b_0 \equiv \delta B_\perp/B_0 \ll 1$, L_0 is longitudinal and δ transverse correlation lengths of the magnetic perturbation). This strong turbulence limit, which is opposite to the quasi-linear one ($R \ll 1$), has certain notable features. The principal result is that the main transport is concentrated in very thin regions, being fractal sets with the dimension d_f, which can range in value from 2 to 2.75, depending on the spectrum of the magnetic perturbation. These regions consist of a small fraction of magnetic lines that percolate, that is, walk from the non-perturbed magnetic flux surfaces to a distance large compared to the transverse correlation length δ. Due to such a strong inhomogeneity of the transport distribution, as well as the long correlations, the standard transport averaging techniques fail, and one should make use of the percolation theory methods. Thus the strong turbulence regime is

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reffered here as *the percolation limit.* In comparison with the quasi-linear limit the percolation limit has several additional intermediate regimes and the expressions for the effective heat conductivity χ_{eff} include the critical exponents of 2-D percolation theory. The estimates of χ_{eff} are obtained both in collisional and collisionless limits, including the the case of non-stationary magnetic perturbations.

I. Introduction

The purpose of the present paper is to extend earlier works on quasi-linear cross-field stochastic transport (Rechester & Rosenbluth 1978, Kadomtsev & Pogutse 1978, Krommes 1978, Krommes, Oberman & Kleva 1983, Isichenko, 1990) for the case of strong magnetic perturbations:

$$R \equiv b_0 \, L_0/\delta \gg 1 \ , \tag{1}$$

where $b_0 \equiv \delta B_\perp / B_0 \ll 1$ is the relative magnitude of the magnetic field perturbation, L_0 is longitudinal and δ is transverse correlation lengths. In this paper we adopt the notation introduced in Part I (Isichenko 1990). The present part of our study has been written as a separate paper because the treatment of the problem in the limit (1) uses new techniques related to the continuum percolation problem (Gruzinov, Isichenko & Kalda 1990). For this reason we shall refer to the limit (1) as *the percolation limit.*

Let us consider a stationary "braided" magnetic field

$$\mathbf{B} = B_0 \hat{z} + \delta \mathbf{B}(x, y, z, t) \ . \tag{2}$$

In the present paper for simplicity we neglect shear effects. With Eq. (2), the equation of a field line takes the form

$$d\mathbf{r}_\perp / dz = \mathbf{b}(\mathbf{r}_\perp, z) \ , \tag{3}$$

where $\mathbf{b} \equiv \delta \mathbf{B}_\perp / B_0$, $\mathbf{r}_\perp \equiv (x, y)$. Equation (3) describes nearly two-dimensional motion of a magnetic line, since the inequality (1) suggests the dependence of RHS of Eq. (3) on z is very slow. Besides, in this approximation one can consider the transverse magnetic perturbation $\mathbf{b}$ to be incompressible. According to $\operatorname{div} \delta \mathbf{B} = 0$ and (1) we have

$$\operatorname{div} \mathbf{b} = \partial b_x / \partial x + \partial b_y / \partial y = -\partial b_z / \partial z \approx b_z / L_0 \ll |\nabla \times \mathbf{b}| \approx b_0 / \delta \ . \tag{4}$$

Equation (4) implies that the compressibility of $\mathbf{b}$ is insignificant. The criterion of the neglect of the compressibility is discussed in more detail in Appendix A. So we can express $\mathbf{b}$ in terms of the longitudinal vector potential $\psi(x, y, z)$ which also depends on z very slowly:

$$\mathbf{b} = \nabla\psi \times \hat{z} \, . \tag{5}$$

Thus, to a first approximation, due to the large parameter R, every field line produces cylindrical screw-type revolutions around a surface of constant ψ. The transverse walk of a line is hence restricted to the size of the corresponding contour of $\psi(x, y, z)$, at a given coordinate $z = z_0$. For a turbulent state of a magnetized plasma one may assume a random distribution of ψ. For simplicity we take the perturbation as statistically isotropic. Among the contours of a random function, most are closed on the correlation scale δ. However, there is a small portion of level lines that are much longer. For example, the Earth's relief exhibits not only lakes and islands, but also continental coastlines. In the limit $R \to \infty$ (exact 2-D case) the integral of motion $\psi(x, y)$, even though a random function, prevents stochastic spreading of magnetic lines. Yet, at large but finite R the magnetic transport should onset, beginning in the first turn from very large contours of $\psi(x, y, z_0)$ which provide "long-correlated jumps" of field lines. The importance of these large contours for plasma transport is due to their coherent contribution to the diffusion of magnetic lines.

So, under condition (1), the effective transport must be long-correlated, due to the important role of transverse walk of magnetic lines to distances large compared to δ. For the treatment of the problem one must study the distribution of isolines of a random function of two variables over their sizes. This problem is closely connected with the percolation problem (c.f. Stouffer 1979, Shklovskii & Efros 1984). Kadomtsev & Pogutse (1978) were the first to point out the relevance of the percolation theory to the limit $R \gg 1$, but the cursory application of that theory in their work lead to an incorrect expression for the magnetic diffusion coefficient.

Gruzinov, Isichenko & Kalda (1990) considered the problem of low-frequency turbulent diffusion in two dimensions (being exactly equivalent to the problem of diffusing magnetic lines described by Eq. (1) with $R \gg 1$). Using recent analytical results in 2-D percolation theory (Saleur & Duplantier 1987) they derived the following scaling of the turbulent diffusion coefficient:

$$D_m \approx b_0 \delta R^{-3/10} , \tag{6}$$

Eq. (6) is written as the diffusivity of magnetic lines. The assessment (6) suggests that in two-dimensional (integrable) case $R = \infty$ magnetic diffusion D_m vanishes, which differs from the previously reported estimate $D_m \approx b_0\delta$ (Kadomtsev & Pogutse 1978, Galeev & Zelenyi 1981, Krommes, Oberman & Kleva 1983).

In the limit $R \gg 1$ the exponentiation of adjacent lines is different from in the quasi-linear limit. In the percolation limit the role of Kolmogorov entropy in average transport is more complicated than in the quasi-linear limit, and requires a more subtle consideration. Specifically, in this paper it is shown that an appropriate test particle decorrelation length is expressed through the length l_m of the convolution of magnetic flux tube (defined in Appendix B), rather than that of field lines exponentiation. Nevertheless, these two processes still remain closely connected.

Perhaps, the most striking feature of stochastic transport in the percolation limit is that the major part the heat and particle fluxes is concentrated in very thin regions occupying infinitesimally small fraction of plasma volume, and, due to their self-similarity, being described in terms of fractal geometry. At the same time, due to the previously discussed role of large contours of ψ, the cross-field flux correlation function decays relatively slowly, up to an anomalously large correlation, or mixing, length $a_m \gg \delta$. These features, on the one hand, leave no hope for applying standard averaging techniques and/or convergence of perturbation series, and, simultaneously, make it extremely difficult to analyze the problem numerically. To study the geometry of stochastic magnetic field in the limit $R \gg 1$, one is forced to employ a direct $\mathbf{x}$-space non-perturbative formalism, like the percolation theory.

In other respects the solution of the effective heat conductivity problem in the percolation limit is based upon the same techniques as the quasi-linear approximation. In what follows, we will use the test particle motion analysis in order to obtain scaling laws for cross-field plasma transport, with particular emphasis on distinguishing physically different transport regimes.

The remainder of the article is organized as follows. In Sec. II we discuss the diffusion and the exponentiation of magnetic lines. In Sec. III the effective perpendicular electron heat conductivity χ_{eff} is expressed through the test particle decorrelation time t_d both for the hydrodynamic ($\nu_e\, t_d > 1$) and the kinetic ($\nu_e\, t_d < 1$) limits, which is intended to generalize the "double diffusion" theory by Krommes, Oberman & Kleva (1983). The very decorrelation time is evaluated in Secs. IV and V, for stationary and non-stationary magnetic perturbations, respectively. In Sec. VI we demonstrate the transition between various anomalous cross-field transport regimes and summarize the results obtained. Lengthy auxiliary arguments are outlined in the Appendices. In Appendix A we take into account the compressibility of the transverse component **b** of the magnetic perturbation in order to establish the limits of applicability of the incompressible approximation. In Appendix B we discuss the stochasticity of magnetic lines and its relation to the decorrelation of a test particle from a specified field line. Appendix C is devoted to the effective heat conductivity in a 2-D random magnetic field ($R = \infty$). For this case the application of the Dykhne (1971) technique is discussed.

