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The statistical dynamics of a two-field model of dissipative drift wave turbulence is investigated using the EDQNM (Eddy Damped Quasi-Normal Markovian) closure method. The analyses include studies of statistical closure equations, derivation of an H-theorem, and its application to formulation of selective decay hypotheses for turbulent relaxation process. The results show that the dynamics of the two-field model is fundamentally different from that of the familiar, one-field Hasegawa-Mima model. In particular, density fluctuations nonlinearly couple to small scales, as does enstrophy. This transfer process is nonlinearly regulated by the dynamics of the density-vorticity cross-correlation. Since density perturbations are not simply related to potential perturbations, as is vorticity, their transfer rate is greater. As a result, turbulent relaxation processes exhibits both dynamic alignment of density and vorticity and coherent vortex formation.

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I. Introduction

Drift wave turbulence has been of considerable interest in plasma fusion research. Despite considerable studies, it still remains rather poorly understood. Most previous studies[1-4] of this subject have utilized a simple, one-field nonlinear fluid model for the potential fluctuation $\tilde{\phi}$ known as the Hasegawa-Mima equation[1]. In its derivation, the electron response is assumed to be adiabatic so that $\tilde{n} = \tilde{\phi}$, where $\tilde{n}$ and $\tilde{\phi}$ are the density fluctuation and the potential fluctuation, respectively. This simple model equation is similar to the two dimensional Navier-Stokes equation, and admits two invariants of the motion, total energy, and total enstrophy. Therefore, based on intuition from two dimensional fluid turbulence, the Hasegawa-Mima model of drift wave turbulence predicts an inverse cascade of total energy, from large $k_\perp$ to small $k_\perp$. Such spectral transfer is qualitatively consistent with experimental observations that the fluctuation spectrum is a decreasing function of $k_\perp$ for $k_\perp > 1$, and with the results of simple computer simulations. However, since it completely ignores the density fluctuation (electron) dynamics by taking the electrons as adiabatic, the Hasegawa-Mima equation is not a <u>general</u> model of drift wave turbulence, and conclusions reached based on it are not universal. In particular, density fluctuation dynamics and their relation to the excitation of turbulence are not addressed by that simple model. Moreover '$i\delta$' models which effectively perturb about the Hasegawa-Mima equation do not address these issues, either. In this paper, we will explore the nonlinear dynamics of dissipative drift wave turbulence beyond the one-field Hasegawa-Mima equation. Since the purpose of this discussion is to elucidate the underlying physical mechanisms, the issues of saturation levels and anomalous transport will not be extensively discussed.

The model equations used in our discussion are the two field ($\tilde{\phi}$, and $\tilde{n}$) nonlinear dissipative drift wave equations[5]. One important parameter characteristic of the model equations is the electron parallel diffusion rate $\chi_e k_\parallel^2$, which determines the linear coupling between $\tilde{n}$ and $\tilde{\phi}$ (χ_e is the electron's parallel diffusivity). When $\chi_e k_\parallel^2$ is much bigger than the linear mode frequency $\omega_{\vec{k}}$ (in the collisionless, or high $k_\parallel$ limit), i.e. $\chi_e k_\parallel^2 \gg \omega_{\vec{k}}$, the electrons behave adiabaticly ($\tilde{n} = \tilde{\phi}$), and the model reduces to the Hasegawa-Mima equation. This regime is called the adiabatic regime. The dynamics of drift wave turbulence can be described by the Hasegawa-Mima equation only in the adiabatic regime. In the opposite limit, i.e. $\chi_e k_\parallel^2 \ll \omega_{\vec{k}}$, the electron response is hydrodynamic, and the model equations reduce to the two dimensional Navier-Stokes equations. This regime is called the hydrodynamic regime. In the hydrodynamic regime, $\tilde{n}$ and $\tilde{\phi}$ are only weakly coupled, and the dynamics of drift wave turbulence is fundamentally different from that described by the Hasegawa-Mima equation. Indeed, as demonstrated by Gang, et al in their equilibrium statistical mechanics analysis[6], the natural tendency of the nonlinear internal energy ($\langle \tilde{n}^2 \rangle$) as well as total energy transfer is toward <u>small</u>, rather than large, scales in the hydrodynamic regime. This is in distinct contrast to the predictions of the Hasegawa-Mima model. It should be noted that many models of practical relevance, such as those for collisionless trapped electron mode or toroidal η_i mode turbulence, have more in common with the two-field hydrodynamic model than with the Hasegawa-Mima model. In order to gain further dynamical insight into drift wave turbulence, we present a statistical theory of a two field model of dissipative drift wave turbulence in this paper. Here, we derive the statistical closure equations, an H-theorem, and describe a selective decay hypothesis for characterizing structures in two-field drift wave turbulence. The results indicate that the nonlinear transfer of internal energy $\langle \tilde{n}^2 \rangle$ toward small scale is dynamically inhibited by the presence of finite cross-correlation $\langle \tilde{n}\tilde{\phi} \rangle$. In the adiabatic regime, $\tilde{n}$ and $\tilde{\phi}$ are strongly correlated, so that the density transfer toward small scales is completely inhibited. In the hydrodynamic limit where $\tilde{n}$ and $\tilde{\phi}$ are only weakly correlated, however, the internal energy transfer toward small scales is rapid, and, in fact, is shown to be faster than enstrophy transfer. This is due to the different role played by local (i.e. comparable scale) interactions in the two processes. As a consequence, turbulent relaxation processes in dissipative drift wave turbulence exhibit both dynamic alignment[7–8], i.e. $\tilde{n} = \alpha_1 \nabla_\perp^2 \tilde{\phi}$, and coherent vortex formation[9], i.e. $\nabla_\perp^2 \tilde{\phi} = \alpha_2 \tilde{\phi}$, where α_1 and α_2 are Lagrange multipliers that are determined by the initial and boundary conditions. The similarities between the two field model of drift wave turbulence and MHD turbulence are also discussed.

At this point, we would like to comment that related studies of dissipative drift wave turbulence using a different approach have been previously carried out by Terry and Diamond[10]. They extended the notion and methodology of phase space density granulation, developed by Dupree[11], to fluid models of dissipative drift wave turbulence. Using the two point correlation theory, it was shown that collective modes and also localized density fluctuations (which are non-wave-like structures generated by <u>electron</u> mode coupling, only) can be driven by expansion free energy. The interaction between the incoherent density 'blob' and the collective mode broadens the mode frequency spectrum and leads to saturation of the instability. In this work, a more consistent treatment of the cross-correlation than that given by Terry and Diamond is presented. In particular, both ion and electron mode coupling are shown to determine the cross-correlation dynamics and are thus treated on an equal footing. Moreover, the cross-correlation dynamics are treated self-consistently with those of the density and potential spectra. Also, a theory of coherent vortex formation in dissipative drift wave turbulence (as opposed to a statistical model of 'blob' formation) is presented.

The remainder of this paper is organized as follows. In Sec.II, the two field model equations are discussed. In Sec.III, the linear theory of the model equations is reviewed. In Sec.IV, the statistical closure equations for the nonlinear evolution of the potential spectrum $\langle \tilde{\phi}^2 \rangle_{\vec{k}}$, density spectrum $\langle \tilde{n}^2 \rangle_{\vec{k}}$, and the cross-correlation spectrum $k^2 \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}}$ are systematically derived, using the $EDQNM$ closure scheme[12]. The implications for the drift wave dynamics are discussed. These closure equations play a similar role in drift wave turbulence to that of the Boltzmann equation in many particle system, and are used to prove a H-theorem in Sec.V. There, an entropy functional is identified and shown to increase with time monotonically due to nonlinear mode couplings (turbulent mixing) until an absolute equilibrium state is reached.. In Sec.VI, we apply the H-theorem and its predictions for relaxation rates to the discussion of constrained turbulent relaxation in dissipative drift wave turbulence. In Sec.VII, we summarize and discuss the results. Possible implications for anomalous transport in tokamaks are discussed.

II. Basic Equations and Models

In this section, the basic model equations are introduced. For simplicity, we consider a plane slab geometry, and take the equilibrium magnetic field to be constant and in the z direction. The equilibrium density is nonuniform only in the x direction. Ions are assumed to be cold, and temperature gradient effects are neglected. The nonlinear evolution of dissipative drift wave turbulence is then described by the following two-field model equations[5] for potential and density fluctuations:

$$\frac{\partial}{\partial t}\nabla_\perp^2\tilde{\phi} - \vec{\nabla}\tilde{\phi}\times\vec{z}\cdot\vec{\nabla}\nabla_\perp^2\tilde{\phi} = \chi_e\nabla_\parallel^2(\tilde{n}-\tilde{\phi}) + \mu\nabla_\perp^4\tilde{\phi} \tag{1a}$$

$$\frac{\partial}{\partial t}\tilde{n} - \vec{\nabla}\tilde{\phi}\times\vec{z}\cdot\vec{\nabla}\tilde{n} = \chi_e\nabla_\parallel^2(\tilde{n}-\tilde{\phi}) + v_e^*\nabla_y\tilde{\phi} \tag{1b}$$

The above equations have been written in dimensionless form. $\tilde{\phi}$ and $\tilde{n}$ are the normalized potential and density fluctuations, respectively. $\chi_e = v_{te}^2/\nu_{ei}\Omega_i$ is the electron parallel thermal diffusivity, $v_{te} = \sqrt{2T_e/m_e}$ is electron thermal speed, ν_{ei} is electron ion collision frequency, Ω_i is ion gyrofrequency, μ is the normalized ion viscosity, and $v_e^* = 1/L_n$ is the normalized electron diamagnetic drift velocity, with L_n the equilibrium density scale length. Eqs.(1a) and (1b) describe the nonlinear evolutions of vorticity fluctuation $\nabla_\perp^2\tilde{\phi}$ and density fluctuation $\tilde{n}$ which are coupled through electron parallel diffusivity χ_e. The parallel wavelength is assumed to be of the order of connection length, $k_\parallel \sim 1/qR$, the perpendicular wavelength is assumed to be of the order of ρ_s (i.e. $k_\perp\rho_s > \rho_s/r$), so that $k_\perp \gg k_\parallel$, i.e. equations (1a) and (1b) describe the nonlinear evolution of a quasi-two-dimensional system.

