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## Statistical Theory of Two-Field Model of Drift Wave Turbulence

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# STATISTICAL THEORY OF TWO-FIELD MODEL OF DRIFT WAVE TURBULENCE

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Statistical dynamics of a two-field (density  $\tilde{n}$  and electrostatic potential  $\tilde{\phi}$ ) model of drift wave turbulence is investigated by using the EDQNM (Eddy Damped Quasi-Normal Markovian) closure scheme. The discussions include the statistical closure equations, a H-theorem, and the turbulent relaxation process. The results show that the dynamics of a two-field model is fundamentally different from that of one-field Hasegawa-Mima model. In particular, the density fluctuation is found to behave like the vorticity fluctuation, and is being transferred toward small scales. This transfer process is nonlinearly regulated by the dynamics of cross-correlation. However, the nonlinear transfer rate is bigger than that of vorticity. As a result, the turbulent relaxation process has the characteristic of both dynamic alignment  $\tilde{n} = \alpha_1 \nabla^2 \tilde{\phi}$  and coherent vortices  $\nabla^2 \tilde{\phi} = \alpha_2 \tilde{\phi}$ , where  $\alpha_1$  and  $\alpha_2$  are the Lagrange multipliers determined by the initial and boundary conditions.

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## I. Introduction

Drift wave turbulence has been of considerable interest in plasma fusion research. Despite of numerous efforts that have been taken, it still remains one of the problems that are poorly understood. Most previous studies<sup>1–4</sup> on this subject have been focused on a simple one-field (potential fluctuation  $\tilde{\phi}$ ) nonlinear fluid model known as Hasegawa-Mima equation<sup>1</sup>. This equation was derived by considering cold ions ( $T_i = 0$ , i.e. ignoring the ion's kinetic effect) and quasi-two-dimensional fluctuations ( $k_{\parallel} \ll k_{\perp}$ , where  $k_{\parallel}$  and  $k_{\perp}$  are the parallel and perpendicular wavenumbers respectively). More crucially, in the derivation, the electron's response is assumed to be adiabatic  $\tilde{n} = \tilde{\phi}$ , where  $\tilde{n}$  and  $\tilde{\phi}$  are the density fluctuation and the potential fluctuation, respectively. This simple model equation is close to the two dimensional Navier-Stokes equation, and permits two invariants of motion: total energy, and total enstrophy. Therefore, based on the intuition from the two dimensional fluid turbulence, the Hasegawa-Mima equation predicts an inverse total energy cascade from large  $k_{\perp}$  to small  $k_{\perp}$ . Such spectral transfer was qualitatively consistent with the experimental observations that the fluctuation spectrum is a decreasing function of  $k_{\perp}$  for  $k_{\perp} > 1$ . However, since it completely ignores the density fluctuation (electron) dynamics by taking the adiabatic electron response, the Hasegawa-Mima equation is not the general model for drift wave turbulence, and the conclusions reached based on this model are not universal. In this paper, we will explore the nonlinear dynamics of drift wave turbulence beyond the one-field Hasegawa-Mima equation. Since the purpose of this discussion is to elucidate the underpinning physics, the issues of saturation level and anomalous transport will not be addressed.

The model equations used in our discussion are the two field ( $\tilde{\phi}$ , and  $\tilde{n}$ ) nonlinear dissipative drift wave equations<sup>5</sup>. One important parameter characteristic of the model equations is the electron's parallel diffusion rate  $\chi_e k_{\parallel}^2$  which determines the linear couplings between  $\tilde{n}$  and  $\tilde{\phi}$ , where  $\chi_e$  is the electron's parallel diffusivity. When  $\chi_e k_{\parallel}^2$  is much bigger than the linear mode frequency  $\omega_{\tilde{k}}$  (in the collisionless or high  $k_{\parallel}$  limit), i.e.  $\chi_e k_{\parallel}^2 \gg \omega_{\tilde{k}}$ , the electrons behave adiabatically  $\tilde{n} = \tilde{\phi}$ , the model equations reduce to the Hasegawa-Mima equation. This regime is called adiabatic regime. Therefore, the dynamics of drift wave turbulence can be appropriately described by the Hasegawa-Mima equation only in the adiabatic regime. In the opposite limit, i.e.  $\chi_e k_{\parallel}^2 \ll \omega_{\tilde{k}}$ , the electrons behave hydrodynamically, and the model equations reduce to the two dimensional Navier-Stokes equations. This regime is called hydrodynamic regime. In the hydrodynamic regime,  $\tilde{n}$  and  $\tilde{\phi}$  are only weakly coupled, the dynamics of drift wave turbulence would be fundamentally different from that described by the Hasegawa-Mima equation. Indeed, as demonstrated by Gang, et al in their equilibrium statistical mechanics analysis<sup>6</sup>, the natural tendency of

the nonlinear internal energy (density fluctuation) as well as total energy transfer is toward small rather than large scales in the hydrodynamic regime, which is in distinct contrast to that of the Hasegawa-Mima model. In order to gain further dynamical insight into the drift wave turbulence, in this paper, we present a statistical theory of two field model of drift wave turbulence. In the discussion, we focus on the statistical closure equations, a H-theorem, and the selective decay process in a two field drift wave turbulence. The results indicate that the nonlinear transfer of density fluctuation  $\langle \tilde{n}^2 \rangle$  toward small scale is dynamically inhibited by the presence of cross-correlation  $\langle \tilde{n} \tilde{\phi} \rangle$ . In the adiabatic regime,  $\tilde{n}$  and  $\tilde{\phi}$  are strongly correlated, the density transfer toward small scales is completely inhibited. In the hydrodynamic limit where  $\tilde{n}$  and  $\tilde{\phi}$  are only weakly correlated, however, the density transfer toward small scales is very intense and is shown to be faster than vorticity transfer due to the different role played by the local (with comparable scales) interaction in the two processes. As a consequence, the turbulent relaxation process in a two field drift wave turbulence has the characteristic of both dynamic alignment<sup>7-8</sup>, i.e.  $\tilde{n} = \alpha_1 \nabla_{\perp}^2 \tilde{\phi}$ , and coherent vortices<sup>9</sup>, i.e.  $\nabla_{\perp}^2 \tilde{\phi} = \alpha_2 \tilde{\phi}$ , where  $\alpha_1$  and  $\alpha_2$  are the Lagrange multipliers that are determined by the initial and boundary conditions. The similarities between the two field drift wave turbulence and *MHD* turbulence are also discussed.

At this point, we want to comment that studies on a similar subject but in a different approach have been carried out by Terry and Diamond<sup>10</sup>. They extended the idea and method of phase space density granulation of Dupree<sup>11</sup> to the fluid model of drift wave turbulence. By using the two point correlation theory, they showed that the relaxation of density gradient can not only excite the collective modes but also induce localized density fluctuations, which are non-wave-like structures. The interaction between the density “blob” and the collective mode broadens the mode spectrum and therefore leads to the saturation of the instability. In this work, it was also predicted that the formation of the localized density fluctuation leads drift wave turbulence more fluid-like in nature.

The remainder of this paper is organized as follows. In Sec.II, the two field model equations are discussed. In Sec.III, the linear theory of the model equations is reviewed. In Sec.IV, the statistical closure equations for the nonlinear evolution of potential spectrum  $\langle \tilde{\phi}^2 \rangle_{\vec{k}}$ , density spectrum  $\langle \tilde{n}^2 \rangle_{\vec{k}}$ , and the cross-correlation spectrum  $k^2 \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}$  are systematically derived by using the *EDQNM* closure scheme<sup>12</sup>. The implications to the drift wave dynamics are discussed. These closure equations play a similar role in drift wave turbulence as does Boltzmann equation in many particle system, and will be used to prove a H-theorem in Sec.V, i.e. an entropy functional is identified and shown to increase with time monotonically due to the nonlinear mode couplings (turbulent mixing) until reaching the absolute equilibrium state. In Sec.VI, we apply the H-theorem to discuss a constrained

turbulent relaxation in a two-field drift wave turbulence. In Sec.VII, we summarize and conclude the results.

## II. Model Equations

In this section, the basic model equations are introduced. For simplicity, we consider a plane slab geometry, and take the equilibrium magnetic field to be constant and in  $z$  direction. Equilibrium density is nonuniform only in  $x$  direction. Ions are assumed to be cold and temperature gradient effects are neglected. The nonlinear evolution of the dissipative drift wave turbulence is then described by the following two-field model equations<sup>5</sup>,

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \tilde{\phi} - \vec{\nabla} \tilde{\phi} \times \vec{z} \cdot \vec{\nabla} \nabla_{\perp}^2 \tilde{\phi} = \chi_e \nabla_{\parallel}^2 (\tilde{n} - \tilde{\phi}) + \mu \nabla_{\perp}^4 \tilde{\phi} \quad (1a)$$

$$\frac{\partial}{\partial t} \tilde{n} - \vec{\nabla} \tilde{\phi} \times \vec{z} \cdot \vec{\nabla} \tilde{n} = \chi_e \nabla_{\parallel}^2 (\tilde{n} - \tilde{\phi}) + v_e^* \nabla_y \tilde{\phi} \quad (1b)$$

The above equations have been set up in a dimensionless form.  $\tilde{\phi}$  and  $\tilde{n}$  are the normalized potential and density fluctuations, respectively,  $\chi_e = \frac{v_{te}^2}{\nu_{ei} \Omega_i}$  is electron parallel thermal diffusivity,  $v_{te} = \sqrt{\frac{2T_e}{m_e}}$  is electron thermal speed,  $\nu_{ei}$  is electron ion collision frequency,  $\Omega_i$  is ion gyrofrequency,  $\mu$  is the normalized ion viscosity,  $v_e^* = \frac{1}{L_n}$  is the normalized electron diamagnetic drift velocity with  $L_n$  being the equilibrium density scale length. Eqs.(1a) and (1b) describe the nonlinear evolutions of vorticity fluctuation  $\nabla_{\perp}^2 \tilde{\phi}$  and density fluctuation  $\tilde{n}$  which are coupled through electron parallel diffusivity  $\chi_e$ . The parallel wavelength is assumed to be of the order of connection length,  $k_{\parallel} \sim \frac{1}{qR}$ , the perpendicular wavelength is assumed to be of the order of minor radius,  $k_{\perp} \sim \frac{1}{r}$ , hence,  $k_{\perp} \gg k_{\parallel}$ , i.e. equations (1a) and (1b) describe the nonlinear evolution of a quasi-two-dimensional system.