In the present paper we make use of the notations and approaches from Part I of the article thereby reducing the need for lengthy explanations.

II. Percolation geometry of stochastic magnetic fields

In this section we relate stochastic magnetic field lines to contours of a random function and discuss the application of the continuum percolation problem to the study of the magnetic field line geometry. Results of Gruzinov, Isichenko & Kalda (1990), which is given a short review, are applied.

As stated in Sec. I, at $R \gg 1$ the (x, y)-projection of a magnetic line nearly follows the contours of the vector potential $\psi(x, y, z)$, with z considered as a slowly varying parameter. Bearing in mind that in the limit involved large contours are of primary importance, one must preface the magnetic diffusion problem with the study of the statistics of random isolines.

The statistical topography of a random relief $\psi(x, y)$ is described by the continuum percolation problem (Shklovskii & Efros 1984). Since this problem can be considered as a limiting case of a lattice percolation problem (see, for instance, the review by Stouffer 1979), and due to the universality of percolation critical exponents (Sykes & Essam 1964), the scaling of the contours distribution function in the limit of large contour size can be determined analytically (Gruzinov, Isichenko & Kalda 1990). Let us briefly summarize the results of the paper that are relevant for our discussion:

(a) Suppose one can ascribe to the random and statistically isotropic function $\psi(x, y)$ a single characteristic oscillatory amplitude $\psi_0 = b_0\delta$, and a single spatial (correlation) scale δ. Then the distribution function of the contours of ψ over their diameters a has the following long-range scaling:

$$F(a) \approx \delta/a \ , \quad a \gg \delta \ . \tag{7}$$

Here $F(a)$ implies the fraction of area occupied by contours with diameters (understood as maximum linear size) from a to $2a$. According to (7), most of contours have sizes

$a \approx \delta$. The space-average of a quantity A depending on the diameter of a contour can be calculated with the help of Eq. (7) as

$$\langle A \rangle = \int_{\delta}^{\infty} F(a)A(a)da/a \ . \tag{8}$$

(b) Any sufficiently long contour with $a \gg \delta$, considered on scales λ, $\delta \ll \lambda \ll a$, is a fractal (i.e. statistically self-similar) curve with the fractal dimension

$$d_h = (\nu + 1)/\nu = 7/4 \ , \tag{9}$$

where $\nu = 4/3$ is the correlation exponent of the 2-D percolation problem. In particular, the length L of a contour is much greater than its diameter a and scales as

$$L(a) \approx \delta(a/\delta)^{(\nu+1)/\nu} \ , \quad a \gg \delta \ . \tag{10}$$

(c) From (a) and (b) it follows that the set of contours with diameters of the order of a (say, a to $2a$) consists of densely packed fractal cells (let us call them "a-type cells" - see Fig. 1) each of which looks like a web with the thread width

$$h(a) \approx a^2 F(a)/L(a) \approx \delta(\delta/a)^{1/\nu} \ , \quad a \gg \delta \ . \tag{11}$$

In what follows, we will call the quantity $h(a)$ the width of the a-type cell. Below we will also refer to the cells composed of the contours of magnetic vector potential ψ as "the magnetic cells".

(d) Let the function ψ include a smooth dependence on a third parameter z, $\psi(x, y, z)$ being a random function of (x, y) at every fixed z. If the characteristic inhomogeneity (correlation) scale over z is L_0, then a-type cells are unrecognizably changed upon the displacement along z,

$$\delta z(a) \approx L_0 h(a)/\delta \ , \quad a \gg \delta \ , \tag{12}$$

which implies the perturbation corresponding to the thickness of the web.

The above results relate to a "single-scale" random function. Perhaps a more interesting case would deal with multiple scales characterized by, say, power Fourier spectrum of $\psi(\mathbf{r})$. The problem of multiple scale random topography has been solved recently in Isichenko & Kalda (1990) using an extended percolation approach. One of the peculiarities of that model is the fractal dimension d_h of long level lines which can take any value between 1 and 7/4, depending on the spectrum of ψ. For the sake of simplicity, the present paper is restricted to the widely used single scale model.

The geometric properties of random contours (a)–(d) are sufficient to calculate the magnetic diffusivity D_m and the Kolmogorov entropy in the percolation limit $R \gg 1$. The mixing length a_m, i.e. the size of a contour performing the most effective contribution to D_m, can be assessed as the maximum transverse correlated walk of a magnetic line. This corresponds to the case when the field line projection performs a complete revolution around the contour, by the longitudinal displacement (12) resulting in the destruction of the magnetic cell via the reconnection of contours. So, we write

$$\delta z(a_m) b_0 = L(a_m) \ . \tag{13}$$

From Eqs. (10)–(13) one readily obtains

$$a_m = \delta \, R^{\nu/(\nu+2)} \ , \tag{14}$$

which corresponds to the longitudinal displacement

$$z_m \equiv \delta z(a_m) = L_0 R^{-1/(\nu+2)} \ , \tag{15}$$

and the magnetic cell width

$$h_m \equiv h(a_m) = \delta R^{-1/(\nu+2)} \ . \tag{16}$$

Now the diffusivity of magnetic lines can be heuristically calculated as

$$D_m = F(a_m) a_m^2 / z_m = b_0 h_m = b_0 \delta \, R^{-1/(\nu+2)} \ . \tag{17}$$

With the numerical value of the percolation exponent $\nu = 4/3$ Eq. (17) yields the expression (6).

Rigorously speaking, to calculate D_m, one should average the magnetic diffusion over all possible scales with the help of (8), namely

$$D_m = \int_\delta^\infty D_m(a) F(a) da/a, \tag{18}$$

with the "partial" diffusivity

$$D_m(a) = \frac{a_\perp^2(a)}{\delta z(a)} \ . \tag{19}$$

Here $a_\perp(a)$ denotes the transverse displacement of a magnetic line corresponding to the longitudinal walk $\delta z(a)$. At $a < a_m$ a magnetic line performs many revolutions around its contour, thus giving

$$a_\perp(a) = a \ , \quad a < a_m \ . \tag{20}$$

In the opposite case, $a > a_m$, the line passes only a small part of the contour resulting in the displacement found from Eq. (10):

$$a_\perp(a) = \delta \left(\frac{\delta z(a) b_0}{\delta} \right)^{\nu/(\nu+1} \quad , \quad a > a_m \ . \tag{21}$$

Combining Eqs. (18)–(21) we simply calculate D_m to obtain the above result (17). This supports the conclusion that the major contribution to magnetic diffusion (i.e. of order of 50%) is made by a small share of magnetic lines occupying the volume fraction

$$F(a_m) = R^{-\nu/(\nu+1)} \ll 1 \ . \tag{22}$$

Another feature of magnetic transport in the percolation limit is its self-similar behavior in the inertial range of transverse scales $[\delta, a_m]$. This is connected with the fractal geometry of random contours (Mandelbrot 1982). Let us introduce the concept of the effective transport region, denoting the region of minimum volume responsible for, say, 50% of transport. Then in the inertial range $[\delta, a_m]$ of scales the effective transport region is a fractal, whose fractal dimension d_f can be calculated by adding unity to the fractal dimension of its plane cross-section (Mandelbrot 1982), being in our case the a_m-type magnetic cell. Hence, in the single scale approximation, using (9) we have

$$d_f = d_h + 1 = 2.75 \ . \tag{23}$$

In a more general case of multiple scale magnetic turbulence $1 \le d_h \le 1.75$ (Isichenko & Kalda 1990), hence

$$2 \le d_f \le 2.75 \ . \tag{24}$$

Analogously to the quasi-linear limit, magnetic lines described by Eq. (1) in the percolation limit also exhibit stochastic exponentiation. However, this behavior is now strongly intermittent. A given couple of infinitesemally close field lines diverge for a very long distance very slowly (namely, linerly with z), but then the distance between the lines increases abrubtly up to a finite value of order of δ. This effect is governed by the dustribution of saddle (elliptic) points of $\psi(x, y, z_0)$ and is more convenient to describe in terms of the elongation of a curve being projected along magnetic lines. In this representation the irregularities of the magnetic lines scattering are smeared out, and the curve undergoes an exponential anfractuous elongation with the growth rate estimated in Gruzinov, Isichenko & Kalda (1990). For the length of the curve we have

$$\mathcal{L}(z) \approx \mathcal{L}(0) \exp(z/l) \ , \quad l \approx \frac{L_0}{\sqrt{R} \log R} \ . \tag{25}$$

The inverse quantity of l could be regarded as the Kolmogorov, or topological entropy for the case under consideration.