It is important to note that the above equations have two linear time scales: $\omega_{\vec{k}}$ (mode frequency) and $\chi_e k_\parallel^2$ (parallel diffusion rate). In the collisionless limit where $\omega_{\vec{k}} \ll \chi_e k_\parallel^2$, we have $\tilde{n} = \tilde{\phi}$, so that electrons are adiabatic. There, eqs.(1a) and (1b) reduce to the single field Hasegawa-Mima[1] or Rossby wave equation[4] (for $\mu = 0$),

$$\frac{\partial}{\partial t}(\tilde{\phi} - \nabla_\perp^2\tilde{\phi}) + \vec{\nabla}\tilde{\phi}\times\vec{z}\cdot\vec{\nabla}\nabla_\perp^2\tilde{\phi} = v_e^*\nabla_y\tilde{\phi} \tag{2}$$

In the strongly collisional limit, $\omega_{\vec{k}} \gg \chi_e k_{\|}^2$, so that $\tilde{n}$ and $\tilde{\phi}$ approximately decouple and evolve quasi-independently. Eqs.(1a) and (1b) reduce to the 2-D Navier-Stokes equations, where the vorticity is actively advected by self-generated velocity field ($\vec{v} = -\vec{\nabla}_\perp \tilde{\phi} \times \vec{z}$) and the density is passively advected by the same velocity field. These two limits are thereafter called the adiabatic and hydrodynamic regimes, respectively. $\tilde{n}$ and $\tilde{\phi}$ are strongly correlated in the adiabatic regime, but need not correlated in the hydrodynamic regime. In both regimes, parallel diffusion induced dissipation is negligible. In the intermediate collisionality regime $\omega_{\vec{k}} \sim \chi_e k_{\|}^2$, the two field model has characteristics of both adiabatic and hydrodynamic regimes, and dissipation is strong.

There are four quadratic quantities which are conserved by the $\vec{E} \times \vec{B}$ convective nonlinearities in eqs.(1a) and (1b), so that their evolution is determined solely by growth and dissipation. They are the fluid kinetic energy E^ϕ, the fluid enstrophy Ω^ϕ, the fluid internal energy E^n, and the cross-helicity Γ, which are

$$E^\phi = \int (\vec{\nabla}_\perp \tilde{\phi})^2 d^2\vec{x} \tag{3a}$$

$$\Omega^\phi = \int (\nabla_\perp^2 \tilde{\phi})^2 d^2\vec{x} \tag{3b}$$

$$E^n = \int \tilde{n}^2 d^2\vec{x} \tag{3c}$$

$$\Gamma = -\int \tilde{n} \nabla_\perp^2 \tilde{\phi} d^2\vec{x} \tag{3d}$$

These four quantities evolve with time as given by:

$$\frac{\partial E^\phi}{\partial t} = 2\chi_e \int d^2\vec{x} \nabla_\| \tilde{\phi} \nabla_\| (\tilde{n} - \tilde{\phi}) - 2\mu \int d^2\vec{x} (\nabla_\perp^2 \tilde{\phi})^2, \tag{4a}$$

$$\frac{\partial \Omega^\phi}{\partial t} = 2\chi_e \int d^3\vec{x} \nabla_\perp^2 \tilde{\phi} \nabla_\|^2 (\tilde{n} - \tilde{\phi}) - 2\mu \int d^2\vec{x} (\vec{\nabla}_\perp^3 \tilde{\phi})^2, \tag{4b}$$

$$\frac{\partial E^n}{\partial t} = -2\chi_e \int d^2\vec{x} \nabla_\| \tilde{n} \nabla_\| (\tilde{n} - \tilde{\phi}) + 2v_e^* \int d^2\vec{x} (\tilde{n} \nabla_y \tilde{\phi}), \tag{4c}$$

$$\frac{\partial \Gamma}{\partial t} = -\chi_e \int d^2\vec{x} (\tilde{n} + \nabla_\perp^2 \tilde{\phi}) \nabla_\| (\tilde{n} - \tilde{\phi}) - \mu \int d^2\vec{x} (\tilde{n} \nabla_\perp^4 \tilde{\phi}). \tag{4d}$$

Generally, these four quantities are not the constants of motion for Eqs.(1a) and (1b), but only conserved by <u>nonlinearities</u>. We define constant of motion so that it denotes evolution on a time scale much longer than that typical for fluctuations. By this definition, these four quadratic quantities are constants of motion of the model equations only in the hydrodynamic limit, where parallel dissipation (χ_e) is negligible (i.e. the first terms on the right side of Eqs.(4a)-(4d) are small). As $\chi_e k_{\|}^2$ increases, the conservation of these four quantities is violated. However, when χ_e is large enough that the system evolves into the adiabatic regime, new constants of motion can be constructed from the combinations of these four quadratic quantities. This is a consequence of the fact that for $\chi_e k_\parallel^2 \gg \omega_{\vec{k}}, \triangle\omega_{\vec{k}}$, $\tilde{n} \to \tilde{\phi}$. These two constants of motion are the total energy E, and the total enstrophy Ω of the Hasegawa-Mima model, which are defined by

$$E = \int [\tilde{\phi}^2 + (\vec{\nabla}_\perp \tilde{\phi})^2] d^2\vec{x} \tag{5a}$$

$$\Omega = \int (\tilde{\phi} - \nabla_\perp^2 \tilde{\phi})^2 d^2\vec{x} \tag{5b}$$

The evolution of these two quantities is determined by:

$$\frac{\partial E}{\partial t} = -2\mu \int (\nabla_\perp^2 \tilde{\phi})^2 d^2\vec{x}, \tag{6a}$$

$$\frac{\partial \Omega}{\partial t} = -2\mu \int [(\nabla_\perp^2 \tilde{\phi})^2 + (\nabla_\perp^3 \tilde{\phi})^2] d^2\vec{x} \tag{6b}$$

In order to facilitate the analysis, we Fourier analyze the fluctuating fields $\tilde{\phi}(\vec{x}, t)$ and $\tilde{n}(\vec{x}, t)$ in $\vec{x}$,

$$\tilde{\phi}(\vec{x}, t) = \sum_{\vec{k}} \tilde{\phi}_{\vec{k}}(t) \exp(i\vec{k} \cdot \vec{x}) \tag{7a}$$

$$\tilde{n}(\vec{x}, t) = \sum_{\vec{k}} \tilde{n}_{\vec{k}}(t) \exp(i\vec{k} \cdot \vec{x}) \tag{7b}$$

Since $\tilde{\phi}(\vec{x}, t)$ and $\tilde{n}(\vec{x}, t)$ are real quantities, we have $\tilde{\phi}^*_{\vec{k}} = \tilde{\phi}_{-\vec{k}}$ and $\tilde{n}^*_{\vec{k}} = \tilde{n}_{-\vec{k}}$. In Fourier space, the model equations can be written as,

$$\left(\frac{\partial}{\partial t} + \mu k_\perp^2 + \frac{\chi_e k_\parallel^2}{k_\perp^2}\right)(k_\perp^2 \tilde{\phi}_{\vec{k}}) - \chi_e k_\parallel^2 \tilde{n}_{\vec{k}} = \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (k_2^2 - k_1^2) \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tag{8a}$$

$$\left(\frac{\partial}{\partial t} + \chi_e k_\parallel^2\right) \tilde{n}_{\vec{k}} - \chi_e k_\parallel^2 \tilde{\phi}_{\vec{k}} + i\omega_e^* \tilde{\phi}_{\vec{k}} = \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (\tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} - \tilde{\phi}_{\vec{k}_2} \tilde{n}_{\vec{k}_1}) \tag{8b}$$

III. Review of Linear Theory

In this section, the linear theory of the dissipative drift mode is reviewed. Special attention is focused on those properties which are pertinent to the nonlinear theory. By linearizing Eqs.(8a) and (8b), and assuming that $\tilde{\phi}$ and $\tilde{n}$ have the following time dependence,

$$\{\tilde{\phi}_{\vec{k}}, \tilde{n}_{\vec{k}}\} = \{\tilde{\phi}_{\substack{\vec{k}\\ \omega}}, \tilde{n}_{\substack{\vec{k}\\ \omega}}\} \exp(-i\omega t)$$

we have

$$(-i\omega k_\perp^2 + \chi_e k_\parallel^2 + \mu k_\perp^4) \tilde{\phi}_{\substack{\vec{k}\\ \omega}} - \chi_e k_\parallel^2 \tilde{n}_{\substack{\vec{k}\\ \omega}} = 0 \tag{9a}$$

$$(i\omega_e^* - \chi_e k_\parallel^2) \tilde{\phi}_{\substack{\vec{k}\\ \omega}} + (-i\omega + \chi_e k_\parallel^2) \tilde{n}_{\substack{\vec{k}\\ \omega}} = 0 \tag{9b}$$

μ is included in the above equations so that the modes are damped at large $k_\perp$ by ion viscosity. At small $k_\perp$, the mode is also damped by ion Landau damping, which can be easily mocked up in the model equations. These two mechanisms provide an energy sink. Therefore, the unstable modes occur primarily in the moderate $k_\perp$ region. In this wavenumber region, μ can be neglected so that the condition that Eqs.(9a) and (9b) have non-zero solution is

$$D(\vec{k},\omega_{\vec{k}})=0 \tag{10}$$

where the linear dielectric function $D(\vec{k},\omega_{\vec{k}})$ is given by

$$D(\vec{k},\omega_{\vec{k}})=-k_\perp^2\omega_{\vec{k}}^2-i\omega_{\vec{k}}\chi_e k_\parallel^2(1+k_\perp^2)+i\omega_e^*\chi_e k_\parallel^2.$$

Eq.(10) has two roots, which correspond to the two modes,

$$\omega_{\vec{k}\pm}=-\frac{i}{2}\chi_e k_\parallel^2(1+\frac{1}{k_\perp^2})\pm\frac{i}{2}[(\chi_e k_\parallel^2(1+\frac{1}{k_\perp^2})^2-4i\omega_e^*\frac{\chi_e k_\parallel^2}{k_\perp^2}]^{\frac{1}{2}} \tag{11}$$

Generally, the expression for $\omega_{\vec{k}}$ are complicated, however, in the two limits mentioned above, simple expressions of the following form can be obtained.

$$\omega_{\vec{k}}=\omega_{\vec{k}}^r+i\gamma_{\vec{k}} \tag{12}$$

where $\omega_{\vec{k}}^r$ is the real frequency, and $\gamma_{\vec{k}}$ is the growth rate.

a. In the adiabatic limit ($\omega_{\vec{k}}<k_\parallel^2\chi_e$):

$$\omega_{\vec{k}}^r=\frac{\omega_e^*}{1+k_\perp^2} \tag{13a}$$

$$\gamma_{\vec{k}}=\frac{k_\perp^2\omega_e^{*2}}{\chi_e k_\parallel^2(1+k_\perp^2)^3} \tag{13b}$$

b. In the hydrodynamic regime ($\omega_{\vec{k}}>k_\parallel^2\chi_e$):

$$\omega_{\vec{k}}^r=(\frac{\chi_e k_\parallel^2\omega_e^*}{2k_\perp^2})^{\frac{1}{2}} \tag{14a}$$

$$\gamma_{\vec{k}}=(\frac{\chi_e k_\parallel^2\omega_e^*}{2k_\perp^2})^{\frac{1}{2}} \tag{14b}$$

Only the real frequency and growth rate of the unstable mode are given in the above equations. From Eqs.(13a)-(14b), we observe that in the adiabatic regime the most unstable modes are short wavelength modes with $k_\perp\sim 1$. Since $\omega_{\vec{k}}\gg\gamma_{\vec{k}}$, and the relevant modes are strongly dispersive, effective wave interactions occur only in the presence of strong turbulence (i.e. $\Delta\omega_{\vec{k}} \geq \omega_{\vec{k}}$). However, in the hydrodynamic regime, the most unstable modes are at long wavelength, with $k_\perp < 1$. In this case, as has been shown previously[6], a significant amount of wave energy will be transferred to short wavelengths regime by a standard cascade, and ultimately damped by ion viscosity. Since $\omega_{\vec{k}} \sim \gamma_{\vec{k}}$, mode coupling can occur even in the weak turbulence regime (i.e. $\omega_{\vec{k}} \sim \gamma_{\vec{k}} \gg \Delta\omega_{\vec{k}}$).