It is important to note that the above equations have two linear time scales:  $\omega_{\tilde{k}}$  (mode frequency) and  $\chi_e k_{\parallel}^2$  (parallel diffusion rate). In the collisionless limit,  $\omega_{\tilde{k}} \ll \chi_e k_{\parallel}^2$ , we have  $\tilde{n} = \tilde{\phi}$ , i.e. electron is adiabatic. Eqs.(1a) and (1b) reduce to the single field Hasegawa-Mima<sup>1</sup> or Rossby wave equation<sup>4</sup> (for  $\mu = 0$ ),

$$\frac{\partial}{\partial t} (\tilde{\phi} - \nabla_{\perp}^2 \tilde{\phi}) + \vec{\nabla} \tilde{\phi} \times \vec{z} \cdot \vec{\nabla} \nabla_{\perp}^2 \tilde{\phi} = v_e^* \nabla_y \tilde{\phi} \quad (2)$$

In the strong collisional limit,  $\omega_{\tilde{k}} \gg \chi_e k_{\parallel}^2$ ,  $\tilde{n}$  and  $\tilde{\phi}$  are decoupled and evolve independently. Eqs.(1a) and (1b) reduce to 2-D Navier-Stokes equations, where the vorticity is actively advected by itself generated velocity field ( $\vec{v} = -\vec{\nabla}_{\perp} \tilde{\phi} \times \vec{z}$ ) and the density is passively advected by the same velocity field. These two limits are thereafter called adiabatic and hydrodynamic regimes, respectively.  $\tilde{n}$  and  $\tilde{\phi}$  are strongly correlated in the adiabatic

regime, but are only weakly correlated in the hydrodynamic regime. In both regimes, parallel diffusion induced dissipation is negligible. In the intermediate collisionality regime  $\omega_{\tilde{k}} \sim \chi_e k_{\parallel}^2$ , the two field model has the characteristic of both adiabatic and hydrodynamic regime, and is strongly dissipative.

There are four quadratic quantities which are conserved by the  $\vec{E} \times \vec{B}$  convective nonlinearities in Eqs.(1a) and (1b). They are the fluid kinetic energy  $E^\phi$ , the fluid enstrophy  $\Omega^\phi$ , the fluid internal energy  $E^n$ , and the cross-correlation  $\Gamma$ , which are defined by

$$E^\phi = \int (\vec{\nabla}_\perp \tilde{\phi})^2 d^2 \vec{x} \quad (3a)$$

$$\Omega^\phi = \int (\nabla_\perp^2 \tilde{\phi})^2 d^2 \vec{x} \quad (3b)$$

$$E^n = \int \tilde{n}^2 d^2 \vec{x} \quad (3c)$$

$$\Gamma = - \int \tilde{n} \nabla_\perp^2 \tilde{\phi} d^2 \vec{x} \quad (3d)$$

These four quantities evolve with time as follows:

$$\frac{\partial E^\phi}{\partial t} = 2\chi_e \int d^2 \vec{x} \nabla_\parallel \tilde{\phi} \nabla_\parallel (\tilde{n} - \tilde{\phi}) - 2\mu \int d^2 \vec{x} (\nabla_\perp^2 \tilde{\phi})^2, \quad (4a)$$

$$\frac{\partial \Omega^\phi}{\partial t} = 2\chi_e \int d^2 \vec{x} \nabla_\perp^2 \tilde{\phi} \nabla_\parallel (\tilde{n} - \tilde{\phi}) - 2\mu \int d^2 \vec{x} (\vec{\nabla}_\perp^3 \tilde{\phi})^2, \quad (4b)$$

$$\frac{\partial E^n}{\partial t} = -2\chi_e \int d^2 \vec{x} \nabla_\parallel \tilde{n} \nabla_\parallel (\tilde{n} - \tilde{\phi}) + 2v_e^* \int d^2 \vec{x} (\tilde{n} \nabla_y \tilde{\phi}), \quad (4c)$$

$$\frac{\partial \Gamma}{\partial t} = -\chi_e \int d^2 \vec{x} (\tilde{n} + \nabla_\perp^2 \tilde{\phi}) \nabla_\parallel (\tilde{n} - \tilde{\phi}) - \mu \int d^2 \vec{x} (\tilde{n} \nabla_\perp^4 \tilde{\phi}). \quad (4d)$$

Generally, these four quantities are not the constants of motion of Eqs.(1a) and (1b). We define constant of motion in such a way that it evolves on a time scale much longer than the typical time scale of fluctuations. By this definition, these four quadratic quantities can only be regarded as constants of motion of the model equations in the hydrodynamic regime, where the dissipations are negligible, since the first term on the right side of Eqs.(4a)-(4d) is vanishingly small. As  $\chi_e k_{\parallel}^2$  increases, the conservation of these four quantities are broken. However, as  $\chi_e$  is large enough that the system evolves into the adiabatic regime, the new constants of motion can be constructed from the combinations of these four quadratic quantities. These two constants of motion are the total energy  $E$ , and the total enstrophy  $\Omega$  of the Hasegawa-Mima equation, which are defined by

$$E = \int [\tilde{\phi}^2 + (\vec{\nabla}_\perp \tilde{\phi})^2] d^2 \vec{x} \quad (5a)$$

$$\Omega = \int (\tilde{\phi} - \nabla_\perp^2 \tilde{\phi})^2 d^2 \vec{x} \quad (5b)$$

The evolutions of these two quantities are determined by:

$$\frac{\partial E}{\partial t} = -2\mu \int (\nabla_{\perp}^2 \tilde{\phi})^2 d^2 \vec{x}, \quad (6a)$$

$$\frac{\partial \Omega}{\partial t} = -2\mu \int [(\nabla_{\perp}^2 \tilde{\phi})^2 + (\nabla_{\perp}^3 \tilde{\phi})^2] d^2 \vec{x} \quad (6b)$$

In order to facilitate the analysis, we Fourier analyze the fluctuating fields  $\tilde{\phi}(\vec{x}, t)$  and  $\tilde{n}(\vec{x}, t)$  in  $\vec{x}$ ,

$$\tilde{\phi}(\vec{x}, t) = \sum_{\vec{k}} \tilde{\phi}_{\vec{k}}(t) \exp(i\vec{k} \cdot \vec{x}) \quad (7a)$$

$$\tilde{n}(\vec{x}, t) = \sum_{\vec{k}} \tilde{n}_{\vec{k}}(t) \exp(i\vec{k} \cdot \vec{x}) \quad (7b)$$

Since  $\tilde{\phi}(\vec{x}, t)$  and  $\tilde{n}(\vec{x}, t)$  are real quantities, we have  $\tilde{\phi}_{\vec{k}}^* = \tilde{\phi}_{-\vec{k}}$  and  $\tilde{n}_{\vec{k}}^* = \tilde{n}_{-\vec{k}}$ . In fourier space, the model equations can be written as,

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \mu k_{\perp}^2 + \frac{\chi_e k_{\parallel}^2}{k_{\perp}^2} \right) (k_{\perp}^2 \tilde{\phi}_{\vec{k}}) - \chi_e k_{\parallel}^2 \tilde{n}_{\vec{k}} \\ &= \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (k_2^2 - k_1^2) \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \end{aligned} \quad (8a)$$

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \chi_e k_{\parallel}^2 \right) \tilde{n}_{\vec{k}} - \chi_e k_{\parallel}^2 \tilde{\phi}_{\vec{k}} + i\omega_e^* \tilde{\phi}_{\vec{k}} \\ &= \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (\tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} - \tilde{\phi}_{\vec{k}_2} \tilde{n}_{\vec{k}_1}) \end{aligned} \quad (8b)$$

### III. Review of Linear Theory

In this section, linear theory of dissipative drift mode is reviewed. Special attention is focused on those properties that are pertinent to our nonlinear discussion. By linearizing Eqs.(8a) and (8b), and assuming that  $\tilde{\phi}$  and  $\tilde{n}$  have the following time dependence,

$$\{\tilde{\phi}_{\vec{k}}, \tilde{n}_{\vec{k}}\} = \{\tilde{\phi}_{\vec{k}}, \tilde{n}_{\vec{k}}\} \exp(-i\omega t)$$

we have

$$(-i\omega k_{\perp}^2 + \chi_e k_{\parallel}^2 + \mu k_{\perp}^4) \tilde{\phi}_{\vec{k}} - \chi_e k_{\parallel}^2 \tilde{n}_{\vec{k}} = 0 \quad (9a)$$

$$(i\omega_e^* - \chi_e k_{\parallel}^2) \tilde{\phi}_{\vec{k}} + (-i\omega + \chi_e k_{\parallel}^2) \tilde{n}_{\vec{k}} = 0 \quad (9b)$$