III. Connection between the Effective Heat Conductivity and the Time of Decorrelation

In this section expressions for $\chi_{\text{eff}}(t_d)$, both for collisional ($\nu_e\, t_d > 1$) and collisionless ($\nu_e\, t_d < 1$) cases are derived. The evaluation of the decorrelation time t_d is addressed in next sections.

The effective diffusivity of a test particle, being the same by the order of magnitude as the effective heat conduction χ_{eff}, is defined through the square-average transverse displacement $\mathbf{r}_\perp$ of an electron at the decorrelation time:

$$\chi_{\text{eff}} \approx \left\langle \mathbf{r}_\perp^2(t_d) \right\rangle / t_d \ , \tag{26}$$

where the averaging is taken over space of initial conditions or, similarly, over magnetic lines.

While moving along a magnetic line, which nearly traces out a spiral, the (x, y)-projection of the point passes the distance

$$L(z) \approx b_0 z \ . \tag{27}$$

If this path does not exceed the transverse correlation length δ, the transverse displacement is equal to $L(z)$, regardless of the magnetic line. For the percolating magnetic lines, and $L(z) \gg \delta$, the displacement $r_\perp(z)$ is defined from Eq. (10), but cannot exceed the diameter of the given magnetic line spiral a:

$$r_\perp(z) \approx \min\left\{\delta(L(z)/\delta)^{\nu/(\nu+1)}, a\right\} \ . \tag{28}$$

If we use the Eqs. (7) and (8), then for this case we obtain

$$\left\langle r_\perp^2(z) \right\rangle \approx \delta^2 (L(z)/\delta)^{\nu/(\nu+1)} \ . \tag{29}$$

Expression (29) is valid until $r_\perp^2$ exceeds the mixing length (14), i.e. while $L(z) < L_m = \delta \cdot R^{(\nu+1)/(\nu+2)}$, and after that the transverse walk of the magnetic line is a diffusion-like one with the diffusivity (6): $\langle \mathbf{r}_\perp^2(z) \rangle \approx D_m |z|$.

Summarizing what has been said above we derive the following expression for the effective transverse heat conduction:

$$\chi_{\rm eff} \approx \begin{cases} b_0^2 \, z_d^2/t_d \ , & z_d < \delta/b_0 \ , \qquad \text{(QD)} \qquad (30a) \\ (b_0 \, z_d/\delta)^{(\nu+1)} \delta^2/t_d \ , & \delta/b_0 < z_d < z_m \ , \qquad \text{(MD)} \qquad (30b) \\ D_m \, z_d/t_d) \ , z_m < z_d \ . & \qquad \text{(MD)} \qquad (30c) \end{cases}$$

Here z_d denotes the path the test particle passes along the magnetic field in the time of decorrelation t_d. Depending on the collision frequency ν_e, it is expressed as follows:

$$z_d(t_d) = \begin{cases} (\chi_{||} \, t_d)^{1/2} \ , & \nu_e \, t_d > 1 \ , \qquad (31a) \\ v_e \, t_d \ , & \nu_e \, t_d < 1 \ , \qquad (31b) \end{cases}$$

which means the hydrodynamic (collisional) and the kinetic (collisionless) limits, respectively. The abbreviations used in (30) distinguish the quick decorrelation regime (QD), intermediate regime (IR), and the regime of magnetic line diffusion (MD). Equation (30) is similar to Eq. (18) of Part I derived for the quasi-linear limit, except for the appearance of a new intermediate regime lying between the QD and MD regimes.

When (31a) is substituted into Eqs. (30a) and (30c), two expressions for $\chi_{\rm eff}$ are obtained corresponding to the "fluid" ($\chi_{\rm eff} = \chi_{||} \, b_0^2$) and the "double diffusion" ($\chi_{\rm eff} = D_m(\chi_{||}/t_d)^{1/2}$) regimes. In the kinetic limit (31b) Eqs. (30a) and (30c) yield the "double streaming" ($\chi_{\rm eff} = b_0^2 \, v_e^2 \, t_d$) and "collisionless" ($\chi_{\rm eff} = D_m \, v_e$) regimes correspondingly. These four regimes also exist in the quasi-linear limit $R \ll 1$ (Krommes, Oberman & Kleva 1983, Isichenko 1990). Thus, we infer that in the percolation limit $R \gg 1$, in addition to all the regimes pertinent to the quasi-linear limit, there exists the new intermediate regime (IR) given by the expression (30b), both in hydrodynamic and kinetic approximations.

IV. Decorrelation in Stationary Stochastic Magnetic Field

In this section we assess the time of test particle decorrelation t_d in a stationary "braided" magnetic field. Among the causes of the decorrelation are either finite transverse diffusivity $\chi_\perp$ (in collisional case) or finite gyroradius r_e (in collisionless limit).

While in the quasi-linear limit the decorrelation time has been defined as the time it would take the test particle to leave a magnetic flux tube with the initial diameter δ, in the percolation limit the decorrelation occurs when the particle leaves the effective transport region responsible for the anomalous transport. Let us now evaluate the width h of the magnetic cell in various regimes.

In the regime of quick decorrelation, $z_d < \delta/b_0$, where the transport loses its long-correlated features, one concludes that $h \approx \delta$. Otherwise, h is defined by Eq. (11), where one must substitute for the diameter a either the transverse displacement $\delta(L(z_d)/\delta)^{\nu/(\nu+1)}$ in the decorrelation time (in the intermediate regime) or the mixing scale (14) (in the regime of magnetic diffusion). This results in the following width of the magnetic cell, depending on the decorrelation time t_d:

$$h(t_d) \approx \begin{cases} \delta\,, & \text{(QD)} \quad (32a) \\ \delta(\delta/b_0\, z_d)^{1/(\nu+1)}\,, & \text{(IR)} \quad (32b) \\ \delta \cdot R^{-1/(\nu+2)}\,, & \text{(MD)} \quad (32c) \end{cases}$$

where the inequalities are correspondingly identical to those of Eq. (30).

One must now define the quantity t_d as the time it takes the particle to leave the magnetic flux tube, whose cross-section $z = 0$ is the magnetic cell of width given by Eq. (32). (Imagine Fig. 1 of Part I when in the cross section $z = 0$ lies not a circle, but a fractal shown in Fig. 1 of this paper.) In a stationary magnetic field and in the collisional case the decorrelation might occur due to the direct cross-field background diffusion with the characteristic time $h^2(t_d)/\chi_\perp$. However, at very small $\chi_\perp$ the electron can decorrelate faster, when first going some distance along the magnetic line, and then diffusing across smaller width of the magnetic tube $\widetilde{h}(z)$ due to the convolution of magnetic flux tube constructed from the magnetic cell. (This effect of stochasticity-driven decorrelation has been pointed out by Rechester & Rosenbluth (1978) for the quasi-linear limit.) The convolution means the thinning of the tube walls due to the area-preserving stochastic stretching of the field-lines-projected magnetic cell. This effect causes a decrease of the flux tube thickness $\widetilde{h}(z)$ which can be described with the model equation

$$\widetilde{h}(z) \approx h \exp(-|z|/l_m) \quad , \quad l_m \approx hL_0/\delta \ . \tag{33}$$

The interconnection of this effect with the stochastic instability, as well as the evaluation of the convolution length l_m for the percolation limit are discussed in more detail in Appendix B.

Now one can propose the equation for t_d, accounting both for direct transverse decorrelation and stochasticity of magnetic lines:

$$h(t_d) \exp\left[-z_d(t_d)\delta/(h(t_d)L_0)\right] = (\chi_\perp \, t_d)^{1/2} \ . \tag{34}$$

Together with (31a) this may be readily solved to obtain

$$t_d \approx \begin{cases} \delta^2/\chi_\perp \ , & \chi_\perp/D_{||} > 1 \quad \text{(QD)} \quad (35a) \\ (\delta^2/\chi_\perp)(\chi_\perp/D_{||})^{1/(\nu+2)} \ , & 1 > \chi_\perp/D_{||} > R^{-2} \quad \text{(IR)} \quad (35b) \\ (z_m^2/\chi_{||}) \ln^2(D_{||}/\chi_\perp R^2) \ , & R^{-2} > \chi_\perp/D_{||} \ . \quad \text{(MD)} \quad (35c) \end{cases}$$

Here for brevity we have introduced the notation $D_{||} \equiv \chi_{||} \, b_0^2$ which means the longitudinal test particle diffusivity projected to the (x, y) plane.