IV. Closure Equations

In the statistical description of turbulence, we are interested in the nonlinear evolution of two-body correlations, since these determine the fluctuation spectra and turbulent transport. For two-field drift wave turbulence models, these two-body correlations are the kinetic energy spectrum $\langle\tilde{v}^2\rangle_{\vec{k}} \equiv k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}$, the internal energy spectrum $\langle n^2\rangle_{\vec{k}}$ and the cross-correlation spectrum $\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} \equiv \langle\tilde{n}_{\vec{k}}\tilde{\phi}_{-\vec{k}}\rangle$. The nonlinear evolution equations for these three spectra are derived in this section.

It is well known that due to the quadratic nonlinearity of the equations, the construction of the evolution equations for the two-body correlations leads to an infinite hierarchy of correlation equations. In order to terminate this hierarchy and obtain a closed set of two-body correlation equations, we employ the $EDQNM$ closure scheme[12]. The basic idea of this closure scheme is that the effect of high order correlations on the evolution of three body correlation is approximated by an eddy damping rate, so that the phase correlation among three distinct modes will be destroyed by turbulent scrambling in an eddy turnover time, which is defined recursively. The application of $EDQNM$ closure scheme, however, is quite restrictive, since it requires that the fluctuating fields have a near Gaussian probability distribution. Hence, it fails to describe turbulence with coherent structures (intermittency)[13–14].

In the following, closure equations for the spectra $\langle\tilde{v}^2\rangle_{\vec{k}}$, $\langle n^2\rangle_{\vec{k}}$, and $\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}$ will be derived. For simplicity, μ will be neglected. From Eqs.(8a) and (8b) the nonlinear evolution equations for the spectra $\langle\tilde{v}^2\rangle_{\vec{k}}$, $\langle n^2\rangle_{\vec{k}}$, and $\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}$ can be constructed:

$$\begin{aligned}
&(k^2\frac{\partial}{\partial t} + 2\chi_e k_\parallel^2)\langle\tilde{\phi}^2\rangle_{\vec{k}} - 2\chi_e k_\parallel^2 Re\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} \\
&= \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} \vec{z}\times\vec{k}_1\cdot\vec{k}_2(k_2^2-k_1^2)Re\langle\tilde{\phi}_{\vec{k}_1}\tilde{\phi}_{\vec{k}_2}\tilde{\phi}_{-\vec{k}}\rangle
\end{aligned} \tag{15a}$$

$$\begin{aligned}
&(\frac{\partial}{\partial t} + 2\chi_e k_\parallel^2)\langle\tilde{n}^2\rangle_{\vec{k}} - 2\chi_e k_\parallel^2 Re\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} + 2\omega_e^* Im\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} \\
&= 2\sum_{\vec{k}_1+\vec{k}_2=\vec{k}} \vec{z}\times\vec{k}_1\cdot\vec{k}_2 Re\langle\tilde{\phi}_{\vec{k}_1}\tilde{n}_{\vec{k}_2}\tilde{n}_{-\vec{k}}\rangle
\end{aligned} \tag{15b}$$

$$(\frac{\partial}{\partial t} + \frac{\chi_e k_\parallel^2}{k_\perp^2} + \chi_e k_\parallel^2)(k^2\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}) - \chi_e k_\parallel^2(\langle\tilde{n}^2\rangle_{\vec{k}} + k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}) + i\omega_e^*(k^2\langle\tilde{\phi}^2\rangle_{\vec{k}})$$

$$= \frac{1}{2} \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{z} \times \vec{k}_1 \cdot \vec{k}_2)[(k_2^2 - k_1^2)\langle \tilde{n}_{\vec{k}} \tilde{\phi}_{-\vec{k}_1} \tilde{\phi}_{-\vec{k}_2} \rangle + 2k^2 \langle \tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle] \tag{15c}$$

In the above equations, the cross-correlation evolution has been treated on an equal footing with the energy evolutions, rather than being ignored or quasi-linearly approximated[2,3]. Thus, both ion and electron enter the determination of the cross-correlation, and thus the particle flux in the strong turbulence regime. Moreover, such a treatment self-consistently couples transport to fluctuation dynamics, in contrast to "iδ" models.

The three body correlations in the above equations like $\langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle$ are determined by the phase correlation or phase coherency among the three modes $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}$. This phase correlation is induced by the nonlinear mode couplings and vanishes as the size of the system increases[15]. To first order, this phase correlation is determined by the direct interaction among these three modes. We denote the perturbation in $\tilde{\phi}_{\vec{k}}$ due to this direct interaction by $\delta\tilde{\phi}_{\vec{k}}$. Then, to first order, the three body correlation $\langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle$ can be approximated by,

$$\langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle \simeq \langle \delta\tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle + \langle \tilde{\phi}_{\vec{k}_1} \delta\tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle + \langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \delta\tilde{\phi}_{-\vec{k}} \rangle \tag{16}$$

The perturbations $\delta\tilde{\phi}_{\vec{k}}$ and $\delta\tilde{n}_{\vec{k}}$ satisfy Eqs.(8a) and (8b) but are driven only by the 'direct interaction' between modes $\vec{k}_1$ and $\vec{k}_2$:

$$(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}})(k^2 \delta\tilde{\phi}_{\vec{k}}) + \chi_e k_\parallel^2 (\delta\tilde{\phi}_{\vec{k}} - \delta\tilde{n}_{\vec{k}}) = S_{1\vec{k}} \tag{17a}$$

$$(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}})\delta\tilde{n}_{\vec{k}} + \chi_e k_\parallel^2 \delta\tilde{n}_{\vec{k}} + (i\omega_e^* - \chi_e k_\parallel^2)\delta\tilde{\phi}_{\vec{k}} = S_{2\vec{k}} \tag{17b}$$

where the source terms are given by:

$$S_{1\vec{k}} = \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (k_2^2 - k_1^2) \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tag{18a}$$

$$S_{2\vec{k}} = \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (\tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} - \tilde{\phi}_{\vec{k}_2} \tilde{n}_{\vec{k}_1}) \tag{18b}$$

Note that S_{1k} and S_{2k} are symmetric in $\vec{k}_1$ and $\vec{k}_2$. In the above equations, $\triangle\omega_{\vec{k}}$ is the eddy damping rate which is introduced to represent the effect of higher order correlations on the evolution of three body correlation. Solutions of equations (17a) and (17b) are given in Appendix A (Eqs. A.12 and A.13). They are

$$\delta\tilde{\phi}_{\vec{k}}(t) = \frac{1}{detA_{\vec{k}}} \int_{-\infty}^{t} dt' \exp\{-(i\omega_{\vec{k}} + \triangle\omega_{\vec{k}})(t-t')\}$$
$$\times [(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_\parallel^2}) S_{1\vec{k}}(t') + S_{2\vec{k}}(t')] \tag{19a}$$

$$\delta\tilde{n}_{\vec{k}}(t) = \frac{1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_\parallel^2}}{detA_{\vec{k}}} \int_{-\infty}^{t} dt' \exp\{-(i\omega_{\vec{k}} + \triangle\omega_{\vec{k}})(t-t')\}$$
$$\times [(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_\perp^2}) S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]. \tag{19b}$$

where $\omega_{\vec{k}}$ is the frequency of the linear eigenmode, and $detA_{\vec{k}}$ is given by,

$$detA_{\vec{k}} = \sqrt{(1+k^2)^2 - 4i\omega_e^* \frac{k^2}{\chi_e k_\parallel^2}}$$

We now proceed with the renormalization procedure. The first term on the right hand side of equation (16) is,

$$\langle \delta\tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle = \frac{1}{detA_{\vec{k}_1}} \int_{-\infty}^{t} dt' \exp\{-(i\omega_{\vec{k}_1} + \triangle\omega_{\vec{k}_1})(t-t')\}$$
$$\times [(1 - \frac{i\omega_{\vec{k}_1}}{\chi_e k_{1\parallel}^2}) \langle S_{1\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle + \langle S_{2\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle] \tag{20}$$

where

$$\begin{aligned} &\langle S_{1\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle \\ \simeq &(\vec{z} \times \vec{k}_1 \cdot \vec{k}_2)(k^2 - k_2^2) \langle \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}_2}(t') \rangle \langle \tilde{\phi}_{\vec{k}}(t') \tilde{\phi}_{-\vec{k}}(t) \rangle \end{aligned} \tag{21a}$$

$$\begin{aligned} &\langle S_{2\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle \\ \simeq &(\vec{z} \times \vec{k}_1 \cdot \vec{k}_2) \times [\langle \tilde{n}_{\vec{k}}(t') \tilde{\phi}_{-\vec{k}}(t) \rangle \langle \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}_2}(t') \rangle \\ &- \langle \tilde{\phi}_{\vec{k}}(t') \tilde{\phi}_{-\vec{k}}(t) \rangle \langle \tilde{\phi}_{\vec{k}_2}(t) \tilde{n}_{-\vec{k}_2}(t') \rangle] \end{aligned} \tag{21b}$$

In the above equations, the two time correlations are assumed to have the following form (for $t > t'$):

$$\langle \tilde{\phi}_{\vec{k}}(t) \tilde{\phi}_{-\vec{k}}(t') \rangle = \langle \tilde{\phi}^2(t) \rangle_{\vec{k}} \exp\{-(i\omega_{\vec{k}} + \triangle\omega_{\vec{k}})(t-t')\} \tag{22a}$$
$$\langle \tilde{n}_{\vec{k}}(t) \tilde{n}_{-\vec{k}}(t') \rangle = \langle \tilde{n}^2(t) \rangle_{\vec{k}} \exp\{-(i\omega_{\vec{k}} + \triangle\omega_{\vec{k}})(t-t')\} \tag{22b}$$
$$\langle \tilde{n}_{\vec{k}}(t) \tilde{\phi}_{-\vec{k}}(t') \rangle = \langle \tilde{n}(t) \tilde{\phi}(t) \rangle_{\vec{k}} \exp\{-(i\omega_{\vec{k}} + \triangle\omega_{\vec{k}})(t-t')\} \tag{22c}$$

Substituting these relations into equations (21a), (21b), and then (20), we obtain:

$$\begin{aligned} \langle \delta\tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle = &\frac{(\vec{z} \times \vec{k}_1 \cdot \vec{k}_2)}{detA_{\vec{k}_1}} \Theta_{\vec{k},\vec{k}_1,\vec{k}_2} [(1 - \frac{i\omega_{\vec{k}_1}}{\chi_e k_\parallel^2})(k^2 - k_2^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \\ &+ \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} - \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}_2}^*] \end{aligned} \tag{23}$$

where

$$\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} = \frac{1}{i(\omega_{\vec{k}_1} + \omega_{\vec{k}_2} - \omega_{\vec{k}}) + \triangle\omega_{\vec{k}_1} + \triangle\omega_{\vec{k}_2} + \triangle\omega_{\vec{k}}} \tag{24}$$

The real part of $\Theta_{\vec{k},\vec{k}_1,\vec{k}_2}$ determines the time scale for three wave nonlinear interaction.