$\mu$  is included in the above equations so that the modes are damped at large  $k_{\perp}$  by ion viscosity. At small  $k_{\perp}$ , the mode can be damped by ion Landau damping which can be easily included into the model equations. These two mechanisms provide the effective energy sink. Therefore the unstable mode can only occur in the moderate  $k_{\perp}$  region. In this wavenumber region,  $\mu$  can be neglected so that the condition that Eqs.(9a) and (9b) have non-zero solution is

$$D(\vec{k}, \omega_{\vec{k}}) = 0 \quad (10)$$

where the linear dielectric function  $D(\vec{k}, \omega_{\vec{k}})$  is given by

$$D(\vec{k}, \omega_{\vec{k}}) = -k_{\perp}^2 \omega_{\vec{k}}^2 - i\omega_{\vec{k}} \chi_e k_{\parallel}^2 (1 + k_{\perp}^2) + i\omega_e^* \chi_e k_{\parallel}^2.$$

Eq.(10) has two roots, which correspond to two modes,

$$\omega_{\vec{k}\pm} = -\frac{i}{2} \chi_e k_{\parallel}^2 \left(1 + \frac{1}{k_{\perp}^2}\right) \pm \frac{i}{2} \left[ (\chi_e k_{\parallel}^2 \left(1 + \frac{1}{k_{\perp}^2}\right))^2 - 4i\omega_e^* \frac{\chi_e k_{\parallel}^2}{k_{\perp}^2} \right]^{\frac{1}{2}} \quad (11)$$

Generally, the expression for  $\omega_{\vec{k}}$  are complicated, however, in the two limiting cases mentioned above, simple expressions of the following form can be obtained.

$$\omega_{\vec{k}} = \omega_{\vec{k}}^r + i\gamma_{\vec{k}} \quad (12)$$

where  $\omega_{\vec{k}}^r$  is the real frequency, and  $\gamma_{\vec{k}}$  is the growth rate.

**a. In the adiabatic regime ( $\omega_{\vec{k}} < k_{\parallel}^2 \chi_e$ ):**

$$\omega_{\vec{k}}^r = \frac{\omega_e^*}{1 + k_{\perp}^2} \quad (13a)$$

$$\gamma_{\vec{k}} = \frac{k_{\perp}^2 \omega_e^{*2}}{\chi_e k_{\parallel}^2 (1 + k_{\perp}^2)^3} \quad (13b)$$

**b. In the hydrodynamic regime ( $\omega_{\vec{k}} > k_{\parallel}^2 \chi_e$ ):**

$$\omega_{\vec{k}}^r = \left( \frac{\chi_e k_{\parallel}^2 \omega_e^*}{2k_{\perp}^2} \right)^{\frac{1}{2}} \quad (14a)$$

$$\gamma_{\vec{k}} = \left( \frac{\chi_e k_{\parallel}^2 \omega_e^*}{2k_{\perp}^2} \right)^{\frac{1}{2}} \quad (14b)$$

From Eqs.(13a)-(14b), we observe that in the adiabatic regime the most unstable modes are short wavelength modes with  $k_{\perp} \sim 1$ . The wave energy will be nonlinearly

transferred to long wavelengths by inverse cascade and damped by ion Landau damping. However, in the hydrodynamic regime, the most unstable modes are long wavelength mode with  $k_{\perp} < 1$ . In this case, as will be shown later, a significant amount of wave energy will be transferred to the short wavelength regime through normal cascade, and damped by ion viscosity. In both cases, the drift waves are strong dispersive so that effective mode couplings can occur only in the presence of strong turbulence.

#### IV. Closure Equations

In the statistical description of turbulence, we are interested in the nonlinear evolution of two-body correlations since these two-body correlations determine the fluctuation spectra, and the turbulent transport. For the two-field drift wave turbulence, these two-body correlations are the kinetic energy spectrum  $\langle \tilde{v}^2 \rangle_{\vec{k}} \equiv k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}$ , the internal energy spectrum  $\langle n^2 \rangle_{\vec{k}}$  and the cross-correlation spectrum  $\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \equiv \langle \tilde{n}_{\vec{k}} \tilde{\phi}_{-\vec{k}} \rangle$ . The nonlinear evolution equations for these three spectra will be derived in this section.

It is well known that due to the intrinsic feature of nonlinear equations, the evolution of two body correlations is determined by three body correlations, and the evolution of three body correlations is determined by four body correlations, and so on. The continuation of this procedure will lead to an infinite hierarchy of correlation equations. In order to terminate this hierarchy and obtain a closed set of two-body correlation equations, we employ the *EDQNM* closure scheme<sup>12</sup>. The basic idea of this closure scheme is that the effect of high order correlations on the evolution of three body correlation is approximated by an eddy damping rate so that the phase correlation among three distinct modes will be destroyed by turbulent scrambling in an eddy turnover time. The application of *EDQNM* closure scheme, however, is quite restrictive since it requires that the fluctuating field has a near Gaussian distribution. Hence it fails to describe turbulence with coherent structures (intermittency)<sup>13-14</sup>.

In the following, closure equations for the spectra  $\langle \tilde{v}^2 \rangle_{\vec{k}}$ ,  $\langle n^2 \rangle_{\vec{k}}$ , and  $\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}$  will be derived. For simplicity,  $\mu$  will be neglected. From Eqs.(8a) and (8b) the nonlinear evolution equations for the spectra  $\langle \tilde{v}^2 \rangle_{\vec{k}}$ ,  $\langle n^2 \rangle_{\vec{k}}$ , and  $\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}$  can be constructed:

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + 2 \frac{\chi_e k_{\parallel}^2}{k^2} \right) (k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}) - 2 \chi_e k_{\parallel}^2 \text{Re} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \\ &= \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (k_2^2 - k_1^2) \text{Re} \langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle \end{aligned} \quad (15a)$$

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + 2 \chi_e k_{\parallel}^2 \right) \langle \tilde{n}^2 \rangle_{\vec{k}} - 2 \chi_e k_{\parallel}^2 \text{Re} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} + 2 \omega_e^* \text{Im} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \\ &= 2 \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 \text{Re} \langle \tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} \tilde{n}_{-\vec{k}} \rangle \end{aligned} \quad (15b)$$

$$\begin{aligned}
& \left( \frac{\partial}{\partial t} + \frac{\chi_e k_{\parallel}^2}{k_{\perp}^2} + \chi_e k_{\parallel}^2 \right) (k^2 \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}) - \chi_e k_{\parallel}^2 (\langle \tilde{n}^2 \rangle_{\vec{k}} + k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}) + i\omega_e^* (k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}) \\
& = \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{z} \times \vec{k}_1 \cdot \vec{k}_2) [(k_2^2 - k_1^2) \langle \tilde{n}_{\vec{k}} \tilde{\phi}_{-\vec{k}_1} \tilde{\phi}_{-\vec{k}_2} \rangle + 2k^2 \langle \tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle] \quad (15c)
\end{aligned}$$

In the above equations, the cross-correlation evolution has been treated on an equal footing with the energy evolutions. The three body correlations in the above equations like  $\langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle$  is determined by the phase correlation or phase coherency among the three modes  $\vec{k}_1, \vec{k}_2$ , and  $\vec{k}$ . This phase correlation is induced by the nonlinear mode couplings and is infinitely small if the size of the system is infinitely large. To the first order, this phase correlation is determined by the direct interaction among these three modes<sup>15</sup>. We denote the perturbation in  $\tilde{\phi}_{\vec{k}}$  due to this direct interaction by  $\delta\tilde{\phi}_{\vec{k}}$ , then to the first order, the three body correlation  $\langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle$  can be approximated by,

$$\langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle \simeq \langle \delta\tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle + \langle \tilde{\phi}_{\vec{k}_1} \delta\tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle + \langle \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \delta\tilde{\phi}_{-\vec{k}} \rangle \quad (16)$$

The perturbations  $\delta\tilde{\phi}_{\vec{k}}$  and  $\delta\tilde{n}_{\vec{k}}$  satisfy Eqs.(8a) and (8b) but are driven only by the direct interaction between modes  $\vec{k}_1$  and  $\vec{k}_2$ :

$$\left( \frac{\partial}{\partial t} + \Delta\omega_{\vec{k}} \right) (k^2 \delta\tilde{\phi}_{\vec{k}}) + \chi_e k_{\parallel}^2 (\delta\tilde{\phi}_{\vec{k}} - \delta\tilde{n}_{\vec{k}}) = S_{1\vec{k}} \quad (17a)$$

$$\left( \frac{\partial}{\partial t} + \Delta\omega_{\vec{k}} \right) \delta\tilde{n}_{\vec{k}} + \chi_e k_{\parallel}^2 \delta\tilde{n}_{\vec{k}} + (i\omega_e^* - \chi_e k_{\parallel}^2) \delta\tilde{\phi}_{\vec{k}} = S_{2\vec{k}} \quad (17b)$$

where the source terms are given by:

$$S_{1\vec{k}} = \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (k_2^2 - k_1^2) \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \quad (18a)$$

$$S_{2\vec{k}} = \vec{z} \times \vec{k}_1 \cdot \vec{k}_2 (\tilde{\phi}_{\vec{k}_1} \tilde{n}_{\vec{k}_2} - \tilde{\phi}_{\vec{k}_2} \tilde{n}_{\vec{k}_1}) \quad (18b)$$

Note that  $S_{1\vec{k}}$  and  $S_{2\vec{k}}$  are symmetric in  $\vec{k}_1$  and  $\vec{k}_2$ . In the above equations,  $\Delta\omega_{\vec{k}}$  is the eddy damping rate which is introduced to represent the effect of higher order correlations on the evolution of three body correlation.