Let us now turn to the kinetic limit $\nu_e \, t_d < 1$. If we take for simplicity, as was done in Part I, the transverse electron position uncertainty of order of its gyroradius r_e, then the collisionless decorrelation time t_d should be given by

$$h(t_d) \exp(-z_d(t_d)/l_m) = r_e \ , \tag{36}$$

together with Eqs. (31b) and (32). At sufficiently small gyroradius, $r_e < h_m \equiv \delta\, R^{-1/(\nu+2)}$, this yields

$$t_d = (z_m/v_e)\ln(h_m/r_e)\ . \tag{37}$$

The kinetic evaluation (37) matches smoothly the hydrodynamic one in (35c) at $\omega_{Be}\,\tau_e = z_m/r_e$.

V. Non-Stationary Decorrelation

Now, let us take $X_\perp = 0$, $r_e = 0$ and consider the decorrelation being the result only of the non-stationarity of magnetic perturbations $\delta\mathbf{B}(x, y, z, t)$ varying with the characteristic frequency ω.

First of all, one can see that the dependence of the perturbation on time simultaneously results in two distinct kinds of decorrelation, which can be referred to as kinematic and dynamic ones.

On the one hand, percolating magnetic lines evolve in such a way that they reconnect with dominating shortening of their transverse spread, since smaller contours are more probable. The problem is similar to the discussed above, concerning the reconnection of contours of $\psi(x, y, z)$ as z is changed. The only difference is that now ψ depends also on time, and the magnetic cells composed of contours of $\psi(x, y, z, t)$ are destroyed (through reconnection) at a fixed z upon the elapsing of time

$$t_h = h(t_h)/(\omega\delta) \; , \tag{38}$$

Equation (36) is quite analogous to Eq. (12) where one substitutes $\delta z \to t_h$, $L_0 \to \omega^{-1}$. The only difference is that now the magnetic cell width h itself depends on the decorrelation time t_h. Thus Eq. (38) should be solved for the kinematic decorrelation time t_h.

On the other hand, non-stationarity leads to a test particle moving not exactly along the magnetic line, even if the latter does not reconnect. This, dynamic, decorrelation can be described by Eq. (3), while accounting for the dependence of $\mathbf{b}$ on time:

$$d\mathbf{r}_\perp/dz = \mathbf{b}(\mathbf{r}_\perp, z, t) \; , \tag{39}$$

together with closure condition (31a) or (31b). Let us consider the time-dependence in Eq. (39) as a small perturbation. Similar to the calculation of Part I we have:

$$d\mathbf{r}_\perp/dz = \mathbf{b}(\mathbf{r}_\perp, z, 0) + t(z)\,\mathbf{b}_1(\mathbf{r}_\perp, z) \; , \tag{40}$$

$$\mathbf{b}_1(\mathbf{r}_\perp, z) \equiv \partial \mathbf{b}(\mathbf{r}_\perp, z, t)/\partial t\Big|_{t=0} \approx \omega b_0 \ ,$$

$$t(z) = \begin{cases} z^2/\chi_{\parallel} \ , & \nu_e t_d > 1 \\ z/v_e \ , & \nu_e t_d < 1 \ . \end{cases}$$

The second term in the right-hand side of Eq. (40) represents a non-correlated (with respect to the first term) slow drift with the correlation length $z = \delta/b_0$, which is also the fall-out length of $\mathbf{b}_1$ correlation function. So the perturbation theory yields the following estimate for the square-average additional displacement $\mathbf{r}_{\perp\omega}$ due to the non-stationarity:

$$\left\langle \mathbf{r}_{\perp\omega}^2(z) \right\rangle = \int_0^z \int_0^z dz' dz'' t(z') t(z'') \left\langle \mathbf{b}_1(z') \mathbf{b}_1(z'') \right\rangle \approx \omega^2 b_0^2 \begin{cases} z^2 t^2(z) \ , & z < \delta/b_0 \\ (\delta/b_0) \int_0^z t^2(z') dz' \ , & z > \delta/b_0 \ . \end{cases} \tag{41}$$

The dynamic decorrelation time t_p may now be estimated from the equation similar to (34):

$$h(t_p) \exp\left[-z(t_p)\delta/(h(t_p) L_0)\right] = \left\langle \mathbf{r}_{\perp\omega}^2(z(t_p)) \right\rangle^{1/2} \ . \tag{42}$$

Resolving Eqs. (38) and (42) in every limit (collisional (31a) and collisionless (31b)) we find the expressions for t_h and t_p, which are not written here. The true decorrelation time is the minimum of them: $t_d = \min(t_h, t_p)$.

Comparing the two times t_h and t_p in every interval of parameters, we finally derive the non-stationary decorrelation time. In the hydrodynamic limit $\nu_e t_d > 1$ the result is

$$t_d \approx \begin{cases} \omega^{-1} \ , & \text{(QD)} & \text{(43a)} \\ (\delta^2/D_{\parallel}) \Omega_{hy}^{-4(\nu+1)/(5\nu+7)} \ , & \text{(IR)} & \text{(43b)} \\ (z_m^2/\chi_{\parallel}) \ln^2\left[\Omega_{hy}^{-1} R^{-(5\nu+7)/(2\nu+4)}\right] \ . & \text{(MD)} & \text{(43c)} \end{cases}$$

The corresponding inequalities are:

$$\begin{cases} \Omega_{hy} > 1 \ , & \text{(QD)} \quad (43a) \\ 1 > \Omega_{hy} > R^{-(5\nu+7)/(2\nu+4)} \ , & \text{(IR)} \quad (43b) \\ R^{-(5\nu+7)/(2\nu+4)} > \Omega_{hy} \ , & \text{(MD)} \quad (43c) \end{cases}$$

In the kinetic limit $\nu_e \, t_d > 1$ we obtain in a similar way:

$$t_d \approx \begin{cases} \omega^{-1} \ , & \text{(QD)} \quad (44a) \\ (\delta/v_{||})\Omega_{hy}^{-2(\nu+1)/(3\nu+5)} \ , & \text{(IR)} \quad (44b) \\ (z_m/v_{||}) \ln \left[\Omega_{ki}^{-1} \, R^{-(3\nu+5)/(2\nu+4)}\right] \ , & \text{(MD)} \quad (44c) \end{cases}$$

$$\begin{cases} \Omega_{ki} > 1 \ , & \text{(QD)} \quad (44a) \\ 1 > \Omega_{ki} > R^{-(3\nu+5)/(2\nu+4)} \ , & \text{(IR)} \quad (44b) \\ R^{-(3\nu+5)/(2\nu+4)} > \Omega_{ki} \ . & \text{(MD)} \quad (44c) \end{cases}$$

In Eqs. (43)–(44) the dimensionless frequencies

$$\Omega_{hy} \equiv \omega \delta^2 / D_{||} \quad , \quad \Omega_{ki} \equiv \omega \delta / v_{||} \ , \tag{45}$$

have been introduced for collisional and collisionless cases, respectively. In addition, $v_{||} \equiv v_e \, b_0$ denotes the projection of longitudinal electron velocity to the (x, y)-plane.

Note that regardless of the collisionality, in the QD regimes the kinematic decorrelation t_h dominates, while in the rest of the regimes (IR and MD) the dynamic decorrelation time t_p is shorter.

VI. Effective Heat Conductivity — Discussion of Results

The formulas (30) and one of the expressions (31a), (35), (43) (in the hydrodynamic limit) or (31b), (37), (44) (in the kinetic limit) solve the problem stated. Among the times of stationary (35), (37), and non-stationary (43), (44), decorrelation one should choose the shorter one.

If one knows the main magnetic perturbation parameters b_0, L_0, δ, ω, and the plasma parameters $\chi_{\|}$, $\chi_{\perp}$, ν_e, v_e, r_e, the effective cross-field heat conductivity can be evaluated with the help of the algorithm shown in Fig. 3.