The above calculations show the general procedure of the renormalization. In the same way, the other three body correlations are also renormalized. After some lengthy but straightforward calculations, we obtain the desired closed set of three coupled spectrum evolution equations. In order to simplify these spectrum evolution equations, we define the coupling coefficients:

$$a_{\vec{k}} = (1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2})b_{\vec{k}} \qquad b_{\vec{k}} = \frac{1}{detA_{\vec{k}}}$$

$$c_{\vec{k}} = (1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_{\parallel}^2})a_{\vec{k}} \qquad d_{\vec{k}} = (1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_{\parallel}^2})b_{\vec{k}}$$

The closure equations are then given by the following:

a) Kinetic energy spectrum $k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}$:

$$\begin{aligned}
&(\frac{\partial}{\partial t} + 2\frac{\chi_e k_{\parallel}^2}{k^2})(k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}) - 2\chi_e k_{\parallel}^2 Re\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} \\
=&Re \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2(k_2^2 - k_1^2)\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} \\
&[2a_{\vec{k}_1}(k^2 - k_2^2)\langle\tilde{\phi}^2\rangle_{\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}} + 2b_{\vec{k}_1}(\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_2} - \langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}}) \\
&+a_{-\vec{k}}(k_2^2 - k_1^2)\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{\phi}^2\rangle_{\vec{k}_2} + b_{-\vec{k}}(\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_2}\langle\phi^2\rangle_{\vec{k}_1} - \langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_1}\langle\phi^2\rangle_{\vec{k}_2})]
\end{aligned} \tag{25a}$$

b) Internal energy spectrum $\langle n^2\rangle_{\vec{k}}$:

$$\begin{aligned}
&(\frac{\partial}{\partial t} + 2\chi_e k_{\parallel}^2)\langle n^2\rangle_{\vec{k}} - 2\chi_e k_{\parallel}^2 Re\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} + 2\omega_e^* Im\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} \\
=&2Re \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} \\
&[a_{\vec{k}_1}(k^2 - k_2^2)\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}}\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2} + b_{\vec{k}_1}(\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle n^2\rangle_{\vec{k}} - \langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}}\langle n^2\rangle_{\vec{k}_2}) \\
&+c_{\vec{k}_2}(k_1^2 - k^2)\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}}\langle\phi^2\rangle_{\vec{k}_1} + d_{\vec{k}_2}(\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}}\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_1} - \langle n^2\rangle_{\vec{k}}\langle\phi^2\rangle_{\vec{k}_1}) \\
&+c_{-\vec{k}}(k_2^2 - k_1^2)\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle\phi^2\rangle_{\vec{k}_1} + d_{-\vec{k}}(\langle n^2\rangle_{\vec{k}_2}\langle\phi^2\rangle_{\vec{k}_1} - \langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_1})]
\end{aligned} \tag{25b}$$

c) Cross-correlation spectrum $\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}$:

$$\begin{aligned}
&(\frac{\partial}{\partial t} + \chi_e k_{\parallel}^2 + \frac{\chi_e k_{\parallel}^2}{k^2})(k^2\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}) - \chi_e k_{\parallel}^2(\langle\tilde{n}^2\rangle_{\vec{k}} + k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}) + i\omega_e^*(k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}) \\
=&\sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2\{\frac{1}{2}(k_2^2 - k_1^2)\Theta^*_{\vec{k},\vec{k}_1,\vec{k}_2} \times \\
&[2a_{-\vec{k}_1}(k^2 - k_2^2)\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_2} + 2b_{-\vec{k}_1}(\langle\tilde{n}^2\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_2} - \langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2})
\end{aligned}$$

$$+c_{\vec{k}}(k_2^2 - k_1^2)\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{\phi}^2\rangle_{\vec{k}_2} + d_{\vec{k}}(\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}_1} - \langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_1}\langle\tilde{\phi}^2\rangle_{\vec{k}_2})]$$
$$+k^2\Theta_{\vec{k},\vec{k}_1,\vec{k}_2}\times$$
$$[a_{\vec{k}_1}(k^2 - k_2^2)\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}} + b_{\vec{k}_1}(\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2} - \langle\tilde{\phi}^2\rangle_{\vec{k}}\langle\tilde{n}^2\rangle_{\vec{k}_2})$$
$$+c_{\vec{k}_2}(k_1^2 - k^2)\langle\tilde{\phi}^2\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_1} + d_{\vec{k}_2}(\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_1}\langle\tilde{\phi}^2\rangle_{\vec{k}} - \langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_1})$$
$$+a_{-\vec{k}}(k_2^2 - k_1^2)\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}_1} + b_{-\vec{k}}(\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{n}^2\rangle_{\vec{k}_2} - \langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_2}\langle\tilde{n}\tilde{\phi}\rangle^*_{\vec{k}_1})]\} \tag{25c}$$

After having obtained the closure equations, we can recursively define the turbulent scrambling rate $\triangle\omega_{\vec{k}}$, introduced previously, in terms of the turbulent fluctuation spectrum, i.e.

$$\triangle\omega_{\vec{k}} = \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 Re\Theta_{\vec{k},\vec{k}_1,\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}_1}$$

Note $\Theta_{\vec{k},\vec{k}_1,\vec{k}_2}$ is a function of $\triangle\omega_{\vec{k}}$.

The above equations provide an appropriate description for the nonlinear spectral evolution and transfer in strong drift wave turbulence. It can be easily shown that the conservation properties of the $\vec{E} \times \vec{B}$ convective nonlinearity are preserved by the closure scheme. Unlike previous studies[2], the cross-correlation dynamics is treated on an equal footing with that of fluctuation energies, so that the dynamic effect of the cross-correlation on the energy spectra evolution and transfer is explicitly retained. Eqs.(25a)-(25c), as they stand, are very difficult to solve analytically. Numerical solutions are needed to obtain detailed, quantitative information about the energy and cross-correlation spectral evolution and transfer. These will be discussed in a future paper.

However, from the above equations, we can note that the nonlinear transfer of kinetic energy and enstrophy is dominated by the nonlocal interactions (i.e. $\vec{k}_1 \neq \vec{k}_2$) due to the presence of the factor $k_2^2 - k_1^2$ in every nonlinear term on the right handside of Eq.(25a). This is because the vorticity $\nabla^2\tilde{\phi}$ is directly related to the potential $\tilde{\phi}$. For the internal energy evolution, however, local interaction (with $k_1 \sim k_2$) is effective, because the density $\tilde{n}$ is <u>not</u> simply related to the potential $\tilde{\phi}$, and thus the factor $k_2^2 - k_1^2$ does not appear in every nonlinear term on the rightside of Eq.(25b). Moreover, we note that in the presence of finite cross-correlation, the usual $\vec{E} \times \vec{B}$ convective nonlinearity in the internal energy evolution equation (without cross-correlation) $\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{n}^2\rangle_{\vec{k}}$ is replaced by $\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{n}^2\rangle_{\vec{k}} - \langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}_1}$, which accounts for the inhibition of the nonlinear transfer of internal energy by finite cross-correlation. Finally, on the right handside of Eq.(25c), the first six terms come from the $\vec{E} \times \vec{B}$ convection of vorticity, while the rest of the terms comes from the $\vec{E} \times \vec{B}$ convection of density, i.e. both ion and electron mode coupling influence cross-correlation evolution.

At this point, it is appropriate to discuss the dynamical effects of cross-correlation on energy transfer. The cross-helicity is defined as $\Gamma \equiv -\langle\tilde{n}\nabla^2\tilde{\phi}\rangle = \langle\vec{\nabla}\tilde{n}\cdot\vec{\nabla}\tilde{\phi}\rangle$, i.e. the correlation between the density and fluid vorticity fluctuations. It also measures the relative orientation between the density and potential contours. Since Γ is conserved by the $\vec{E} \times \vec{B}$ convective nonlinearity, it imposes an extra constraint on the nonlinear dynamics of energy transfer. Dynamically, the cross-correlation inhibits internal energy transfer to small scales. In order to see this, we examine the closure equation (Eq.(25b)) for the internal energy evolution. The dominant nonlinear terms in this equation have the following structure: $\langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{n}^2 \rangle_{\vec{k}} - \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}} \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}_1}$. The other nonlinear terms are not significant, since large cancellation occurs between contributions from $k_1 < k$ and $k_1 > k$. This will be much more apparent if we look at the density evolution equation in the configuration space where the $\vec{E} \times \vec{B}$ convective nonlinearity $\vec{\nabla}\tilde{\phi} \times \vec{z} \cdot \vec{\nabla}\tilde{n}$ has the property that the identity $(\vec{\nabla}\tilde{\phi} \times \vec{z} \cdot \vec{\nabla}\tilde{n})^2 = (\vec{\nabla}\tilde{\phi})^2 (\vec{\nabla}\tilde{n})^2 - (\vec{\nabla}\tilde{\phi} \cdot \vec{\nabla}\tilde{n})^2$ follows. This relation clearly indicates that the cross-correlation (between $\tilde{n}$ and $\nabla_\perp^2 \tilde{\phi}$) dynamically inhibits the transfer of internal energy (mean square density fluctuation) to small scales by reducing the transfer rate. In the adiabatic limit, the cross-correlation is strong, i.e. $\langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}} \sim \langle \tilde{n}^2 \rangle_{\vec{k}} \sim \langle \tilde{\phi}^2 \rangle_{\vec{k}}$, so that density fluctuation transfer to small scale is completely inhibited. In the hydrodynamic regime, the density fluctuation and the potential fluctuation need only be weakly correlated, namely $\langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}}^2 < \langle \tilde{n}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}}$. Hence, the nonlinear transfer of internal energy to small scales can be very efficient. Therefore, in the hydrodynamic regime, significant amount of energy can be transferred to small scales through the 'density interaction' channel, thus leading to total energy transfer to small scales in certain cases, as well.