Solutions of equations (17a) and (17b) have been obtained in Appendix A. They are (Eqs. A.12 and A.13):

$$\begin{aligned}
\delta\tilde{\phi}_{\vec{k}}(t) &= \frac{1}{\det A_{\vec{k}}} \int_{-\infty}^t dt' \exp\{-(i\omega_{\vec{k}} + \Delta\omega_{\vec{k}})(t - t')\} \\
&\times \left[ \left(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2}\right) S_{1\vec{k}}(t') + S_{2\vec{k}}(t') \right] \quad (19a)
\end{aligned}$$

$$\begin{aligned}
\delta\tilde{n}_{\vec{k}}(t) &= \frac{1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_{\parallel}^2}}{\det A_{\vec{k}}} \int_{-\infty}^t dt' \exp\{-(i\omega_{\vec{k}} + \Delta\omega_{\vec{k}})(t - t')\} \\
&\times \left[ \left(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\perp}^2}\right) S_{1\vec{k}}(t') + S_{2\vec{k}}(t') \right]. \quad (19b)
\end{aligned}$$

where  $\omega_{\vec{k}}$  is the frequency of the linear eigenmode, and  $\det A_{\vec{k}}$  is given by,

$$\det A_{\vec{k}} = \sqrt{(1 + k^2)^2 - 4i\omega_e^* \frac{k^2}{\chi_e k_{\parallel}^2}}$$

We now proceed the renormalization procedure. The first term on the right hand side of equation (16) is,

$$\begin{aligned} & \langle \delta \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle \\ &= \frac{1}{\det A_{\vec{k}_1}} \int_{-\infty}^t dt' \exp\{-(i\omega_{\vec{k}_1} + \Delta\omega_{\vec{k}_1})(t - t')\} \\ & \times [(1 - \frac{i\omega_{\vec{k}_1}}{\chi_e k_{1\parallel}^2}) \langle S_{1\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle + \langle S_{2\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle] \end{aligned} \quad (20)$$

where

$$\begin{aligned} & \langle S_{1\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle \\ &= (\vec{z} \times \vec{k}_1 \cdot \vec{k}_2)(k^2 - k_2^2) \langle \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}_2}(t') \rangle \langle \tilde{\phi}_{\vec{k}}(t') \tilde{\phi}_{-\vec{k}}(t) \rangle \end{aligned} \quad (21a)$$

$$\begin{aligned} & \langle S_{2\vec{k}_1}(t') \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}}(t) \rangle \\ &= (\vec{z} \times \vec{k}_1 \cdot \vec{k}_2) \times [\langle \tilde{n}_{\vec{k}}(t') \tilde{\phi}_{-\vec{k}}(t) \rangle \langle \tilde{\phi}_{\vec{k}_2}(t) \tilde{\phi}_{-\vec{k}_2}(t') \rangle \\ & - \langle \tilde{\phi}_{\vec{k}}(t') \tilde{\phi}_{-\vec{k}}(t) \rangle \langle \tilde{\phi}_{\vec{k}_2}(t) \tilde{n}_{-\vec{k}_2}(t') \rangle] \end{aligned} \quad (21b)$$

In the above equations, the two time correlations are assumed to be of the following form (for  $t > t'$ ):

$$\langle \tilde{\phi}_{\vec{k}}(t) \tilde{\phi}_{-\vec{k}}(t') \rangle = \langle \tilde{\phi}^2(t) \rangle_{\vec{k}} \exp\{-(i\omega_{\vec{k}} + \Delta\omega_{\vec{k}})(t - t')\} \quad (22a)$$

$$\langle \tilde{n}_{\vec{k}}(t) \tilde{n}_{-\vec{k}}(t') \rangle = \langle \tilde{n}^2(t) \rangle_{\vec{k}} \exp\{-(i\omega_{\vec{k}} + \Delta\omega_{\vec{k}})(t - t')\} \quad (22b)$$

$$\langle \tilde{n}_{\vec{k}}(t) \tilde{\phi}_{-\vec{k}}(t') \rangle = \langle \tilde{n}(t) \tilde{\phi}(t) \rangle_{\vec{k}} \exp\{-(i\omega_{\vec{k}} + \Delta\omega_{\vec{k}})(t - t')\} \quad (22c)$$

Substitute these relations into equations (21a), (21b), and then (20), we obtain:

$$\begin{aligned} \langle \delta \tilde{\phi}_{\vec{k}_1} \tilde{\phi}_{\vec{k}_2} \tilde{\phi}_{-\vec{k}} \rangle &= \frac{(\vec{z} \times \vec{k}_1 \cdot \vec{k}_2)}{\det A_{\vec{k}_1}} \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} [(1 - \frac{i\omega_{\vec{k}_1}}{\chi_e k_{\parallel}^2})(k^2 - k_2^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \\ & + \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} - \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2}^*] \end{aligned} \quad (23)$$

where

$$\Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} = \frac{1}{i(\omega_{\vec{k}_1} + \omega_{\vec{k}_2} - \omega_{\vec{k}}) + \Delta\omega_{\vec{k}_1} + \Delta\omega_{\vec{k}_2} + \Delta\omega_{\vec{k}}} \quad (24)$$

The real part of  $\Theta_{\vec{k}, \vec{k}_1, \vec{k}_2}$  determines the time scale of three wave nonlinear interaction.

The above calculations show the general procedure of renormalization. In the same way, the other three body correlations can also be renormalized. After some lengthy but straightforward calculation, we obtain the desired closed set of three coupled spectrum evolution equations. In order to simplify these spectrum evolution equations, we define the coupling coefficients:

$$\begin{aligned} a_{\vec{k}} &= (1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2}) b_{\vec{k}} & b_{\vec{k}} &= \frac{1}{\det A_{\vec{k}}} \\ c_{\vec{k}} &= (1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_{\parallel}^2}) a_{\vec{k}} & d_{\vec{k}} &= (1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_{\parallel}^2}) b_{\vec{k}} \end{aligned}$$

The closure equations are then given by the following.

For the kinetic energy spectrum  $k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}$ :

$$\begin{aligned} & (\frac{\partial}{\partial t} + 2\frac{\chi_e k_{\parallel}^2}{k^2})(k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}) - 2\chi_e k_{\parallel}^2 \text{Re} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \\ &= \text{Re} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 (k_2^2 - k_1^2) \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \\ & [2a_{\vec{k}_1} (k^2 - k_2^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} + 2b_{\vec{k}_1} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2}^* \langle \tilde{\phi}^2 \rangle_{\vec{k}}) \\ & + a_{-\vec{k}} (k_2^2 - k_1^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} + b_{-\vec{k}} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2}^* \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1}^* \langle \tilde{\phi}^2 \rangle_{\vec{k}_2})] \end{aligned} \quad (25a)$$

For the internal energy spectrum  $\langle n^2 \rangle_{\vec{k}}$ :

$$\begin{aligned} & (\frac{\partial}{\partial t} + 2\chi_e k_{\parallel}^2) \langle n^2 \rangle_{\vec{k}} - 2\chi_e k_{\parallel}^2 \text{Re} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} + 2\omega_e^* \text{Im} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \\ &= 2\text{Re} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \\ & [a_{\vec{k}_1} (k^2 - k_2^2) \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}^* \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} + b_{\vec{k}_1} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle n^2 \rangle_{\vec{k}} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}^* \langle n^2 \rangle_{\vec{k}_2}) \\ & + c_{\vec{k}_2} (k_1^2 - k^2) \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}^* \langle \phi^2 \rangle_{\vec{k}_1} + d_{\vec{k}_2} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}^* \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1}^* - \langle n^2 \rangle_{\vec{k}} \langle \phi^2 \rangle_{\vec{k}_1}) \\ & + c_{-\vec{k}} (k_2^2 - k_1^2) \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle \phi^2 \rangle_{\vec{k}_1} + d_{-\vec{k}} (\langle n^2 \rangle_{\vec{k}_2} \langle \phi^2 \rangle_{\vec{k}_1} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1}^*)] \end{aligned} \quad (25b)$$

For the cross-correlation spectrum  $\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}$ :

$$(\frac{\partial}{\partial t} + \chi_e k_{\parallel}^2 + \frac{\chi_e k_{\parallel}^2}{k^2})(k^2 \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}) - \chi_e k_{\parallel}^2 (\langle \tilde{n}^2 \rangle_{\vec{k}} + k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}) + i\omega_e^* (k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}})$$

$$\begin{aligned}
&= \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \left\{ \frac{1}{2} (k_2^2 - k_1^2) \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2}^* \times \right. \\
&\quad [2a_{-\vec{k}_1} (k^2 - k_2^2) \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} + 2b_{-\vec{k}_1} (\langle \tilde{n}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2}) \\
&\quad + c_{\vec{k}} (k_2^2 - k_1^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} + d_{\vec{k}} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2})] \\
&\quad + k^2 \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \times \\
&\quad [a_{\vec{k}_1} (k^2 - k_2^2) \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} + b_{\vec{k}_1} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} - \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{n}^2 \rangle_{\vec{k}_2}) \\
&\quad + c_{\vec{k}_2} (k_1^2 - k^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} + d_{\vec{k}_2} (\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1}^* \langle \tilde{\phi}^2 \rangle_{\vec{k}} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1}) \\
&\quad \left. + a_{-\vec{k}} (k_2^2 - k_1^2) \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} + b_{-\vec{k}} (\langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{n}^2 \rangle_{\vec{k}_2} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_2} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1}^*)] \right\} \quad (25c)
\end{aligned}$$

The above equations provide an appropriate description for the nonlinear spectra evolution and transfer in strong drift wave turbulence. It can be easily shown that the conservation properties of the  $\vec{E} \times \vec{B}$  convective nonlinearity are preserved by the closure scheme. Unlike previous studies<sup>10</sup>, the cross-correlation dynamics is treated on an equal footing with that of the energetics, and the dynamic effect of the cross-correlation on the energy spectra evolution and transfer is explicitly retained. Eqs.(25a)-(25c) as they stand are very difficult to solve analytically. Numerical solutions are needed to obtain detailed information about the energy and cross-correlation spectra evolution and transfer.