Let us write down here the expressions for the effective heat conduction in the most obvious limits. Firstly, consider the stationary limit ($\omega = 0$) at $r_e < h_m$. Under such conditions we have

$$\chi_{\text{eff}} \approx \begin{cases} \chi_{\perp}\,, & \chi_{\perp}/D_{\|} > 1 \qquad \text{(QD)} \qquad (46a) \\ (D_{\|}\,\chi_{\perp})^{1/2}\,, & 1 > \chi_{\perp}/D_{\|} > R^{-2} \qquad \text{(IR)} \qquad (46b) \\ (D_{\|}/R)\ln^{-1}\left[D_{\|}/\chi_{\perp}R^2\right]\,, & R^{-2} > \chi_{\perp}/D_{\|} > (r_e/b_0\,z_m)^2 \qquad \text{(MD)} \qquad (46c) \\ D_m\,v_e\,, & (r_e/b_0\,z_m)^2 > \chi_{\perp}/D_{\|}\,. \qquad \text{(MD)} \qquad (46d) \end{cases}$$

The first three regimes (46a–c) are hydrodynamic while the last one (46d) is a kinetic one.

In the case of strictly two-dimensional magnetic perturbations ($R = \infty$) the result is given by expression (46b). The same estimate has been obtained by Kadomtsev & Pogutse (1978) in two different ways – one of them using the Dykhne (1971) method incorrectly. The essence of this misunderstanding and the application of the Dykhne technique are discussed in Appendix C.

It is interesting that expressions (46b) and (46c), related to the percolation limit, do not include the percolation exponent ν, and hence could be obtained using a simplified independent approach, as demonstrated Kadomtsev & Pogutse (1978). (The only difference between (46c) and their result lies in the logarithmic denominator.) However, the mixing length a_m does include ν (see Appendix C).

Similar to the quasi-linear limit, non-trivial quick decorrelation regimes; i.e. QD-regimes with $\chi_{\text{eff}} \gg \chi_\perp$, such as "fluid" ($\chi_{\text{eff}} = D_\|$) and "double streaming" regimes become possible only in a non-stationary stochastic magnetic field. Taking $\chi_\perp = 0$, $r_e = 0$, and accounting only for finite frequency ω of the perturbations, one gets from Eqs. (30), (43), and (44):

$$\chi_{\text{eff}} \approx \begin{cases} D_\| \,, & \text{(QD)} \quad (47a) \\ D_\| \, \Omega_{hy}^{2(\nu+2)/(5\nu+7)} \,, & \text{(IR)} \quad (47b) \\ (D_\|/R) \ln^{-1}\left[\Omega_{hy}^{-1} \, R^{-(5\nu+7)/(2\nu+4)}\right] \,, & \text{(MD)} \quad (47c) \end{cases}$$

for the collisional limit $\nu_e \, t_d > 1$; and

$$\chi_{\text{eff}} \approx \begin{cases} v_\|^2/\omega \,, & \text{(QD)} \quad (48a) \\ \delta \, v_\| \, \Omega_{ki}^{2/(3\nu+5)} \,, & \text{(IR)} \quad (48b) \\ D_m \, v_e \,, & \text{(MD)} \quad (48c) \end{cases}$$

in the collisionless case $\nu_e \, t_d < 1$, where the inequalities for (47) and (48) are the same as those in expressions (40) and (41), respectively.

One can follow the transition between different regimes when the characteristic frequency ω varies. Figure 4 demonstrates this transition for the case $\delta/b_0 < \lambda_e < z_m$, where $\lambda_e = v_e/\nu_e$ is the mean free path of electrons.

In conclusion, we restate the key points of the analysis:

(i) The theory of anomalous transport in a "braided" magnetic field in the strong turbulence limit $R \gg 1$, which is opposite to the quasi-linear limit, must use the percolation theory methods.

(ii) In the percolation limit $R \gg 1$, excluding quick decorrelation regimes, the main transport is concentrated on fractals, consisting of a small fraction of percolating magnetic lines.

(iii) In addition to all the quasi-linear regimes or their direct analogs, in the percolation limit there arise a number of new intermediate regimes of the anomalous electron heat conductivity.

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Appendix A Effect of Compressibility of Transverse Magnetic Perturbation

The compressibility of **b** takes place only in the case of non-zero longitudinal component of the magnetic perturbation: $b_z(x, y, z) \equiv \delta B_z/B_0 \neq 0$. Then the transverse magnetic component can be written in the form

$$\mathbf{b} = \nabla_\perp \psi(x, y, z) \times \hat{z} + \nabla_\perp \varphi(x, y, z) \ . \tag{A1}$$

Taking the divergence of Eq. (A1) we obtain the relation for φ:

$$\nabla_\perp^2 \varphi = -\partial b_z/\partial z \ . \tag{A2}$$

The field line motion due to the magnetic perturbation (A1) consists of two parts: the incompressible motion $\nabla_\perp \psi \times \hat{z}$ (approximately along the isolines of ψ), plus a small drift on account of the potential correction. The drift part of the displacement can be calculated as

$$\mathbf{r}_{\perp\varphi}(z) = \int_0^z \nabla_\perp \varphi(\mathbf{r}_\perp(z'), z') dz' \ , \tag{A3}$$

where $\mathbf{r}(z')$ is the solution of the non-perturbed Equation (3) (i.e. at $\varphi = 0$). Since the quantity φ changes its sign randomly along the isolines of ψ, the drift (A3) can be described in terms of diffusion: $r_{\perp\varphi}(z) \approx (D_\varphi z)^{1/2}$, where the diffusion coefficient is the product of characteristic speed and the correlation length: $D_\varphi \approx (\varphi/\delta)(\delta/b_0)$. Using the estimate $\varphi \approx b_z \delta^2/L_0$, which follows from (A2), we obtain additional displacement caused by the compressible correction:

$$r_{\perp\varphi} \approx (z b_z \delta/R)^{1/2} \ . \tag{A4}$$

Comparing this quantity with the minimum width of magnetic cell h_m (32c), at z, corresponding to the maximum longitudinal mixing length z_m (15), we arrive at the following condition, under which one may neglect compressibility effects:

$$b_z/b_0 < R^{-1/(\nu+2)} \ . \tag{A5}$$

The inequality (A5) is the criterion for the neglect of the compressibility of **b** in MD regimes of anomalous transport. In IR regimes the condition (A5) is sufficient, however not necessary, since in that case the effective transport width h given by Eq. (32b) is greater than h_m, while the longitudinal mixing length z_d is less than z_m. In QD regimes the compressibility of **b** is irrelevant and does not affect the transport.

Note that in a strong magnetic field the longitudinal perturbation of **B** is energetically much more expensive than the transverse one and hence must be much smaller. This makes the condition (A5) not too restrictive.

Appendix B Convolution of Magnetic Flux Tube

The map

$$\mathbf{r}_\perp(0) \longrightarrow \mathbf{r}_\perp(z) \; , \tag{B1}$$

given by the initial value problem solution of Eq. (3), may be thought of as an incompressible, and consequently, as a Hamiltinian one. The corresponding Hamiltonian $\psi(x, y, z)$ depending on "time" z, admits the stochastic behavior; moreover, such a behavior is typical for a generic Hamiltonian (c.f. Arnold 1978). It means that every curve in the phase space (x, y), consisting of points evolving according to the equation of motion (3), elongates in time[1] exponentially, as every two close points exponentiate from each other. The mean growth rate of this stochastic instability (the Kolmogorov entropy) has been calculated in the quasi-linear limit $R = v/(\delta\omega) \ll 1$ by Krommes (1978), Rechester,Rosenbluth & White (1979), and Krommes, Oberman & Kleva (1983) for the case of a strong shear. For the opposite limit without shear the Kolmogorov entropy at $R \ll 1$ has been estimated in Kadomtsev & Pogutse (1978):

$$\gamma_s \approx \omega R^2 \quad , \quad R \ll 1 \; . \tag{B2}$$

In the percolation limit $R \gg 1$ γ_s has been calculated in Gruzinov, Isichenko & Kalda (1990):

$$\gamma_s \approx \omega R^{1/2} \ln R \quad , \quad R \gg 1 \; . \tag{B3}$$

Rigorously speaking, the result (B3) is not the Kolmogorov entropy, being the mean growth rate of close orbits scattering, but rather the topological entropy, or maximum growth rate, defining the elongation of a liquid curve.