Numerical evidence for the inhibition of spectral density transfer by finite density-vorticity cross-correlation is illustrated in Fig.1. The data is obtained from a numerical simulation of the two-field model equations in the hydrodynamic limit, with no sources, sinks or mean density gradient. The figure shows a measure of the timescale for density relaxation, T_n, as a function of time for four values of the cross-correlation. T_n is defined as $T_n \equiv (d\bar{S}_n/dt)^{-1}$ where $\bar{S}_n$ is related to the density-entropy functional used in the next section, i.e. $\bar{S}_n \equiv -\sum_{\vec{k}} \ln \langle n^2 \rangle_{\vec{k}}$. The initial spectra for all four runs consisted of a band of low-k modes with random phases, using the phase information to adjust the cross-correlation. In all four runs the initial density spectrum was the same as the vorticity $(\nabla^2\phi)$ spectrum. The curves are labeled by the value of the normalized cross-correlation, defined as $\eta \equiv \Gamma^2/(E_n \Omega_\phi)$. The top (solid) curve, corresponding to the smallest relaxation time, was for the uncorrelated run ($\eta = 0$). The bottom curve was for complete correlation ($\eta = 1$ or $n = \nabla^2\phi$). The middle curves were for $\eta \approx 0.9$ and $\eta \approx 0.95$. Note that T_n seems to be particularly sensitive to η near $\eta = 1$. That is, the spectral relaxation rate is sensitive to slight deviations away from complete phase correlation. The figure also shows that T_n decreases monotonically for $t > 0.08$ which is symptomatic of the tendency to approach absolute equilibrium.

It is interesting to note that the cross-helicity $\langle \tilde{n}\nabla^2\tilde{\phi} \rangle$ in drift wave turbulence plays a similar role as the cross-helicity $\langle \vec{v} \cdot \vec{B} \rangle$ does in MHD turbulence. In MHD turbulence, the nonlinear evolution of the magnetic field is determined by equation,

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = 0$$

The nonlinearity in the above equation has the property that the identity $(\vec{v} \times \vec{B})^2 = v^2 B^2 - (\vec{v} \cdot \vec{B})^2$ is true. Therefore, the cross-helicity $\langle \vec{v} \cdot \vec{B} \rangle$ in MHD turbulence inhibits the nonlinear transfer of magnetic energy. This similarity becomes much more apparent if we define a vector field $\vec{N} = \vec{\nabla} \times (\tilde{n}\vec{z})$. The evolution of $\vec{N}$ can be easily determined from Eq.(1b), and is given by:

$$\frac{\partial \vec{N}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{N}) = \chi_e \nabla_{\parallel}^2 (\vec{N} - \vec{v}) + \kappa \nabla_y \vec{v}$$

with $\vec{v} = \vec{z} \times \vec{\nabla} \tilde{\phi}$. The nonlinear structure of the above equation is identical to that of the magnetic field evolution equation. Also, $\vec{N}$ and n are clearly analogous of the magnetic field, and the parallel component of vector potential, respectively.

Clearly, a novel feature of this work is the identification of the role of the cross-helicity in the nonlinear dynamics of drift wave turbulence, and the treatment of cross-helicity evolution on an equal footing with the internal and kinetic energy evolution. In this way, the proper relation between nonlinear dynamics and transport is developed in the theory. This is in distinct contrast to "iδ" models, which ignore electron dynamics and cross-correlation, except in the quasilinear calculation of transport fluxes. In a recent more detailed study of dissipative drift wave turbulence[10], the theory of phase space density granulation was applied to a fluid-like system. In that case, the drag on incoherent density fluctuations, generated by electron mode coupling (via the $\tilde{n}$ equation), was identified as a novel relaxation mechanism. This mechanism triggered growth of localized density "blobs", as well as waves, and so yielded frequency linewidth broadening. By way of contrast, both electron ($\tilde{n}$ equation) and ion ($\nabla^2 \tilde{\phi}$ equation) mode coupling are incorporated in the treatment of the nonlinear dynamics of cross-helicity presented here. Moreover, the role of cross-helicity in kinetic and internal energy dynamics is represented, as well, in contrast to Ref.(10). Thus, this work represents a significant improvement over the treatment of dissipative drift wave turbulence given in Ref.(10). Necessarily, the closure model is considerably more complex, though.

V. H-Theorem

In the kinetic theory of many particle systems, statistical evolution is described by the Boltzmann equation. Through the H-theorem, the Boltzmann equation implies monotonic relaxation toward an absolute canonical equilibrium. In the statistical closure theory of turbulence, a similar feature is exhibited, i.e. the statistical evolution of turbulence is described by coupled spectrum evolution equations. These statistical closure equations play a similar role as does Boltzmann equation, namely, they imply that turbulence will monotonically relax toward an absolute equilibrium state in the absence of sources and sinks. The driving force for this relaxation (or entropy production) is the nonlinear mode coupling. The concepts of entropy and the H-theorem from the statistical mechanics of many particle systems have been extended to Navier-Stokes turbulence by Carnevale, et al[16−17]. In this section, similar entropy functionals for drift wave turbulence are introduced. The corresponding H-theorems are proved in the limiting cases of adiabatic and hydrodynamic turbulence.

The general prescription for an entropy functional of a dynamical system is given by[16]:

$$S = \frac{1}{2} \ln detX \tag{26}$$

where X is a matrix with its ith row and jth column elements being the two body correlations $\langle x_i x_j \rangle$ of a finite set of dynamical variables: $x_1, x_2, \cdots, x_n$. The entropy functional S defined in this way has a simple interpretation, namely that $-S$ can be regarded as the amount of information contained in the knowledge of two body correlations $\langle x_i x_j \rangle$[18]. The relation between the entropy functional introduced here and the Gibbs entropy functional[19] has been discussed in Ref.16.

In the formalism described above, the notion that the dynamical variables are independent is crucial. For drift wave turbulence, $\tilde{n}$ and $\tilde{\phi}$ can be regarded as independent dynamic variables only in the hydrodynamic regime where $\chi_e k_\parallel^2$ is small. As $\chi_e k_\parallel^2$ increases, dynamical coupling between $\tilde{n}$ and $\tilde{\phi}$ grows, so that $\tilde{n}$ and $\tilde{\phi}$ are dependent. In the limit where $\chi_e k_\parallel^2 = \infty$, $\tilde{n} = \tilde{\phi}$, and $\tilde{\phi}$ is the only dynamic variable. Therefore, the above definition of entropy is only strictly valid in the two limiting cases, where either $\chi_e k_\parallel^2 \to \infty$ or $\chi_e k_\parallel^2 \to 0$. The following discussion is thus restricted to these two limiting cases.

From equation (26), the entropy functionals for drift wave turbulence in the two limiting cases, apart from an additive constant, are given by:

$$S = \sum_{\vec{k}} \ln \langle \tilde{\phi}^2 \rangle_{\vec{k}}, \qquad \text{in adiabatic regime,} \tag{27a}$$

$$S = \sum_{\vec{k}} \ln(\langle \tilde{n}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}} - \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}}^2), \quad \text{in hydrodynamic regime.} \tag{27b}$$

We now discuss the dynamic properties of S. We first show that the entropy functional S increases monotonically with time. The equilibrium state is then determined by the maximum of the entropy functional. Equilibrium spectra are calculated from this condition.

a.) Adiabatic regime

In the adiabatic regime, only $\tilde{\phi}$ is an independent dynamical variable. The statistical closure equations (25a)-(25c) reduce to:

$$\frac{\partial\langle\tilde{\phi}^2\rangle_{\vec{k}}}{\partial t} = \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1\times\vec{k}_2)^2\frac{k_2^2-k_1^2}{1+k^2}Re\Theta_{\vec{k},\vec{k}_1,\vec{k}_2}$$
$$\times[\frac{k^2-k_2^2}{1+k_1^2}\langle\tilde{\phi}^2\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_2}+\frac{k_1^2-k^2}{1+k_2^2}\langle\tilde{\phi}^2\rangle_{\vec{k}}\langle\tilde{\phi}^2\rangle_{\vec{k}_1}+\frac{k_2^2-k_1^2}{1+k^2}\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{\phi}^2\rangle_{\vec{k}_2}] \tag{28}$$

The entropy production rate is then given by,

$$\frac{dS}{dt}=\sum_{\vec{k}}\frac{1}{\langle\tilde{\phi}^2\rangle_{\vec{k}}}\frac{\partial\langle\tilde{\phi}^2\rangle_{\vec{k}}}{\partial t}=\frac{1}{3}\sum_{\vec{k}_1+\vec{k}_2+\vec{k}=0}(\vec{k}_1\times\vec{k}_2)^2Re\Theta_{-\vec{k},\vec{k}_1,\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}_1}\langle\tilde{\phi}^2\rangle_{\vec{k}_2}\langle\tilde{\phi}^2\rangle_{\vec{k}}$$
$$\times(\frac{(k_2^2-k_1^2)}{(1+k^2)\langle\tilde{\phi}^2\rangle_{\vec{k}}}+\frac{(k_1^2-k^2)}{(1+k_2^2)\langle\tilde{\phi}^2\rangle_{\vec{k}_2}}+\frac{(k^2-k_2^2)}{(1+k_1^2)\langle\tilde{\phi}^2\rangle_{\vec{k}_1}})^2 \tag{29}$$

Thus, we clearly have,

$$\frac{dS}{dt}\geq 0 \tag{30}$$

Next we calculate the equilibrium spectrum by maximizing S subject to the invariance of total energy E and total enstrophy Ω,

$$\delta S-\alpha\delta E-\beta\delta\Omega=0 \tag{31}$$

where α and β are the Lagrange multipliers associated with E and Ω. Noting Eq.(5a) and (5b), the above equation is easily solved to obtain,

$$\langle\tilde{\phi}^2\rangle_{\vec{k}}=\frac{1}{(1+k^2)[\alpha+\beta(1+k^2)]} \tag{32}$$

Eq.(32) shows that the equilibrium spectrum obtained by maximizing S is identical to that obtained from the Gibbs ensemble average[6]. It is easily shown that this spectrum is the only stationary solution of equation $dS/dt=0$.

b.) Hydrodynamic regime.