At this point, it is appropriate to discuss the dynamic effect of cross-correlation on the energy transfer. The cross-correlation is defined as  $\Gamma \equiv -\langle \tilde{n} \nabla^2 \tilde{\phi} \rangle = \langle \vec{\nabla} \tilde{n} \cdot \vec{\nabla} \tilde{\phi} \rangle$ , i.e. the correlation (or closeness) between the density and fluid vorticity fluctuations. It also measures the similarity between the density and potential contours. Since,  $\Gamma$  is conserved by  $\vec{E} \times \vec{B}$  convective nonlinearity, it provides an extra constraint on the nonlinear transfer of energies. Dynamically, the cross-correlation inhibits the internal energy transfer to small scales. In order to see this, let's look at the closure equation (Eq.(25b)) for the internal energy evolution. The dominant nonlinear terms in this equation have the following structure:  $\langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{n}^2 \rangle_{\vec{k}} - \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}_1}$ . The other terms are not significant since large cancellation occurs between contributions from  $k_1 < k$  and  $k_1 > k$  part of the spectrum. This will be much more apparent if we look at the density evolution equation in the configuration space where the  $\vec{E} \times \vec{B}$  convective nonlinearity:  $\vec{\nabla} \tilde{\phi} \times \vec{z} \cdot \vec{\nabla} \tilde{n}$  has the property that  $(\vec{\nabla} \tilde{\phi} \times \vec{z} \cdot \vec{\nabla} \tilde{n})^2 = (\vec{\nabla} \tilde{\phi})^2 (\vec{\nabla} \tilde{n})^2 - (\vec{\nabla} \tilde{\phi} \cdot \vec{\nabla} \tilde{n})^2$ . This relation clearly indicates that the cross-correlation (between  $\tilde{n}$  and  $\nabla_{\perp}^2 \tilde{\phi}$ ) dynamically inhibits the internal energy (density fluctuation) transfer to small scales by reducing the transfer rate. In the adiabatic limit, the cross-correlation is strong, i.e.  $\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} \sim \langle \tilde{n}^2 \rangle_{\vec{k}} \sim \langle \tilde{\phi}^2 \rangle_{\vec{k}}$ , the density fluctuation transfer to small scale is completely inhibited. In the hydrodynamic regime, the density fluctuation and the potential fluctuation is only weakly correlated, namely  $\langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}} < \langle \tilde{n}^2 \rangle_{\vec{k}}, \langle \tilde{\phi}^2 \rangle_{\vec{k}}$ . Hence, the nonlinear transfer of internal energy to small scales is

very efficient. Therefore, in the hydrodynamic regime, significant amount of energy can be transferred to small scales through density channel which may leads to the total energy transfer to small scales as well in certain cases.

The numerical evidence of the inhibition of spectral density transfer by finite density-vorticity cross-correlation is illustrated in Fig.1. The data comes from a numerical simulation of our two-field model equations in the hydrodynamic limit with no sources, sinks and mean density gradient. The figure shows a measure of the timescale for the density relaxation,  $T_n$ , as a function of time for four values of the cross-correlation.  $T_n$  is defined as:

$$T_n \equiv \left( \frac{d\bar{S}_n}{dt} \right)^{-1},$$

where  $\bar{S}_n$  is defined as the minus density-entropy functional (see next section):

$$\bar{S}_n \equiv - \sum_{\vec{k}} \ln \langle n^2 \rangle_{\vec{k}}.$$

The initial spectra for all four runs consisted of a band of low- $k$  modes with random phases, using the phase information to adjust the cross-correlation. In all four runs the initial density spectrum was the same as the vorticity ( $\nabla^2 \phi$ ) spectrum. The curve are labeled by the value of the normalized cross-correlation, defined as  $\eta \equiv \frac{\Gamma^2}{E_n \Omega_\phi}$ . The top (solid) curve, corresponding to the smallest relaxation time, was for the uncorrelated run ( $\eta = 0$ ). The bottom curve was for complete correlation ( $\eta = 1$  or  $n = \nabla^2 \phi$ ). The middle curves were for  $\eta \approx 0.9$  and  $\eta \approx 0.95$ . Note that  $T_n$  seems to be particularly sensitive to  $\eta$  near  $\eta = 1$ . That is, the spectral relaxation rate is sensitive to slight deviations away from complete phase correlation. The figure also shows that  $T_n$  decreases monotonically for  $t > 0.08$  which indicates the tendency approaching absolute equilibrium.

It is interesting to note that the cross-correlation  $\langle \tilde{n} \nabla^2 \tilde{\phi} \rangle$  in drift wave turbulence play a similar role as the cross-helicity  $\langle \vec{v} \cdot \vec{B} \rangle$  does in *MHD* turbulence. In *MHD* turbulence, the nonlinear evolution of the magnetic field fluctuation is determined by equation,

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = 0$$

The nonlinearity in the above equation also has the property that  $(\vec{v} \times \vec{B})^2 = v^2 B^2 - (\vec{v} \cdot \vec{B})^2$ . Therefore, the cross-helicity  $\langle \vec{v} \cdot \vec{B} \rangle$  in *MHD* turbulence inhibits the nonlinear transfer of magnetic fluctuations. This similarity becomes much more apparent if we construct a new vector called  $\vec{B}^n = \vec{\nabla} \times (\tilde{n} \vec{z})$ . The evolution of  $\vec{B}^n$  can be easily derived from Eq.(1b),

$$\frac{\partial \vec{B}^n}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}^n) = \chi_e \nabla_{\parallel}^2 (\vec{B}^n - \vec{v}) + \kappa \nabla_y \vec{v}$$

with  $\vec{v} = \vec{z} \times \vec{\nabla} \tilde{\phi}$ . The nonlinear structure in the above equation is exact the same as that determining the nonlinear evolution of the magnetic field fluctuation,  $\vec{B}^n$  and  $n$  are analogies of the magnetic field, and the parallel component of vector potential.

## V. H-Theorem

In the kinetic theory of many particle systems, the statistical evolution is described by the Boltzmann equation. Through the H-theorem, the Boltzmann equation implies a monotonic relaxation toward an absolute canonical equilibrium. In the statistical closure theory of turbulence, a similar feature is exhibited, i.e. the statistical evolution of turbulence is described by coupled spectrum evolution equations. These statistical closure equations play a similar role as does Boltzmann equation, namely, they imply that turbulence will monotonically relax toward an absolute equilibrium state in the absence of energy source and sink. The driving force for this relaxation (or entropy production) is the nonlinear mode couplings. The concept of entropy and H-theorem from the statistical mechanics of many particle systems has been extended to Navier-Stokes turbulence by Carnevale, et al<sup>16-17</sup>. In this section, similar entropy functionals for drift wave turbulence are introduced. The corresponding H-theorem is proved explicitly in the two limiting cases.

The general prescription of entropy functional for a dynamical system is given by<sup>16</sup>:

$$S = \frac{1}{2} \ln \det X \quad (26)$$

where  $X$  is a matrix with its element in the  $i$ th row and  $j$ th column being the two body correlations  $\langle x_i x_j \rangle$  of a finite set of dynamical variables:  $x_1, x_2, \dots, x_n$ . The entropy functional  $S$  defined in this way has a simple interpretation that  $-S$  can be regarded as the amount of information contained in just a knowledge of two body correlations  $\langle x_i x_j \rangle$ <sup>18</sup>. The relation between the entropy functional introduced here and the Gibbs entropy functional<sup>19</sup> has been discussed in Ref.16.

In the above formalism, the concept that the dynamical variables are independent is very important. For drift wave turbulence,  $\tilde{n}$  and  $\tilde{\phi}$  can be regarded as independent dynamic variables only in the hydrodynamic regime where  $\chi_e k_{\parallel}^2$  is small. As  $\chi_e k_{\parallel}^2$  increases, the dynamical coupling between  $\tilde{n}$  and  $\tilde{\phi}$  develops. Thus,  $\tilde{n}$  and  $\tilde{\phi}$  become dependent. In the limit where  $\chi_e k_{\parallel}^2 = \infty$ ,  $\tilde{n} = \tilde{\phi}$ ,  $\tilde{\phi}$  is the only dynamic variable. Therefore, the above definition of entropy is only meaningful in the two limiting cases where either  $\chi_e k_{\parallel}^2 = \infty$  or  $\chi_e k_{\parallel}^2 = 0$  so that the dissipation by parallel diffusion is zero. The following discussion is thus restricted to these two limiting cases.

From equation (26), the entropy functionals for drift wave turbulence in the two



limiting cases, apart from an additive constant, are given by:

$$S = \sum_{\vec{k}} \ln \langle \tilde{\phi}^2 \rangle_{\vec{k}}, \quad \text{in adiabatic regime,} \quad (27a)$$

$$S = \sum_{\vec{k}} \ln (\langle \tilde{n}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}} - \langle (\tilde{n}\tilde{\phi})^2 \rangle_{\vec{k}}), \quad \text{in hydrodynamic regime.} \quad (27b)$$

We now discuss the dynamic properties of  $S$ . We first show that the entropy functional  $S$  increases monotonically with time. The equilibrium state is then characterized by the maximum of the entropy functional. Equilibrium spectra will be calculated from this condition.