The decorrelation of test particles in the magnetic field is not directly related to the

[1] In this Appendix, for the sake of clarity, we take $z \to t$, $\mathbf{b} \to \mathbf{v}$, $L_0 \to \omega^{-1}$, thus transfering the magnetic line problem to the passive scalar problem in the random 2-D incompressible, weakly non-stationary flow $\mathbf{v}(x, y, t) = \nabla\psi(x, y, t) \times \hat{z}$, varying in time with the small characteristic frequency $\omega \ll v/\delta$. In this representation the magnetic line diffusion corresponds to the "turbulent diffusion" in the flow.

rate of stretching of a curve, but rather to the convolution of flux tube constructed from a magnetic cell. This effect is connected with the evolution of the characteristic width $\tilde{h}(t)$ of the Lagrangian convection cell, which in hydrodynamic terms corresponds to the magnetic cell. This width can be defined as the shortest distance between a point, situated at $t = 0$ somewhere in the middle of the cell (i.e. $\tilde{h}(0) \approx h$), and the cell's boundary. The "Lagrangian convection cell" represents the flow-driven image of the convection cell (effective transport region).

There is a definite connection between stochastic instability of orbits and the Lagrangian stretching of the convection cell. However, this connection is quite different in quasi-linear and percolation limits. The difference begins with the appearance of the cells: while in the quasi-linear limit a circle with the diameter δ can be considered as a magnetic cell (if any sense of this notion at all), in the percolation limit this is a fractal a_m-type cell of the contours of vector potential ψ (see Fig. 1). Furthermore, even for similar 2-D domains one can imagine two kinds of area-preserving maps with exponentially elongated curves. The type I map stretches all the region at once (see Fig. 2a). The map of the second type affects for some time only those parts of the domain which are very close to its initial boundary (see Fig. 2b). One can easily understand that neither of the two contradicts a global stochastic instability. At the same time in the first case the characteristic width $\tilde{h}(t)$ decreases inversely proportional to the perimeter, and in the second case much slower.

The type I stretching takes place in the quasi-linear limit due to the fast variation of the velocity field, which gives rise to destruction of the flow "memory". The type II is characteristic for the percolation limit. In the low-frequency limit $R \gg 1$ the elongation of a Lagrangian curve results from its hooking the saddle points of the flow and its dragging in the channels between the separatrices. (A separatrix means an "eight-like" stream line coming through a saddle.) The exponentiation with the rate (B3) occurs as a result of the reconnection of adjacent separatrices, moving with velocity of order of $\omega\delta$, due to the flow non-stationarity. The inverse growth rate (B3) corresponds to the time it takes a separatrix of the length L to pass the distance δ^2/L to the nearest separatrix, under an optimum choice of L (Gruzinov, Isichenko & Kalda 1990). Yet, during this small time $t_s = \delta/(\omega L)$ the velocity field remains almost unchanged, as the convection cell contains $h/(\delta^2/L) \approx a/\delta \gg 1$ separatrices. The life time of the convection cell, corresponding to the intersection of the most remote separatrices, is much longer:

$$t_h \approx h/(\omega\delta) \gg t_s \; . \tag{B4}$$

(Compare with Eq. (12).) This means that during time t_s the Lagrangian convection cell is nearly unchanged except for narrow channels of the width δ^2/L in the vicinity of its boundaries. As the separatrices of the convection cell keep on reconnecting, the Lagrangian convection cell grows new exponentiating "whiskers" (see Fig. 2b). Finally, near the end of the life time (B4) all the domain is subject to the intensive stretching with the rate given by Eq. (B3).

Hence, the life time (B4) of the convection cell is also the characteristic time of the Lagrangian convection cell stretching. In terms of three-dimensional stationary magnetic fields the time (B4) corresponds to the following length l_m of the convolution of flux tube constructed from a magnetic cell:

$$l_m = L_0 h/\delta \; , \tag{B5}$$

where h means the width of the magnetic cell. However, as it is seen from above, this process is rather complicated and has its own stages. Consequently, the formula (33) is a model one, and the results (35c), (43c), (44c), (46c), following from it, are valid up to a logarithmic accuracy only.

Appendix C Effective Transport in Two-Dimensional Anisotropic Random Media

Let us consider a two-dimensional anisotropic medium in which the direction of the anisotropy $\mathbf{n}(x, y)$ is a function of coordinates, $|\mathbf{n}| = 1$. Let us suggest that along this direction the electric conductivity (or heat conductivity, diffusivity, etc.) equals σ_1 while in the perpendicular direction it is equal to σ_2. So, the local Ohm's law takes the form

$$\mathbf{j} = \sigma_1 \mathbf{E}_{\|} + \sigma_2 \mathbf{E}_{\perp} \ , \tag{C1}$$

$$\mathbf{E}_{\|} = \mathbf{n}(\mathbf{E}\mathbf{n}) \quad , \quad \mathbf{E}_{\perp} = \mathbf{E} - \mathbf{E}_{\|} \ , \tag{C2}$$

$$(\nabla \mathbf{j}) = 0 \quad , \quad \nabla \times \mathbf{E} = 0 \ . \tag{C3}$$

Suppose further that the medium is a self-averaged one, and the mean conductivity is isotropic; i.e. from (C1)–(C3) it follows that

$$\langle \mathbf{j} \rangle = \sigma_{\text{eff}} \left(\sigma_1, \sigma_2; \mathbf{n}(x, y) \right) \langle \mathbf{E} \rangle \ . \tag{C4}$$

Here the angle brackets mean space-averaging over a domain large compared to some mixing length a_m.

Using the anzatz $\mathbf{j}' = C_1 \, \hat{z} \times \mathbf{E}$, $\mathbf{E}' = C_2 \, \mathbf{j} \times \hat{z}$ and comparing (C4) with the resulting "Ohm's law" for $\mathbf{j}', \mathbf{E}'$, Dykhne (1971) has shown that the effective conductivity satisfies the relation

$$\sigma_{\text{eff}} \left(\sigma_1, \sigma_2; \mathbf{n}(x, y) \right) \, \sigma_{\text{eff}} \left(\sigma_2, \sigma_1; \mathbf{n}'(x, y) \right) = \sigma_1 \sigma_2 \ , \tag{C5}$$

where $\mathbf{n}'(x, y) = \hat{z} \times \mathbf{n}(x, y)$ is the perpendicular direction field. From Eq. (C5) it follows that the effective conductivity of a two-dimensional polycrystal with random directions of the main axes of crystallites ($\mathbf{n}(x, y)$ is uniform inside every crystallite and discontinuous on boundaries between them) is equal to $(\sigma_1 \sigma_2)^{1/2}$. In that case the two fields $\mathbf{n}$ and $\mathbf{n}'$ are statistically equivalent, σ_{eff} is an even function of σ_1 and σ_2, which proves the Dykhne (1971) result.

In a 2-D magnetic field (2) (with $\partial/\partial z = 0$) the problem of heat conductivity is equivalent to (C1)-(C3), the field of anisotropy direction being given by a smooth function

$$\mathbf{n} \propto \mathbf{b}(x, y) = \boldsymbol{\nabla}\psi \times \hat{z} \ . \tag{C6}$$

Further, one should put

$$\sigma_1 = \left(\chi_{\|}\mathbf{b}^2 + \chi_{\perp}\right)/(1 + \mathbf{b}^2) \quad , \quad \sigma_2 = \chi_{\perp} \ . \tag{C7}$$

In such a problem the fields $\mathbf{n}$ and $\mathbf{n}' \propto \nabla\psi$ are not statistically equivalent, as one of them is proportional to a solenoidal field and the other to a potential one. These field are by no means statistically equivalent. At first glance, this makes it impossible to apply the Dykhne method to this problem in the way it has been done by Kadomtsev & Pogutse (1978). Nevertheless, their result

$$\chi_{\text{eff}} = (\sigma_1\sigma_2)^{1/2} \approx b_0(\chi_{\|}\chi_{\perp})^{1/2} = (D_{\|}\chi_{\perp})^{1/2} \tag{C8}$$

turned out to be correct, which is due to the following simple observation: *the two media* $(\sigma_1, \sigma_2; \mathbf{n})$ *and* $(\sigma_2, \sigma_1; \mathbf{n}')$ *are in fact identical.* Consequently, regardless of the statistical equivalence of the two fields $\mathbf{n}$ and $\mathbf{n}'$ the two factors in the left-hand side of Eq. (C5) are equal.