In the hydrodynamic regime, we have two independent variables, $\tilde{\phi}$ and $\tilde{n}$. The proof of the H-theorem is analytically formidable for the general case of arbitrary cross-correlation, because of the complexity in the structure of the closure equations. Therefore, in the following, we prove the H-theorem for the special case of zero cross-correlation ($\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}}=0$). The entropy functional in the case of zero cross-correlation has a simple form,

$$S=S^{\phi}+S^{n} \tag{33}$$

where the entropy functionals of ϕ field and n field are given by,

$$S^{\phi} = \sum_{\vec{k}} \ln \langle \tilde{\phi}^2 \rangle_{\vec{k}} \tag{34a}$$

$$S^{n} = \sum_{\vec{k}} \ln \langle \tilde{n}^2 \rangle_{\vec{k}} \tag{34b}$$

The statistical closure equations reduce to,

$$\begin{aligned} \frac{\partial \langle \tilde{\phi}^2 \rangle_{\vec{k}}}{\partial t} =& \frac{1}{2} \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \frac{k_2^2 - k_1^2}{k^2} Re\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} [\frac{k^2-k_2^2}{k_1^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \\ &+ \frac{k_1^2 - k^2}{k_2^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} + \frac{k_2^2 - k_1^2}{k^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2}] \end{aligned} \tag{35a}$$

$$\frac{\partial \langle \tilde{n}^2 \rangle_{\vec{k}}}{\partial t} = \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 Re\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} [\langle \tilde{n}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} - \langle \tilde{n}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1}] \tag{35b}$$

The total entropy production rate is,

$$\frac{dS}{dt} = \frac{dS^{\phi}}{dt} + \frac{dS^{n}}{dt}$$

where the entropy production rate for the ϕ field and n field are given by,

$$\begin{aligned} \frac{dS^{\phi}}{dt} =& \sum_{\vec{k}} \frac{1}{\langle \tilde{\phi}^2 \rangle_{\vec{k}}} \frac{\partial \langle \tilde{\phi}^2 \rangle_{\vec{k}}}{\partial t} = \frac{1}{6} \sum_{\vec{k}_1+\vec{k}_2+\vec{k}=0} (\vec{k}_1 \times \vec{k}_2)^2 Re\Theta_{-\vec{k},\vec{k}_1,\vec{k}_2} \\ &\times \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} (\frac{k_2^2 - k_1^2}{k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}} + \frac{k_1^2 - k^2}{k_2^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}} + \frac{k^2 - k_2^2}{k_1^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}_1}})^2 \end{aligned} \tag{36a}$$

$$\begin{aligned} \frac{dS^{n}}{dt} =& \sum_{\vec{k}} \frac{1}{\langle \tilde{n}^2 \rangle_{\vec{k}}} \frac{\partial \langle \tilde{n}^2 \rangle_{\vec{k}}}{\partial t} = \frac{1}{2} \sum_{\vec{k}_1+\vec{k}_2+\vec{k}=0} (\vec{k}_1 \times \vec{k}_2)^2 Re\Theta_{-\vec{k},\vec{k}_1,\vec{k}_2} \\ &\times \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{n}^2 \rangle_{\vec{k}_2} \langle \tilde{n}^2 \rangle_{\vec{k}} (\frac{1}{\langle \tilde{n}^2 \rangle_{\vec{k}}} - \frac{1}{\langle \tilde{n}^2 \rangle_{\vec{k}_2}})^2 \end{aligned} \tag{36b}$$

Hence, we have,

$$\frac{dS}{dt} \geq 0$$

since dS^{ϕ}/dt and dS^{n}/dt can be written as perfect squares. As in the adiabatic regime, the equilibrium spectra are determined from the variational equation:

$$\delta S - \alpha \delta E^{\phi} - \beta \delta E^{n} - \gamma \delta \Omega^{\phi} - \epsilon \delta \Gamma = 0 \tag{37}$$

where α,β,γ and ϵ are the Lagrange multipliers or the inverse "temperatures" associated with E^ϕ, E^n, Ω^ϕ, and Γ, respectively. Noting Eqs.(3a)-(3d), the solution of equation (37) is easily obtained, i.e.

$$\langle\tilde{\phi}^2\rangle_{\vec{k}} = \frac{4\alpha}{k^2[4\alpha(\beta+\gamma k^2)-\epsilon^2 k^2]} \tag{8a}$$

$$\langle\tilde{n}^2\rangle_{\vec{k}} = \frac{4(\beta+\gamma k^2)}{4\alpha(\beta+\gamma k^2)-\epsilon^2 k^2} \tag{38b}$$

$$\langle\tilde{n}\tilde{\phi}\rangle_{\vec{k}} = -\frac{2\epsilon}{4\alpha(\beta+\gamma k^2)-\epsilon^2 k^2} \tag{38c}$$

It is interesting to note[20] that while finite cross-helicity may change the equilibrium internal energy spectrum $\langle\tilde{n}^2\rangle_{\vec{k}}$, it has no effect on the equilibrium potential spectrum $\langle\tilde{\phi}^2\rangle_{\vec{k}}$. This is because we can rewrite $\langle\tilde{\phi}^2\rangle_{\vec{k}}$ in Eq.(8a) as:

$$\langle\tilde{\phi}^2\rangle_{\vec{k}} = \frac{1}{k^2(A+Bk^2)},$$

where $A=\beta$ and $B=\gamma-\epsilon^2/4\alpha$. Since A and B are completely determined by the constants of the motion, i.e. fluid kinetic energy E^ϕ and enstrophy Ω^ϕ, through: $E^\phi=\sum_{\vec{k}} k^2\langle\tilde{\phi}^2\rangle_{\vec{k}}$ and $\Omega^\phi=\sum_{\vec{k}} k^4\langle\tilde{\phi}^2\rangle_{\vec{k}}$, the potential spectrum will not be effected by the presence of finite cross-helicity.

We remark that in the hydrodynamic regime, even though the H-theorem is proved for the special case of zero cross-correlation, it is expected to be true generally. This expectation is supported by the fact that the maximization of the entropy functional S produce the correct equilibrium spectrum, i.e. the equilibrium spectra in Eqs.(38a)-(38c) are exactly the same as that predicted by the Gibbs ensemble theory[6]. The effect of finite cross-correlation is to reduce the entropy production rate, but not change the final state. The entropy functionals introduced in this section not only tell us the direction in $\vec{k}$ space that the turbulence will relax toward due to nonlinear mode couplings, but also $\underline{quantify}$ the relaxation rate according to the entropy production rate dS/dt. This entropy production rate is especially useful for determining relative rates of relaxation in formulating selective decay hypotheses.

VI. Selective Decay Processes

Turbulent relaxation is a common phenomenon in plasmas and fluids. It refers to a decay of an initially turbulent state in the presence of dissipation. There is considerable evidence that long-lived, well-defined "coherent structures" can emerge from a turbulent background through self-organization. These structures have appeared in the numerical simulations of the two-dimensional Navier-Stokes equation carried out by McWilliams[13], and elsewhere. These long-lived structures are viewed theoretically as localized equilibria emerging due to selective decay in constrained turbulent relaxation. In a selective decay process, several (otherwise conserved) quantities are dissipated at different rates. The quantities that undergo the most rapid decay are said to relax, subject to the constraint of conservation of quantities that decay more slowly, or not at all. While the detailed process of the relaxation may be very complicated, the final state of the decay or relaxation can be simply described as a constrained minimum. Mathematically, this minimum can be obtained by minimizing the most rapidly decaying quantity (the 'relax-er') while holding others (the constraints) constant. For example, in 2-D Navier-stokes turbulence (with viscosity acting only at very small scales), the enstrophy Ω which cascades to small scales experiences the most rapid dissipation, and thus is the 'relax-er'. The energy E which inverse cascades to large scales experiences negligible dissipation, and thus is the constraint. The final state of the relaxation is thus described by minimizing enstrophy at constant energy, i.e. by the variational equation[9] $\delta\Omega - \lambda\delta E = 0$, where λ is the Lagrange multiplier associated with E. The solution of this equation is used as a model structure for coherent vortices. In MHD turbulence with conserved cross-helicity $\langle \vec{v} \cdot \vec{B} \rangle$ and magnetic helicity $\langle \vec{A} \cdot \vec{B} \rangle$ ($\vec{A}$ is the vector potential), selective decay of energy relative to either helicity leads to dynamical alignment[7] between $\vec{v}$ and $\pm\vec{B}$, i.e. as in the generation of a relaxed, force free magnetic field, like the Taylor state in the Reversed Field Pinch[8].

The selective decay process in a one-field Hasegawa-Mima model of drift wave turbulence has been discussed in Ref.(21). Therefore, in this section, we focus on selective decay processes in the hydrodynamic regime with finite cross-correlation. This case is very interesting because the selective decay of drift wave turbulence in this regime appears to be a <u>two</u> stage process, and has the features of both dynamic alignment and coherent vortex formation.

In the hydrodynamic regime, we have four inviscid invariants, the fluid kinetic energy E^{ϕ}, the internal energy E^{n}, the fluid enstrophy Ω^{ϕ}, and the cross-correlation Γ. Due to the nonlinear mode couplings, the fluid kinetic energy E^{ϕ} is transferred to large scales through inverse cascade, and experiences negligible dissipation. Thus, it is a 'constraint'. The other three inviscid invariants E^{n}, Ω^{ϕ}, and Γ are all transferred to small scales and thus are dissipated. Therefore, all these three quantities are possible relaxers. Thus, we need to determine which of these candidates experiences the most rapid decay. In order to do this, we need to compare the entropy production rate for the $\tilde{n}$ field and the $\tilde{\phi}$ field evolution. In the hydrodynamic regime, the density and potential fluctuations are weakly correlated. The entropy production rates for the n field and ϕ field are (see Eqs.(36a)-(36b)) given approximately by:

$$\frac{dS^{n}_{\vec{k}}}{dt} \simeq \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 Re\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \tag{39a}$$

$$\frac{dS^{\phi}_{\vec{k}}}{dt} \simeq \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 Re\Theta_{\vec{k},\vec{k}_1,\vec{k}_2} \frac{(k_1^2-k_2^2)(k_2^2-k^2)}{k_2^2 k^2} \langle\tilde{\phi}^2\rangle_{\vec{k}_1} \tag{39b}$$

It is easily seen that

$$\frac{dS^{n}_{\vec{k}}}{dt} > \frac{dS^{\phi}_{\vec{k}}}{dt} \tag{40}$$

which implies that internal energy transfer to small scales is faster than the enstrophy transfer, so that the former is dissipated faster. This is because local interactions (with comparable wavenumbers $k_1 \approx k$ or $k_2 \approx k$) which appear in the nonlinear internal energy transfer process do not participate in nonlinear enstrophy transfer (i.e. $dS^{\phi}/dt \rightarrow 0$ for $k \simeq k_1$). More simply, $\tilde{n}$ is a 'passive' scalar while $\nabla^2\tilde{\phi}$ is necessarily related self-consistently to velocity $\vec{v}$.