#### a.) Adiabatic regime

In the adiabatic regime, only  $\tilde{\phi}$  is the independent dynamical variable. The statistical closure equations (25a)-(25c) reduce to:

$$\begin{aligned} \frac{\partial \langle \tilde{\phi}^2 \rangle_{\vec{k}}}{\partial t} &= \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \frac{k_2^2 - k_1^2}{1 + k^2} \text{Re} \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \\ &\times \left[ \frac{k^2 - k_2^2}{1 + k_1^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} + \frac{k_1^2 - k^2}{1 + k_2^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} + \frac{k_2^2 - k_1^2}{1 + k^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \right] \end{aligned} \quad (28)$$

The entropy production rate is then given by,

$$\begin{aligned} \frac{dS}{dt} &= \sum_{\vec{k}} \frac{1}{\langle \tilde{\phi}^2 \rangle_{\vec{k}}} \frac{\partial \langle \tilde{\phi}^2 \rangle_{\vec{k}}}{\partial t} = \frac{1}{3} \sum_{\vec{k}_1 + \vec{k}_2 + \vec{k} = 0} (\vec{k}_1 \times \vec{k}_2)^2 \text{Re} \Theta_{-\vec{k}, \vec{k}_1, \vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \\ &\times \left( \frac{(k_2^2 - k_1^2)}{(1 + k^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}}} + \frac{(k_1^2 - k^2)}{(1 + k_2^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}_2}} + \frac{(k^2 - k_2^2)}{(1 + k_1^2) \langle \tilde{\phi}^2 \rangle_{\vec{k}_1}} \right)^2 \end{aligned} \quad (29)$$

Thus, we have,

$$\frac{dS}{dt} \geq 0 \quad (30)$$

Next we calculate the equilibrium spectrum by maximizing  $S$  subject to the invariance of total energy  $E$  and total enstrophy  $\Omega$ ,

$$\delta S - \alpha \delta E - \beta \delta \Omega = 0 \quad (31)$$

where  $\alpha$  and  $\beta$  are the Lagrange multipliers associated with  $E$  and  $\Omega$ . Noting Eq.(5a) and (5b), the above equation can be easily solved to obtain,

$$\langle \tilde{\phi}^2 \rangle_{\vec{k}} = \frac{1}{(1 + k^2)[\alpha + \beta(1 + k^2)]} \quad (32)$$

Eq.(32) shows that the equilibrium spectrum we obtained by maximizing  $S$  is exactly the same as that obtained from Gibbs ensemble average<sup>6</sup>. It is easily shown that this spectrum is the only stationary solution of equation  $dS/dt = 0$ .

**b.) Hydrodynamic regime.**

In the hydrodynamic regime, we have two independent variables  $\tilde{\phi}$  and  $\tilde{n}$ . The proof of H-theorem is analytically formidable in the general case of arbitrary cross-correlation because of the complexity in the structures of the closure equations. Therefore, in the following, we are going to prove it in the special case of zero cross-correlation ( $\langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}} = 0$ ). The entropy functional in the case of zero cross-correlation has a simple form,

$$S = S^\phi + S^n \quad (33)$$

where the entropy functionals of  $\phi$  field and  $n$  field are given by,

$$S^\phi = \sum_{\vec{k}} \ln \langle \tilde{\phi}^2 \rangle_{\vec{k}} \quad (34a)$$

$$S^n = \sum_{\vec{k}} \ln \langle \tilde{n}^2 \rangle_{\vec{k}} \quad (34b)$$

The statistical closure equations reduce to,

$$\begin{aligned} \frac{\partial \langle \tilde{\phi}^2 \rangle_{\vec{k}}}{\partial t} = & \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \frac{k_2^2 - k_1^2}{k^2} \text{Re} \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \left[ \frac{k^2 - k_2^2}{k_1^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \right. \\ & \left. + \frac{k_1^2 - k^2}{k_2^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} + \frac{k_2^2 - k_1^2}{k^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \right] \end{aligned} \quad (35a)$$

$$\frac{\partial \langle \tilde{n}^2 \rangle_{\vec{k}}}{\partial t} = \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \text{Re} \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} [\langle \tilde{n}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} - \langle \tilde{n}^2 \rangle_{\vec{k}} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1}] \quad (35b)$$

The total entropy production rate is,

$$\frac{dS}{dt} = \frac{dS^\phi}{dt} + \frac{dS^n}{dt}$$

where the entropy production rate in the  $\phi$  field and  $n$  field are given by,

$$\begin{aligned} \frac{dS^\phi}{dt} = & \sum_{\vec{k}} \frac{1}{\langle \tilde{\phi}^2 \rangle_{\vec{k}}} \frac{\partial \langle \tilde{\phi}^2 \rangle_{\vec{k}}}{\partial t} = \frac{1}{6} \sum_{\vec{k}_1 + \vec{k}_2 + \vec{k} = 0} (\vec{k}_1 \times \vec{k}_2)^2 \text{Re} \Theta_{-\vec{k}, \vec{k}_1, \vec{k}_2} \\ & \times \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{\phi}^2 \rangle_{\vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}} \left( \frac{k_2^2 - k_1^2}{k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}} + \frac{k_1^2 - k^2}{k_2^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}} + \frac{k^2 - k_2^2}{k_1^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}_1}} \right)^2 \end{aligned} \quad (36a)$$

$$\begin{aligned} \frac{dS^n}{dt} = & \sum_{\vec{k}} \frac{1}{\langle \tilde{n}^2 \rangle_{\vec{k}}} \frac{\partial \langle \tilde{n}^2 \rangle_{\vec{k}}}{\partial t} = \frac{1}{2} \sum_{\vec{k}_1 + \vec{k}_2 + \vec{k} = 0} (\vec{k}_1 \times \vec{k}_2)^2 \text{Re} \Theta_{-\vec{k}, \vec{k}_1, \vec{k}_2} \\ & \times \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \langle \tilde{n}^2 \rangle_{\vec{k}_2} \langle \tilde{n}^2 \rangle_{\vec{k}} \left( \frac{1}{\langle \tilde{n}^2 \rangle_{\vec{k}}} - \frac{1}{\langle \tilde{n}^2 \rangle_{\vec{k}_2}} \right)^2 \end{aligned} \quad (36b)$$

Hence, we have,

$$\frac{dS}{dt} \geq 0$$

As in the adiabatic regime, the equilibrium spectra are determined from the variational equation:

$$\delta S - \alpha \delta E^\phi - \beta \delta E^n - \gamma \delta \Omega^\phi - \epsilon \delta \Gamma = 0 \quad (37)$$

where  $\alpha, \beta, \gamma$  and  $\epsilon$  are the Lagrange multipliers or the inverse “temperatures” associated with  $E^\phi$ ,  $E^n$ ,  $\Omega^\phi$ , and  $\Gamma$ , respectively. Noting Eqs.(3a)-(3d), the solutions of equation (37) is easily obtained,

$$\langle \tilde{\phi}^2 \rangle_{\vec{k}} = \frac{4\alpha}{k^2[4\alpha(\beta + \gamma k^2) - \epsilon^2 k^2]} \quad (8a)$$

$$\langle \tilde{n}^2 \rangle_{\vec{k}} = \frac{4(\beta + \gamma k^2)}{4\alpha(\beta + \gamma k^2) - \epsilon^2 k^2} \quad (38b)$$

$$\langle \tilde{n}\tilde{\phi} \rangle_{\vec{k}} = -\frac{2\eta}{4\alpha(\beta + \gamma k^2) - \epsilon^2 k^2} \quad (38c)$$

We want to remark that in the hydrodynamic regime, even though the H-theorem is proved in the special case of zero cross-correlation, it is expected to be true generally. This expectation is supported by the fact that the maximization of the entropy functional  $S$  produce the right equilibrium spectrum, i.e. the equilibrium spectra in Eqs.(38a)-(38c) are exactly the same as those predicted from Gibbs ensemble theory<sup>6</sup>. The effect of finite cross-correlation will be to reduce the entropy production rate. The entropy functionals that we introduced in this section not only tell us the direction that the turbulence will relax to due to the nonlinear mode couplings, but also quantify the relaxation rate by the entropy production rate  $dS/dt$ . This entropy production rate is especially useful when we try to determine which quantity decays faster in a turbulent relaxation process.

## VI. Selective Decay Process

Turbulent relaxation is a common phenomenon in plasmas and fluids. It refers to a decay process of an initially developed turbulent state in the presence of dissipation. There is considerable evidence that long-lived, well-defined structures (coherent structures) can emerge in an incoherent background through turbulent relaxation (i.e. self-organization). These structures have appeared in the numerical simulation of two-dimensional Navier-Stokes equation carried out by McWilliams<sup>13</sup>. Theoretically, these long-lived, well-defined structures are described as localized equilibria due to selective decay or constrained turbulent relaxation process. In the selective decay process, several (otherwise conserved) quantities are dissipated at different rate. The quantities that experience the most rapid

decay are called ‘relaxiers’. The other quantities that experience slow or no decay are called ‘constrainer’, since they constrain the decay evolutions of the relaxiers. The detailed process of the relaxation may be very complicated, however, the final state of the decay or relaxation can be simply described as the minimum state of the relaxiers. Mathematically, this can be obtained by minimizing the relaxiers while holding the constrainers as invariants. For example, in 2-D Navier-stokes turbulence with viscosity acting at small scales, the enstrophy  $\Omega$  which cascades to small scales experiences the most rapid dissipation, thus is the relaxier. The energy  $E$  which inverse cascades to large scales experiences negligible dissipation, hence is the constrainer. The final state of the relaxation is described by a variational equation<sup>9</sup>  $\delta\Omega - \lambda\delta E = 0$ , where  $\lambda$  is the Lagrange multiplier associated with  $E$ . The solution of this equation is usually used as a model structure for coherent vortices. In MHD turbulence with conserved cross-helicity  $\langle \vec{v} \cdot \vec{B} \rangle$  and magnetic helicity  $\langle \vec{A} \cdot \vec{B} \rangle$  where  $\vec{A}$  is the vector potential, the selective decay often leads to the dynamical alignment<sup>7</sup> between  $\vec{v}$  and  $\vec{B}$  and the generation of the force free magnetic field, like the Taylor state in Reversed Field Pinch<sup>8</sup>.