Thus, when the effective conductivity of a two-dimensional locally anisotropic self-averaging medium is isotropic, then it equals exactly

$$\sigma_{\text{eff}} = (\sigma_1\sigma_2)^{1/2} \ . \tag{C9}$$

It is rather instructive to obtain an assessment of the exact result (C9) in another way together with the evaluation of the mixing length a_m. Here it is more convenient to argue in terms of diffusivity. Let σ_1 be much greater than σ_2. The characteristic **b** lines with the length L responsible for the effective transport are defined by that in the mixing time $\tau_m = L^2/\sigma_1$ needed for the longitudinal particle diffusion, the particle leaves the percolation cell width h on account of the transverse diffusivity σ_2:

$$\tau_m \approx L^2/\sigma_1 \approx h^2/\sigma_2 \ . \tag{C10}$$

Taking (8) and (9) into account, this yields the mixing length

$$a_m = \delta(\sigma_1/\sigma_2)^{\nu/(2\nu+4)} \ . \tag{C11}$$

The effective diffusion is defined according to $\sigma_{\text{eff}} \approx F(a_m)a_m^2/\tau_m$, where $F(a_m) = Lh/a_m^2$ is the share of the percolating **b** lines. While taking into consideration (C10), this results in the formula (C9).

Note that the feature of the large mixing lengths ($a_m \gg \delta$) is typical for the percolation-like transport problems and at shorter scales the transport processes are non-diffusive.

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Figure Captions

Fig. 1. Percolation magnetic cell. Dashed area is occupied by contours of ψ with diameters $[a, 2a]$, $a \gg \delta$.

Fig. 2. Two types of stretching maps: Quasi-linear limit (a). Percolation limit (b), where the fractality and the malti-connectedness of the cell are ignored for the sake of simplicity.

Fig. 3. Flow chart of effective heat conductance evaluation. In round brackets indicated are the formulas numbers to be used.

Fig. 4. Dependence $\chi_{\text{eff}}(\omega)$ at $R \gg 1$, $\delta/b_0 \ll \lambda_e \ll z_m$.

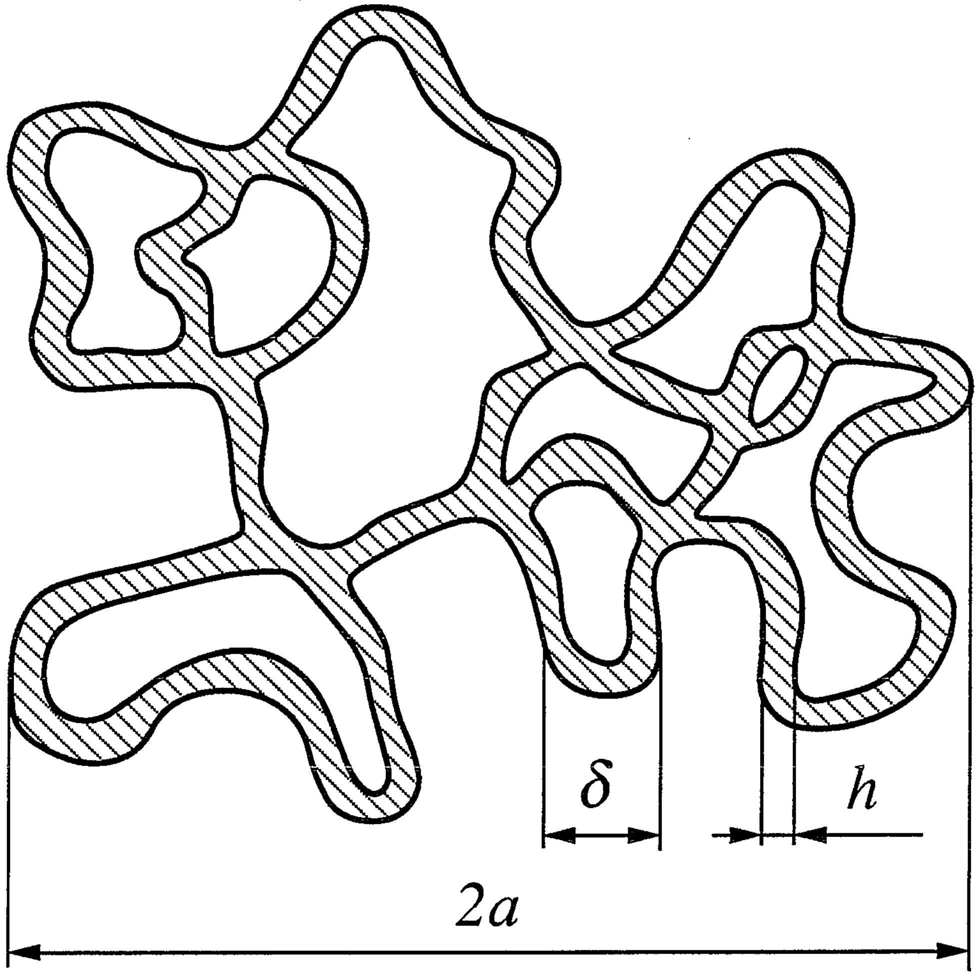


Figure 1.