These observations lead us to propose that turbulent relaxation in weakly correlated, hydrodynamic, two-field models of drift wave turbulence is a two stage selective decay process, occurring on two principal time scales. On the fast time scale, the internal energy E^n decays while the fluid kinetic energy E^{ϕ}, the fluid enstrophy Ω^{ϕ}, and the cross-helicity Γ do not. The reason why the cross-correlation does not decay on the fast time scale can be understood in the following way. Let's assume that the density fluctuation is dissipated by viscous diffusion $\nu\nabla^2 n$ at small scales. The conservation laws for internal energy and cross-helicity described in Sec.II can be rewritten as (for $v_e^* = \chi_e = 0$):

$$\frac{\partial E^n}{\partial t} = -\nu \int d^2\vec{x}(\vec{\nabla}\tilde{n})^2 \tag{41a}$$

$$\frac{\partial \Gamma}{\partial t} = -(\nu+\mu)\int d^2\vec{x}\tilde{n}\nabla^4\tilde{\phi} \tag{41b}$$

Now, we assume that both density fluctuations and vorticity fluctuations are initially loaded at wavenumber k_0. We define the average wavenumber of the internal energy spectrum (in k) k_n by:

$$k_n^2(t) \equiv \frac{\int d^2\vec{x}(\vec{\nabla}\tilde{n})^2}{\int d^2\vec{x}\tilde{n}^2} = \frac{\sum_{\vec{k}} k^2\langle\tilde{n}^2\rangle_{\vec{k}}}{\sum_{\vec{k}}\langle\tilde{n}^2\rangle_{\vec{k}}} \tag{42}$$

so that $k_n(0) = k_0$. The dissipation term on the right side of equation (41a) can be expressed in terms of k_n, namely $-\nu k_n^2 \int d^2\vec{x}\tilde{n}^2 = -\nu k_n^2 E^n$, so that the dissipation rate for the internal energy is given by $\gamma_n = \nu k_n^2$. Since the internal energy is transferred to small scales faster than the enstrophy is, we expect at a later time $t > t_0$ that $k_n(t) > k_0$, while the enstrophy is peaked near wavenumber k_0. The dissipation term on the right side of Eq.(41b) is approximately, $(\nu+\mu)k_0^2\int d^2\vec{x}\tilde{n}\nabla^2\tilde{\phi} = -(\nu+\mu)k_0^2\Gamma$, so that the dissipation rate for the cross-helicity is $\gamma_\Gamma = (\nu+\mu)k_0^2$. Therefore, we have $\gamma_n > \gamma_\Gamma$ provided $\nu \sim \mu$, i.e. the internal energy decays faster than the cross-helicity. The underlying physics is again related to the absence of self-consistency constraints on density evolution, as opposed to vorticity evolution. Hence, density fluctuations couple to small scales faster than the vorticity fluctuations do.

The final state of selective decay on this fast time scale is then described by the following variational equation,

$$\delta[\int d^2\vec{x}(\tilde{n})^2 - 2\alpha_1 \int d^2\vec{x}\tilde{n}\nabla_\perp^2\tilde{\phi}] = 0 \tag{43}$$

which has the solution,

$$\tilde{n}(\vec{x}) = \alpha_1 \nabla_\perp^2 \tilde{\phi}(\vec{x}) \tag{44}$$

i.e. the density fluctuation tends to 'align' with the vorticity fluctuation. It is interesting to note that Eq.(44) implies that density fluctuations tend to accumulate or become 'trapped' in regions with large vorticity concentrations (i.e. coherent vortices). Here, α_1 is the Lagrange multiplier, and is determined by the initial value of enstrophy Ω_0 and cross-helicity Γ_0, i.e. $\alpha_1 = \Gamma_0/\Omega_0$. Note that the internal energy and the cross-helicity are both proportional to the fluid vorticity as a result of this fast relaxation, i.e. $E^n = \alpha_1^2\Omega^\phi$ and $\Gamma = -\alpha_1\Omega^\phi$.

On the slower time scale, the fluid enstrophy decays while the fluid kinetic energy does not due to the inverse cascade. The final state is described by the constrained variational principal:

$$\delta[\int d^2\vec{x}(\nabla_\perp^2\tilde{\phi})^2 + \alpha_2 \int d^3\vec{x}(\nabla_\perp\tilde{\phi})^2] = 0 \tag{45}$$

the solution of which is

$$\nabla^2\tilde{\phi} = \alpha_2\tilde{\phi} \tag{46}$$

where the Lagrange multiplier α_2 is determined by the initial value of the fluid kinetic energy E^ϕ and the boundary conditions of the flow, i.e. $E_0^\phi = \alpha_2 \int \tilde{\phi}^2 d^2\vec{x}$ and $\alpha_2 > 0$. The solution of equation (46) corresponds to the simplest model of a coherent vortex. In cylindrical coordinates, the solution of equation (46) is,

$$\tilde{\phi}(r,\theta) = J_m(\frac{r}{\sqrt{\alpha_2}})\cos(m\theta) \tag{47}$$

where $m = 0$ corresponds to a monopole structure, and $m = 1$ corresponds to a dipole structure, etc. Note that the density and vorticity fluctuation are given by:

$$\tilde{n} = \alpha_1\alpha_2 J_m(\frac{r}{\sqrt{\alpha_2}})\cos(m\theta) \tag{48}$$

$$\nabla^2\tilde{\phi} = \alpha_2 J_m(\frac{r}{\sqrt{\alpha_2}})\cos(m\theta) \tag{49}$$

Eqs.(47)-(49) satisfy the relations

$$\vec{\nabla}\tilde{\phi} \times \vec{z} \cdot \vec{\nabla}\nabla^2\tilde{\phi} = 0$$
$$\nabla\tilde{\phi} \times \vec{z} \cdot \vec{\nabla} n = 0$$

therefore correspond to stationary solution of the primitive nonlinear equations. It is interesting to note that in the hydrodynamic regime, the density tends to align with vorticity, while in the adiabatic regime, the density is completely 'aligned' with the potential, $\tilde{n} = \tilde{\phi}$. In the intermediate regimes, the density is expected to have the characteristic of each. Finally, it is worthwhile to remind the reader that in this non-trivial instance, the H-theorem was crucial to the formulation of the selective decay hypothesis, in that it provided a means to quantify decay rates of different quantities.

VII. Summaries and Conclusions

In this paper, a statistical theory of a two-field model of drift wave turbulence was presented. The principal results are summarized below:

i.) Through the *EDQNM* closure scheme, we have systematically derived a closed set of spectrum evolution equations for dissipative drift wave turbulence. These equations describe the nonlinear evolution of the internal energy spectrum $\langle \tilde{n}^2 \rangle_{\vec{k}}$, the kinetic energy spectrum $k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}$, and the cross-helicity spectrum $k^2 \langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}}$. Unlike previous studies, cross-helicity dynamics are treated on an equal footing with the other spectra. Similarities between the effects and dynamics of the cross-helicity $\langle \tilde{n}\nabla^2\tilde{\phi} \rangle$ in drift wave turbulence and the cross-helicity $\langle \vec{v} \cdot \vec{B} \rangle$ in *MHD* turbulence have been noted. In particular, the cross-helicity dynamically inhibits the nonlinear transfer of internal energy to small scale.

ii.) Using the statistical closure equations, a H-theorem which allows the determination of the rate of turbulent relaxation in drift wave turbulence is proved. The appropriate entropy functionals are identified. These entropy functionals have the following properties: (a) due to the nonlinear mode couplings, entropy increases monotonically with time until an absolute equilibrium state is reached. (b) The maximization of the entropy subject to the constraint of conserved invariants of the motion yields the absolute equilibrium spectra, which are identical to those obtained from the Gibbs ensemble theory. One of the implications of the H-theorem is that in the hydrodynamic regime, internal energy transfer to small scales is faster than enstrophy transfer is.

iii.) Turbulent relaxation processes are is discussed for the hydrodynamic regime. As a result of disparity in transfer rates between internal energy and enstrophy, the turbulent relaxation (selective decay) occurs on two time scales. On the fast time scale, the internal energy decays while the cross-helicity, the fluid enstrophy, and the fluid kinetic energy do not. On the slow time scale, the fluid enstrophy decays while the fluid kinetic energy does not. As a consequence, the turbulent relaxation exhibits both "dynamic alignment" between density and vorticity $\tilde{n} = \alpha_1 \nabla_\perp^2 \tilde{\phi}$ and the formation of force-free coherent vortices for which $\nabla_\perp^2 \tilde{\phi} = \alpha_2 \tilde{\phi}$. This implies that in hydrodynamic, dissipative drift wave turbulence with nonzero cross-helicity, density fluctuations tend to remain (trapped) inside regions of large vorticity (coherent vortices).

Although the primary focus of this paper is on fundamental physics rather than anomalous transport, it is nevertheless intriguing to speculate on some of the possible implications of this work for tokamak confinement. First and foremost, it is clear that the self-consistent treatment of the cross-correlation given here indicates that in the 'strong turbulence' regime, the effective density-potential phase shift is determined by a complicated interplay between electron and ion mode coupling. Such interaction may reverse the sign of the density-potential phase in certain parts of the spectrum, thus generating inward particle flows. This would be most fortuitous, since the inward pinch is a universal, robust phenomenon of transport, but has emerged in simple, quasi-linear analyses only in certain, highly specialized regimes. Second, the apparent relation between cross-correlation, transport and nonlinear transfer may indicate a more subtle connection between fluctuations and transport than that which is predicted by quasi-linear theory. Indeed, the concepts developed here certainly question the blind application of quasi-linear theory to strong turbulence, as well as exercises in which measured spectra are plugged into quasi-linear formulae which are then used to estimate transport.

On-going work is focused on computational studies of the primitive dissipative drift wave equations and of the closure model thereof. These will be reported in a future publication.

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Appendix A: Solution of Driven Equations.

In this appendix, we solve equations (17a) and (17b) to obtain $\delta\tilde{\phi}_{\vec{k}}$ and $\delta\tilde{n}_{\vec{k}}$. First we introduce the following column vectors:

$$\delta\Psi_{\vec{k}} = \begin{pmatrix} \delta\tilde{\phi}_{\vec{k}} \\ \delta\tilde{n}_{\vec{k}} \end{pmatrix} \qquad S = \begin{pmatrix} S_{1\vec{k}}/k^2 \\ S_{2\vec{k}} \end{pmatrix}$$

and the square matrix:

$$H = \begin{pmatrix} \chi_e k_\parallel^2/k^2 & -\chi_e k_\parallel^2/k^2 \\ (i\omega_e^* - \chi_e k_\parallel^2) & \chi_e k_\parallel^2 \end{pmatrix}$$

then equations (4.32a) and (4.32b) can be written into a simple matrix form:

$$(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}})\delta\Psi + H\delta\Psi = S \tag{A1}$$

We then make a linear transformation $\delta\Psi = A\delta\hat{\Psi}$ where the new dynamical variables are represented by $\delta\hat{\Psi}$, and A is the transformation matrix. The above equation then becomes:

$$A(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}})\delta\hat{\Psi} + HA\delta\hat{\Psi} = S \tag{A2}$$

or

$$(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}})\delta\hat{\Psi} + A^{-1}HA\delta\hat{\Psi} = A^{-1}S \tag{A3}$$

where A^{-1} is the inverse of A. We choose A such that $A^{-1}HA$ is a diagonal matrix. Such A can be constructed from the eigenvectors of H.