The selective decay process in a one-field Hasegawa-Mima model of drift wave turbulence has been discussed in Ref.(20). Therefore, in this section, we concentrate on discussing the case of hydrodynamic regime with finite crosscorrelations. This case is very interesting because the selective decay of drift wave turbulence is a two stage process and has the characteristic of both dynamic alignment and coherent vortices.

In the hydrodynamic regime, we have four inviscid invariants, the fluid kinetic energy  $E^\phi$ , the internal energy  $E^n$ , the fluid enstrophy  $\Omega^\phi$ , and the cross-correlation  $\Gamma$ . Due to the nonlinear mode couplings, the fluid kinetic energy  $E^\phi$  is transferred to large scales through inverse cascade and experiences negligible dissipation, hence is the constrainer. The other three inviscid invariants  $E^n$ ,  $\Omega^\phi$ , and  $\Gamma$  are all transferred to small scales and experience dissipations. Therefore, all these three quantities are generally the relaxiers. Now, we need to determine that among these relaxiers, which one experiences the most rapid decay. In order to do this, we need to compare the entropy production rate for  $\tilde{n}$  field and  $\tilde{\phi}$  field. In the hydrodynamic regime, the density and potential fluctuations are only weakly correlated. The entropy production rate for the  $n$  field and  $\phi$  field are approximately (see Eqs.(36a)-(36b)) given by,

$$\frac{dS_{\vec{k}}^n}{dt} \simeq \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \text{Re} \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \quad (39a)$$

$$\frac{dS_{\vec{k}}^\phi}{dt} \simeq \sum_{\vec{k}_1 + \vec{k}_2 = \vec{k}} (\vec{k}_1 \times \vec{k}_2)^2 \text{Re} \Theta_{\vec{k}, \vec{k}_1, \vec{k}_2} \frac{(k_1^2 - k_2^2)(k_2^2 - k^2)}{k_2^2 k^2} \langle \tilde{\phi}^2 \rangle_{\vec{k}_1} \quad (39b)$$

It can be easily shown that

$$\frac{dS_{\vec{k}}^n}{dt} > \frac{dS_{\vec{k}}^\phi}{dt} \quad (40)$$

which implies that the density fluctuation transfer to small scales is faster than the vorticity fluctuation transfer, and thus is dissipated faster. This is because the local interactions (with comparable wavenumbers  $k_1 \approx k$  or  $k_2 \approx k$ ) which appear in the nonlinear density transfer process are excluded from the nonlinear vorticity transfer.

These observations lead us to speculate that the turbulent relaxation in two-field weakly correlated drift wave turbulence is a two stage selective decay process and has two major time scales. On the fast time scale, the internal energy  $E^n$  decays while the fluid kinetic energy  $E^\phi$ , the fluid enstrophy  $\Omega^\phi$ , and the cross-correlation  $\Gamma$  do not decay. The reason why the cross-correlation does not decay on the fast time scale can be understood in the following way. Let's assume that the density fluctuation is dissipated by viscous diffusion  $\nu \nabla^2 n$  at small scales. The conservation laws for internal energy and cross-correlation described in Sec.II can be rewritten as (for  $v_e^* = \chi_e = 0$ )

$$\frac{\partial E^n}{\partial t} = -\nu \int d^2 \vec{x} (\vec{\nabla} \tilde{n})^2 \quad (41a)$$

$$\frac{\partial \Gamma}{\partial t} = -(\nu + \mu) \int d^2 \vec{x} \tilde{n} \nabla^4 \tilde{\phi} \quad (41b)$$

Now, we assume that both density fluctuation and vorticity fluctuation are initially (time  $t_0$ ) injected at wavenumber  $k_0$ . We define the average wavenumber of internal energy distribution (in  $k$ )  $k_n$  by,

$$k_n^2(t) \equiv \frac{\int d^2 \vec{x} (\vec{\nabla} \tilde{n})^2}{\int d^2 \vec{x} \tilde{n}^2} = \frac{\sum_{\vec{k}} k^2 \langle \tilde{n}^2 \rangle_{\vec{k}}}{\sum_{\vec{k}} \langle \tilde{n}^2 \rangle_{\vec{k}}} \quad (42)$$

so that  $k_n(0) = k_0$ . The dissipation term on the right side of equation (41a) can be expressed in terms of  $k_n$ , namely,  $-\nu k_n^2 \int d^2 \vec{x} \tilde{n}^2 = -\nu k_n^2 E^n$ , thus the dissipation rate for the internal energy is given by  $\gamma_n = \nu k_n^2$ . Since the density fluctuation is transferred to small scales faster than the vorticity fluctuation is, at a later time  $t > t_0$ , we expect  $k_n(t) > k_0$ , while the vorticity fluctuation still remains at wavenumber  $k_0$ . The dissipation term on the right side of Eq.(41b) is approximately,  $(\nu + \mu) k_0^2 \int d^2 \vec{x} \tilde{n} \nabla^2 \tilde{\phi} = -(\nu + \mu) k_0^2 \Gamma$ , and the dissipation rate for the cross-correlation is  $\gamma_\Gamma = (\nu + \mu) k_0^2$ . Therefore, we have  $\gamma_n > \gamma_\Gamma$  provided  $\nu \sim \mu$ , i.e. the internal energy decays faster than the cross-correlation does, and the underpinning physics is that the density fluctuation flows to small scales faster than the vorticity fluctuation does.

The final state of the selective decay on this fast time scale is then described by the following variational equation,

$$\delta[\int d^2\vec{x}(\tilde{n})^2 - 2\alpha_1 \int d^2\vec{x}\tilde{n}\nabla_{\perp}^2\tilde{\phi}] = 0 \quad (43)$$

which has the solution,

$$\tilde{n}(\vec{x}) = \alpha_1 \nabla_{\perp}^2 \tilde{\phi}(\vec{x}) \quad (44)$$

i.e. the density fluctuation tends to be aligned with the vorticity fluctuation. It is interesting to note that Eq.(44) implies that the density fluctuation tend to accumulate or be trapped in regions with large vorticity concentrations (i.e. coherent structures).  $\alpha_1$  is the Lagrange multiplier, and is determined by the initial value of vorticity  $\Omega_0$  and cross-correlation  $\Gamma_0$ ,  $\alpha_1 = \Gamma_0/\Omega_0$ . The internal energy and the cross-correlation are all proportional to the fluid vorticity as a result of this fast relaxation, i.e.  $E^n = \alpha_1^2 \Omega^{\phi}$  and  $\Gamma = -\alpha_1 \Omega^{\phi}$ .

On the slow time scale, the fluid enstrophy decays while the fluid kinetic energy does not decay due to the existence of an inverse cascade. The final state is described by:

$$\delta[\int d^2\vec{x}(\nabla_{\perp}^2\tilde{\phi})^2 + \alpha_2 \int d^3\vec{x}(\nabla_{\perp}\tilde{\phi})^2] = 0 \quad (45)$$

the solution of which is

$$\nabla^2\tilde{\phi} = \alpha_2\tilde{\phi} \quad (46)$$

where the Lagrange multiplier  $\alpha_2$  is determined by the initial value of the fluid kinetic energy  $E^{\phi}$  and boundary conditions of the structure, i.e.  $E_0^{\phi} = \alpha_2 \int \tilde{\phi}^2 d^2\vec{x}$  and  $\alpha_2 > 0$ . The solution of equation (46) corresponds to the simplest model of coherent vorticies. In cylindrical coordinate system, the solution of equation (46) is,

$$\tilde{\phi}(r, \theta) = J_m\left(\frac{r}{\sqrt{\alpha_2}}\right) \cos(m\theta) \quad (47)$$

where  $m = 0$  corresponds to monopole structure, and  $m = 1$  corresponds to dipole structure, etc.

Eqs.(4.61), (4.59) satisfy the relations

$$\begin{aligned} \vec{\nabla}\tilde{\phi} \times \vec{z} \cdot \vec{\nabla}\nabla^2\tilde{\phi} &= 0 \\ \nabla\tilde{\phi} \times \vec{z} \cdot \vec{\nabla}n &= 0 \end{aligned}$$

therefore correspond to the stationary solution of the model equations. It is interesting to note that in the hydrodynamic regime, the density tends to be ‘aligned’ with the vorticity in

contrast to the adiabatic regime where the density is completely ‘aligned’ with the potential  $\tilde{n} = \tilde{\phi}$ . In the intermediate regime, the density is expected to have the characteristic of both potential and vorticity.