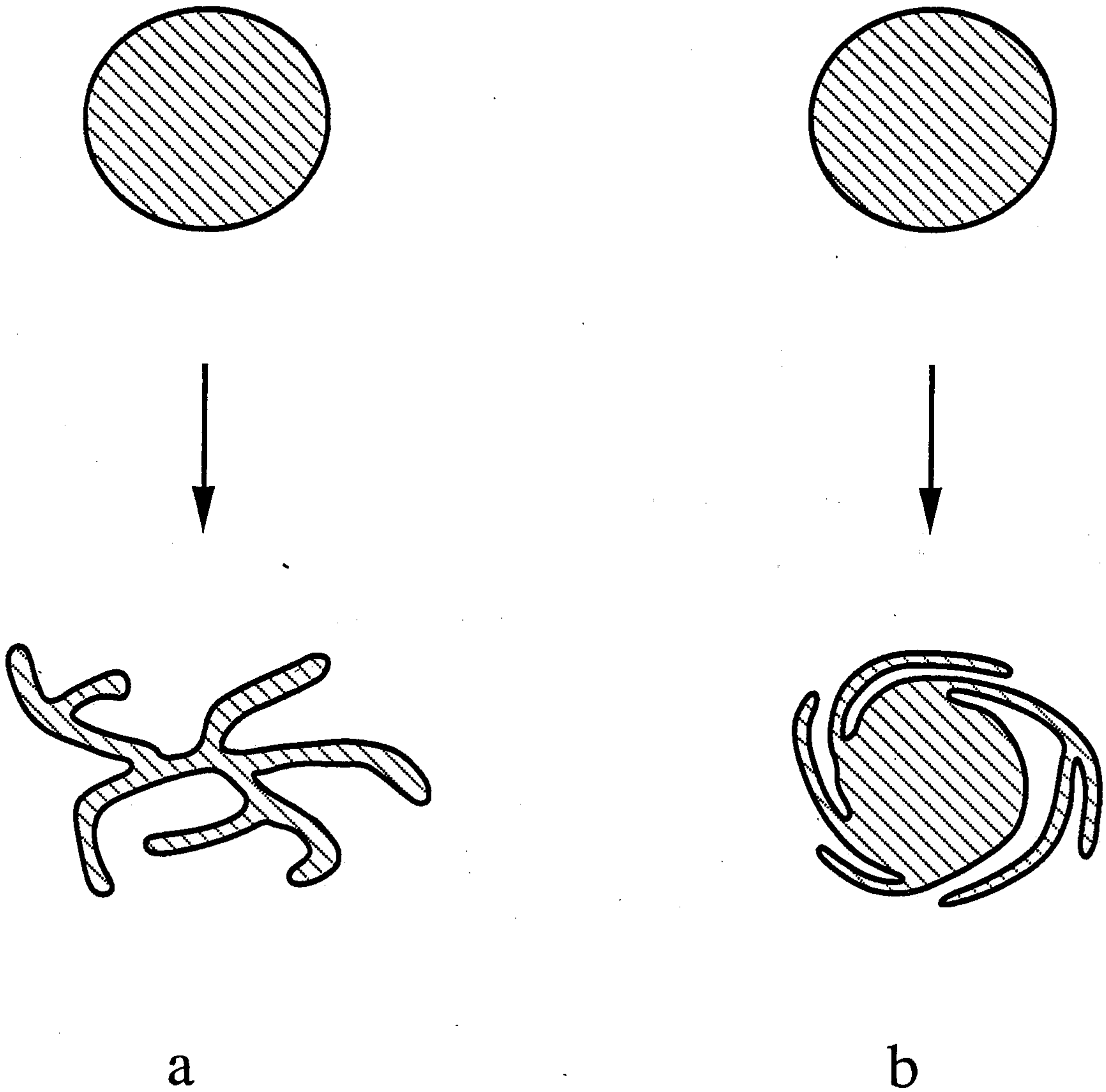
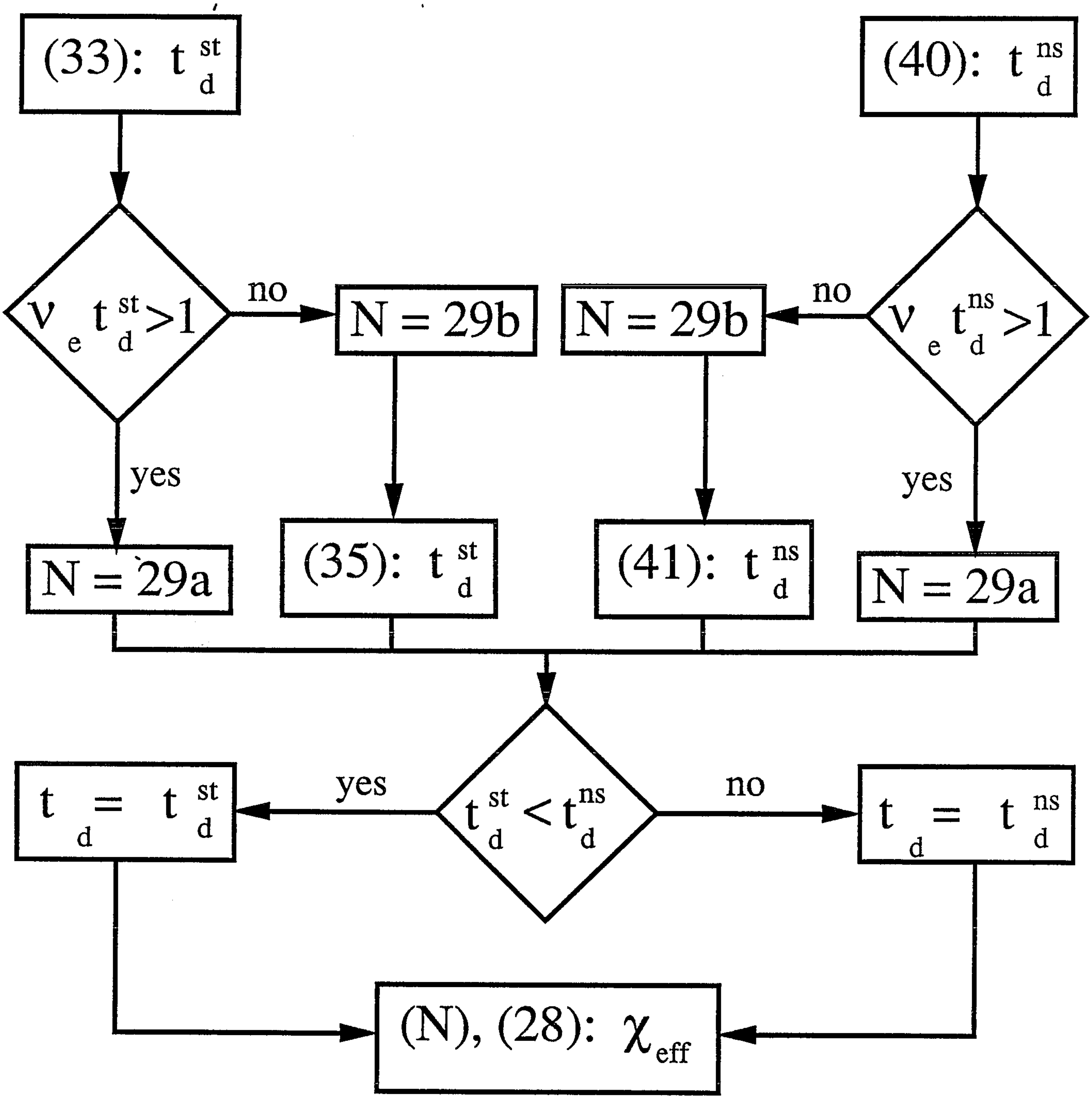
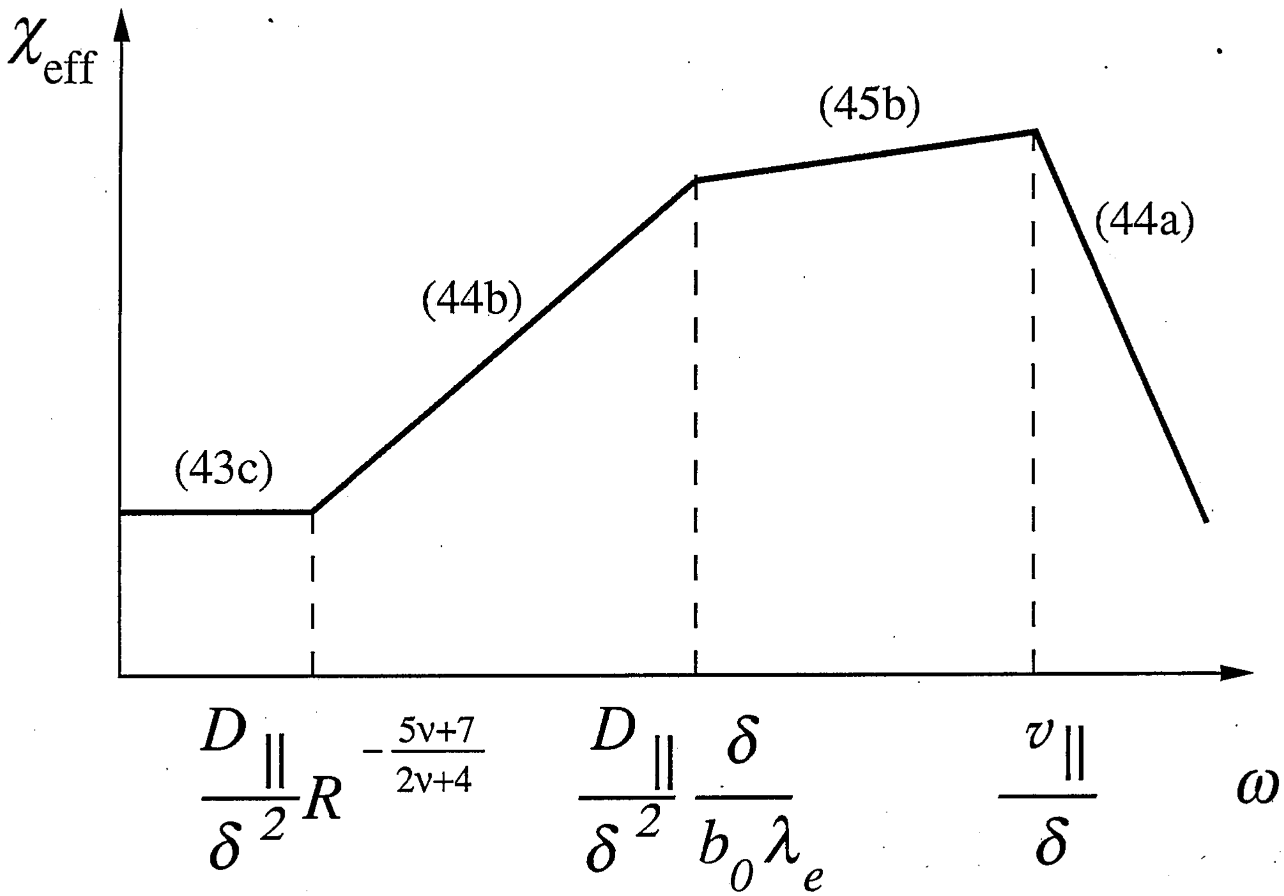


Figure 2.



(33): t_d^{st}
(40): t_d^{ns}
$\nu_e t_d^{st} > 1$
no
N = 29b
N = 29b
no
$\nu_e t_d^{ns} > 1$
yes
yes
N = 29a
(35): t_d^{st}
(41): t_d^{ns}
N = 29a
$t_d^{st} < t_d^{ns}$
yes
no
$t_d = t_d^{st}$
$t_d = t_d^{ns}$
(N), (28): χ_{eff}

Figure 3.



χ_{eff}
(45b)
(44a)
(44b)
(43c)
$\frac{D_{\|}}{\delta^2} R^{-\frac{5\nu+7}{2\nu+4}}$
$\frac{D_{\|}}{\delta^2} \frac{\delta}{b_0 \lambda_e}$
$\frac{v_{\|}}{\delta}$
ω

Figure 4.