The eigenvalue λ of H are determined from equation: $det(H - \lambda I) = 0$, where I is the unit matrix. This equation reduces to

$$\lambda^2 - \chi_e k_{\parallel}^2(1 + \frac{1}{k^2})\lambda + i\omega_e^* \frac{\chi_e k_{\parallel}^2}{k^2} = 0 \tag{A4}$$

with two solutions λ_1 and λ_2 given by:

$$\lambda_{1,2} = \frac{1}{2}\chi_e k_{\parallel}^2(1 + \frac{1}{k^2}) \pm \frac{1}{2}[(\chi_e k_{\parallel}^2(1 + \frac{1}{k^2}))^2 - 4i\omega_e^* \frac{\chi_e k_{\parallel}^2}{k^2}]^{\frac{1}{2}} \tag{A5}$$

The eigenvectors belonging to the eigenvalues λ_1 and λ_2 are respectively

$$\begin{pmatrix} 1 \\ 1 - \frac{\lambda_1 k^2}{\chi_e k_{\parallel}^2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 - \frac{\lambda_2 k^2}{\chi_e k_{\parallel}^2} \end{pmatrix}$$

so the desired transformation matrix A and its inverse A^{-1} are,

$$A = \begin{pmatrix} 1 & 1 \\ 1 - \frac{\lambda_1 k^2}{\chi_e k_{\parallel}^2} & 1 - \frac{\lambda_2 k^2}{\chi_e k_{\parallel}^2} \end{pmatrix}$$

and

$$A^{-1} = \frac{1}{detA}\begin{pmatrix} 1 - \frac{\lambda_2 k^2}{\chi_e k_{\parallel}^2} & -1 \\ -1 + \frac{\lambda_1 k^2}{\chi_e k_{\parallel}^2} & 1 \end{pmatrix}$$

where the determinant of the matrix A is

$$detA = \frac{(\lambda_1 - \lambda_2)k^2}{\chi_e k_{\parallel}^2}$$

With a straightforward manipulation, we obtain

$$A^{-1}HA = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The equation (A3) then becomes:

$$(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}} + \lambda_1)\delta\hat{\Psi}_1 = (A^{-1}S)_1 \tag{A6}$$

$$(\frac{\partial}{\partial t} + \triangle\omega_{\vec{k}} + \lambda_2)\delta\hat{\Psi}_2 = (A^{-1}S)_2 \tag{A7}$$

The solutions of these equations are:

$$\delta\hat{\Psi}_1(t) = \int_{-\infty}^{t} dt' \exp\{-(\triangle\omega_{\vec{k}} + \lambda_1)(t-t')\}(A^{-1}S)_1(t') \tag{A8}$$

$$\delta\hat{\Psi}_2(t) = \int_{-\infty}^{t} dt' \exp\{-(\triangle\omega_{\vec{k}} + \lambda_2)(t-t')\}(A^{-1}S)_2(t') \tag{A9}$$

where

$$(A^{-1}S)_1 = -\frac{1}{detA}[(1 - \frac{\lambda_1}{\chi_e k_{\parallel}^2})S_{1\vec{k}} + S_{2\vec{k}}]$$

$$(A^{-1}S)_2 = \frac{1}{detA}[(1 - \frac{\lambda_2}{\chi_e k_{\parallel}^2})S_{1\vec{k}} + S_{2\vec{k}}]$$

the λ_1 and λ_2 are related to the linear frequency:

$$\lambda_1 = i\omega_{1\vec{k}} \qquad \text{and} \qquad \lambda_2 = i\omega_{2\vec{k}}$$

Therefore the perturbations in the original dynamical variables are given by:

$$\begin{aligned}\delta\tilde{\phi}_{\vec{k}}(t) =& \delta\hat{\Psi}_1(t) + \delta\hat{\Psi}_2(t) \\ =& \sum_{l=1}^{2}\frac{(-1)^l}{detA}\int_{-\infty}^{t} dt' \exp\{-(\triangle\omega_{\vec{k}} + i\omega_{l\vec{k}})(t-t')\} \\ & \times [(1 - \frac{i\omega_{l\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]\end{aligned} \tag{A10}$$

$$\begin{aligned}\delta n_{\vec{k}}(t) =& (1 - \frac{i\omega_{1\vec{k}}k^2}{\chi_e k_{\parallel}^2})\delta\hat{\Psi}_1(t) + (1 - \frac{i\omega_{2\vec{k}}k^2}{\chi_e k_{\parallel}^2})\delta\hat{\Psi}_2(t) \\ =& \sum_{l=1}^{2}\frac{(-1)^l}{detA}(1 - \frac{i\omega_{l\vec{k}}k^2}{\chi_e k_{\parallel}^2})\int_{-\infty}^{t} dt' \exp\{-(\triangle\omega_{\vec{k}} + i\omega_{l\vec{k}})(t-t')\} \\ & \times [(1 - \frac{i\omega_{l\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]\end{aligned} \tag{A11}$$

As pointed out in the discussion of linear theory, out of two linear modes, one is growing while another is strongly damped (at least in the two limiting cases of interest). Therefore, we only retain the growing mode (represented by $\omega_{2\vec{k}}$ in our notation) in the above expression for the driven fields. Dropping the subscript 2, we finally obtain:

$$\begin{aligned}
\delta\tilde{\phi}_{\vec{k}}(t) =& \frac{1}{detA}\int_{-\infty}^{t} dt' \exp\{-(\triangle\omega_{\vec{k}} + i\omega_{\vec{k}})(t-t')\} \\
&\times [(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')] \qquad (A12)
\end{aligned}$$

$$\begin{aligned}
\delta\tilde{n}_{\vec{k}}(t) =& \frac{1 - \frac{i\omega_{\vec{k}}k^2}{\chi_e k_{\parallel}^2}}{detA}\int_{-\infty}^{t} dt' \exp\{-(\triangle\omega_{\vec{k}} + i\omega_{\vec{k}})(t-t')\} \\
&\times [(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')] \qquad (A13)
\end{aligned}$$

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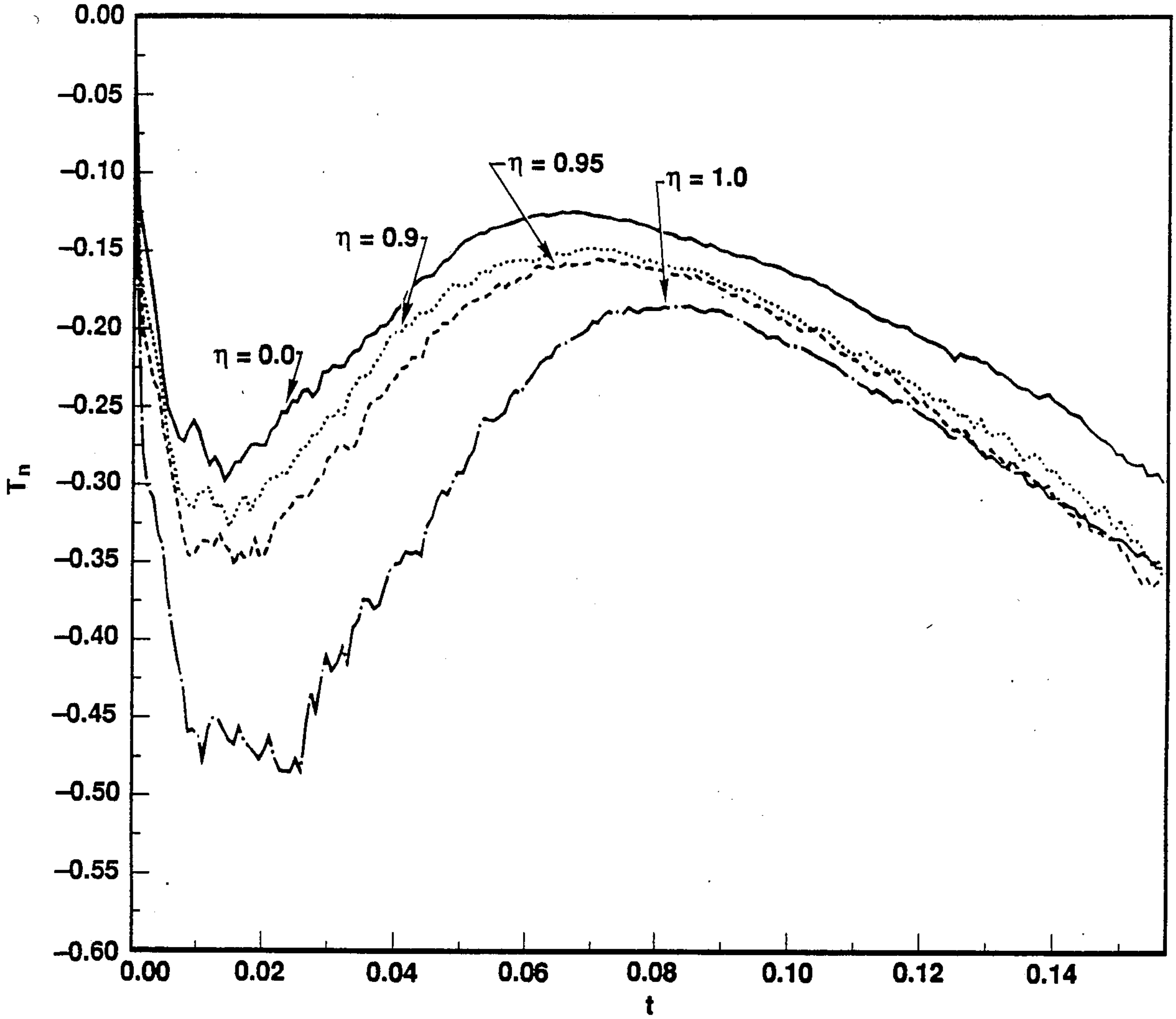
Figures

FIG. 1. Timescale T_n for density relaxation as a function of time for different values of the density-potential cross-correlation. The solid curve $\eta = 0.0$ represents no correlation, whereas $\eta = 1.0$ is complete correlation.


0.00
-0.05
-0.10
-0.15
-0.20
-0.25
-0.30
-0.35
-0.40
-0.45
-0.50
-0.55
-0.60
T_n
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
t
η = 0.0
η = 0.9
η = 0.95
η = 1.0