## VII. Summaries and Conclusions

In this chapter, a statistical theory of two-field model of drift wave turbulence is presented. The principal results are summarized as follows:

- i.) By using the *EDQNM* closure scheme, we have systematically derived a closed set of spectrum evolution equations for drift wave turbulence. These equations describe the nonlinear evolution of internal energy spectrum  $\langle \tilde{n}^2 \rangle_{\vec{k}}$ , the kinetic energy spectrum  $k^2 \langle \tilde{\phi}^2 \rangle_{\vec{k}}$ , and the cross-correlation spectrum  $k^2 \langle \tilde{n} \tilde{\phi} \rangle_{\vec{k}}$ . Unlike the previous studies, the cross-correlation dynamics is treated on an equal footing with the energetics. Similarities between the cross-correlation  $\langle \tilde{n} \nabla^2 \tilde{\phi} \rangle$  in drift wave turbulence and the cross-helicity  $\langle \vec{v} \cdot \vec{B} \rangle$  in *MHD* turbulence have been found. In particular, the cross-correlation dynamically inhibits the nonlinear transfer of density fluctuations to small scales.
- ii.) based on the statistical closure equations, a *H*-theorem which underpins the turbulent relaxation process in a two-field drift wave turbulence is proved. The appropriate entropy functionals for drift wave turbulence are identified. These entropy functionals have the following properties: (a) due to the nonlinear mode couplings, these entropy functionals increase with time monotonically until an absolute equilibrium state is reached. (b) The maximization of the entropy functionals subject to the constraint of the invariants of the motion leads to the absolute equilibrium spectra, which are exact the same as those obtained from the Gibbs ensemble theory. One of the implications from the *H*-theorem is that in the hydrodynamic regime, the density (internal energy) transfer to small scales is faster than the vorticity (enstrophy) transfer is.
- iii.) constrained turbulent relaxation process is discussed in the hydrodynamic regime. As a result of the unequal transfer rates between the density and the vorticity fluctuations, the turbulent relaxation (selective decay) has two major time scales. On the fast time scale, the internal energy decays while the cross-correlation, the fluid enstrophy, and the fluid kinetic energy do not decay. On the slow time scale, the fluid enstrophy decays while the fluid kinetic energy does not decay. As a consequence, the turbulent relaxation has the characteristic of both “dynamic alignment”  $\tilde{n} = \alpha_1 \nabla_{\perp}^2 \tilde{\phi}$  and coherent vortices  $\nabla_{\perp}^2 \tilde{\phi} = \alpha_2 \tilde{\phi}$ . This implies that in two field drift wave turbulence with nonzero cross-correlation, the density fluctuations tend to remain inside regions with large vorticity concentration (coherent vortices).

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## Appendix B: Solution of Driven Equations.

In this appendix, we solve equations (17a) and (17b) to obtain  $\delta\tilde{\phi}_{\vec{k}}$  and  $\delta\tilde{n}_{\vec{k}}$ . First we introduce the following column vectors:

$$\delta\Psi_{\vec{k}} = \begin{pmatrix} \delta\tilde{\phi}_{\vec{k}} \\ \delta\tilde{n}_{\vec{k}} \end{pmatrix} \quad S = \begin{pmatrix} S_{1\vec{k}}/k^2 \\ S_{2\vec{k}} \end{pmatrix}$$

and the square matrix:

$$H = \begin{pmatrix} \chi_e k_{\parallel}^2/k^2 & -\chi_e k_{\parallel}^2/k^2 \\ (i\omega_e^* - \chi_e k_{\parallel}^2) & \chi_e k_{\parallel}^2 \end{pmatrix}$$

then equations (4.32a) and (4.32b) can be casted into a simple matrix form:

$$\left(\frac{\partial}{\partial t} + \Delta\omega_{\vec{k}}\right)\delta\Psi + H\delta\Psi = S \quad (A1)$$

We then make a linear transformation  $\delta\Psi = A\delta\hat{\Psi}$  where the new dynamical variables are represented by  $\delta\hat{\Psi}$ , and A is the transformation matrix. The above equation then becomes:

$$A\left(\frac{\partial}{\partial t} + \Delta\omega_{\vec{k}}\right)\delta\hat{\Psi} + HA\delta\hat{\Psi} = S \quad (A2)$$

or

$$\left(\frac{\partial}{\partial t} + \Delta\omega_{\vec{k}}\right)\delta\hat{\Psi} + A^{-1}HA\delta\hat{\Psi} = A^{-1}S \quad (A3)$$

where  $A^{-1}$  is the inverse of A. We choose A such that  $A^{-1}HA$  is a diagonal matrix. Such A can be constructed from the eigenvectors of H.

The eigenvalue  $\lambda$  of H are determined from equation:  $\det(H - \lambda I) = 0$ , where  $I$  is the unit matrix. This equation reduces to

$$\lambda^2 - \chi_e k_{\parallel}^2\left(1 + \frac{1}{k^2}\right)\lambda + i\omega_e^* \frac{\chi_e k_{\parallel}^2}{k^2} = 0 \quad (A4)$$

with two solutions  $\lambda_1$  and  $\lambda_2$  given by:

$$\lambda_{1,2} = \frac{1}{2}\chi_e k_{\parallel}^2\left(1 + \frac{1}{k^2}\right) \pm \frac{1}{2}\left[(\chi_e k_{\parallel}^2\left(1 + \frac{1}{k^2}\right))^2 - 4i\omega_e^* \frac{\chi_e k_{\parallel}^2}{k^2}\right]^{\frac{1}{2}} \quad (A5)$$



The eigenvectors belonging to the eigenvalues  $\lambda_1$  and  $\lambda_2$  are respectively

$$\begin{pmatrix} 1 \\ 1 - \frac{\lambda_1 k^2}{\chi_e k_{\parallel}^2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 - \frac{\lambda_2 k^2}{\chi_e k_{\parallel}^2} \end{pmatrix}$$

so the desired transformation matrix  $A$  and its inverse  $A^{-1}$  are,

$$A = \begin{pmatrix} 1 & 1 \\ 1 - \frac{\lambda_1 k^2}{\chi_e k_{\parallel}^2} & 1 - \frac{\lambda_2 k^2}{\chi_e k_{\parallel}^2} \end{pmatrix}$$

and

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1 - \frac{\lambda_2 k^2}{\chi_e k_{\parallel}^2} & -1 \\ -1 + \frac{\lambda_1 k^2}{\chi_e k_{\parallel}^2} & 1 \end{pmatrix}$$

where the determinant of matrix  $A$  is

$$\det A = \frac{(\lambda_1 - \lambda_2)k^2}{\chi_e k_{\parallel}^2}$$

With a straightforward manipulation, we obtain

$$A^{-1} H A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The equation (A3) then becomes:

$$\left( \frac{\partial}{\partial t} + \Delta\omega_{\vec{k}} + \lambda_1 \right) \delta\hat{\Psi}_1 = (A^{-1}S)_1 \quad (\text{A6})$$

$$\left( \frac{\partial}{\partial t} + \Delta\omega_{\vec{k}} + \lambda_2 \right) \delta\hat{\Psi}_2 = (A^{-1}S)_2 \quad (\text{A7})$$

The solutions of these equations are:

$$\delta\hat{\Psi}_1(t) = \int_{-\infty}^t dt' \exp\{-(\Delta\omega_{\vec{k}} + \lambda_1)(t - t')\} (A^{-1}S)_1(t') \quad (\text{A8})$$

$$\delta\hat{\Psi}_2(t) = \int_{-\infty}^t dt' \exp\{-(\Delta\omega_{\vec{k}} + \lambda_2)(t - t')\} (A^{-1}S)_2(t') \quad (\text{A9})$$

where

$$(A^{-1}S)_1 = -\frac{1}{\det A} \left[ \left(1 - \frac{\lambda_1}{\chi_e k_{\parallel}^2}\right) S_{1\vec{k}} + S_{2\vec{k}} \right]$$

$$(A^{-1}S)_2 = \frac{1}{\det A} \left[ \left(1 - \frac{\lambda_2}{\chi_e k_{\parallel}^2}\right) S_{1\vec{k}} + S_{2\vec{k}} \right]$$

the  $\lambda_1$  and  $\lambda_2$  are related to the frequency of linear modes but with a turbulent enhanced parallel diffusivity:

$$\lambda_1 = i\omega_{1\vec{k}} \quad \text{and} \quad \lambda_2 = i\omega_{2\vec{k}}$$

therefore the perturbations in the original dynamical variables are given by:

$$\begin{aligned}
\delta\tilde{\phi}_{\vec{k}}(t) &= \delta\hat{\Psi}_1(t) + \delta\hat{\Psi}_2(t) \\
&= \sum_{l=1}^2 \frac{(-1)^l}{\det A} \int_{-\infty}^t dt' \exp\{-(\Delta\omega_{\vec{k}} + i\omega_{l\vec{k}})(t-t')\} \\
&\quad \times [(1 - \frac{i\omega_{l\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]
\end{aligned} \tag{A10}$$

$$\begin{aligned}
\delta n_{\vec{k}}(t) &= (1 - \frac{i\omega_{1\vec{k}} k^2}{\chi_e k_{\parallel}^2})\delta\hat{\Psi}_1(t) + (1 - \frac{i\omega_{2\vec{k}} k^2}{\chi_e k_{\parallel}^2})\delta\hat{\Psi}_2(t) \\
&= \sum_{l=1}^2 \frac{(-1)^l}{\det A} (1 - \frac{i\omega_{l\vec{k}} k^2}{\chi_e k_{\parallel}^2}) \int_{-\infty}^t dt' \exp\{-(\Delta\omega_{\vec{k}} + i\omega_{l\vec{k}})(t-t')\} \\
&\quad \times [(1 - \frac{i\omega_{l\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]
\end{aligned} \tag{A11}$$

As being pointed out in the linear theory, out of two linear modes, one of them is growing while another is strongly damped. Therefore, without loss of generality, we only keep the growing mode, which is represented by  $\omega_{2\vec{k}}$  in our notation, in the above expression as the driven fields. Dropping the subscript 2, we finally get

$$\begin{aligned}
\delta\tilde{\phi}_{\vec{k}}(t) &= \frac{1}{\det A} \int_{-\infty}^t dt' \exp\{-(\Delta\omega_{\vec{k}} + i\omega_{\vec{k}})(t-t')\} \\
&\quad \times [(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]
\end{aligned} \tag{A12}$$

$$\begin{aligned}
\delta\tilde{n}_{\vec{k}}(t) &= \frac{1 - \frac{i\omega_{\vec{k}} k^2}{\chi_e k_{\parallel}^2}}{\det A} \int_{-\infty}^t dt' \exp\{-(\Delta\omega_{\vec{k}} + i\omega_{\vec{k}})(t-t')\} \\
&\quad \times [(1 - \frac{i\omega_{\vec{k}}}{\chi_e k_{\parallel}^2})S_{1\vec{k}}(t') + S_{2\vec{k}}(t')]
\end{aligned} \tag{A13}$$

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