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On the Origin of Cosmological Magnetic Fields

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ON THE ORIGIN OF COSMOLOGICAL MAGNETIC FIELDS

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ABSTRACT

It is shown that a plasma with temperature T sustains fluctuations of electromagnetic fields and particle density even if it is assumed to be in a thermal equilibrium. The level of fluctuations in the plasma for a given wavelength and frequency of electromagnetic fields is rigorously computed by the fluctuation-dissipation theorem. A large zero frequency peak of electromagnetic fluctuations is discovered. We show that the energy contained in this peak is complementary to the energy "lost" by the plasma cutoff effect. The level of the zero (or nearly zero) frequency magnetic fields is computed as $\langle B^2\rangle^0/8\pi = \dfrac{1}{2\pi^3}T(\omega_p/c)^3$, where T and ω_p are the temperature and plasma frequency. This is the theoretical minimum magnetic field strength spontaneously generated, as no turbulence is assumed. The size of the fluctuations is $\lambda \sim (c/\omega_p)(\eta/\omega)^{1/2}$, where η and ω are the collision frequency and the (nearly zero) frequency of magnetic fields oscillations. These results are not in contradiction with the conventional black-body radiation spectra but its extension, and as such, do not contradict the observed lack of structure in the cosmic microwave background. Our computer particle simulation shows the support of the theory and in fact exhibits a peaking of the magnetic energy spectrum at zero frequency. The level of magnetic fields is significant at the early radiation epoch of the Universe. Implications of these magnetic fields in the early Universe ($t = 10^{-2} - 10^{13}$ sec) are discussed.

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I. Introduction

Around the time of 1 second after the big bang, the standard cosmological theory says (Weinberg, 1972) that the weak (neutrino) interaction detached from reacting with the rest of the radiation and matter which are in thermal equilibrium. Around the time of 10 seconds ($z \sim 10^9$) to 10^3 seconds ($z \sim 10^8$) the strong (nuclear) interaction ceased to play a role in the evolution of the Universe. According to the standard theory "The Universe will go on expanding and cooling, but not much of interest will occur for 700,000 years ($\sim 10^{13}$ sec; $z \sim 10^3$). At that time the temperature will drop to the point where electrons and nuclei can form stable atoms" (recombination) (Weinberg, 1977). During this epoch from $z = 10^{10}$ to 10^3 the radiation couples strongly with matter and thus has been called the radiation epoch. The main constituent of matter of this period is a plasma and the main interaction of this period is that of plasma dynamics, including that of radiation-plasma coupling. Thus this epoch of $10^0 - 10^{13}$ seconds ($z = 10^{10} - 10^3$) of the evolution of the Universe may be called the plasma epoch, the time during which the electromagnetic interaction among the four fundamental forces (weak, strong, EM, and gravitational) dominated. We first assume that at the dawn of the radiation (plasma) epoch ($t \sim 10^0$ sec; $z \sim 10^{10}$) photons and charged particles were in thermal equilibrium. In a wider sense of the word this epoch may include also the period of $10^{-2} - 10^0$ seconds ($z = 10^{11} - 10^{10}$) in which the Universe was dominated by photons, electrons, and positrons, accompanied by nuclei and neutrinos. We may call this the first period of the plasma epoch, and the period $t = 10^0 - 10^{13}$ sec ($z = 10^{10} - 10^3$) the second period of the plasma epoch. Although once again it is not traditional to consider this epoch as characterized by the dominance of plasma interaction, the most important interaction in this period is the coupling between photons and leptons, collective or individual. However, we shall show that in each of these epochs the presence of plasma plays an important role in shaping the radiation spectrum.

For the primordial Universe to be treated as a (gaseous) plasma, the collection of charged particles have to satisfy a certain condition. In the epoch of $t = 10^{-2} - 10^{0}$ sec the typical plasma density is $10^{28} - 10^{34}\,\mathrm{cm}^{-3}$. The plasma parameter $g = 1/(n\,\lambda_{De}^{3})$ is much less than unity (for example, Ichimaru 1973) and is approximately 10^{-3} in the epoch of $10^{0} - 10^{13}$ sec it is about 10^{-7}, where n is the density of electrons, λ_{De} is the electron Debye length. The Debye length is equal to c/ω_{pe} in the relativistic plasma in the first epoch ($z = 10^{11} - 10^{10}$), where c is the speed of light and ω_{pe} is the plasma frequency and the quantity c/ω_{pe} is usually referred to the collisionless skin depth. In both of the above epochs the mean distance between particles is much smaller than the typical collective length (the Debye length = the collisionless skin depth), which in turn is much smaller than the mean free path of electrons colliding with photons

$$\frac{1}{n^{1/3}} < \frac{c}{\omega_{pe}} < (n\,\sigma_{\mathrm{KN}})^{-1}\,, \tag{1}$$

where σ_{KN} is the Klein-Nishina cross-section of electron-photon collisions. When $T \gg \mathrm{mc}^2$, the cross-section should be, instead of the Thompson cross-section, the Klein-Nishina formula:

$$\sigma_{\mathrm{KN}} = \frac{3}{8}\left(\frac{\mathrm{mc}^2}{\hbar\omega}\right)\sigma_{\mathrm{T}} \qquad (\text{for } \hbar\omega \gg \mathrm{mc}^2)\,, \tag{2}$$

while

$$\sigma_{\mathrm{KN}} = \sigma_{\mathrm{T}} \qquad (\text{for } \hbar\omega \ll \mathrm{mc}^2)\,, \tag{3}$$

where the Thompson cross-section $\sigma_{\mathrm{T}} = \frac{8\pi}{3}\,(e^2/\mathrm{mc}^2)^2$.

In this plasma of $t = 10^{-2} - 10^{0}$ sec the (average) photon energy is $\hbar\omega \sim T \gg \mathrm{mc}^2$ and we have

$$T \sim \hbar\omega \gg \mathrm{mc}^2 > \hbar\omega_p\,. \tag{4}$$

In the description of fluid behavior the Reynolds number sometimes plays an important role. For wavelengths much larger than $(n\sigma_{\mathrm{KN}})^{-1}$, the plasma behaves like a usual fluid and the Reynolds number may be expressed as $\mathcal{R}_e = \lambda^2\eta/\mu$, which can be much larger than unity, where η is the collision frequency or the effective collision frequency replaced by the Landau damping rate or other collisionless mechanisms such as the chaotic orbit effect. On the other hand, for wavelengths $\lambda \sim c/\omega_{pe} \ll (n\sigma_{\mathrm{KN}})^{-1}$, the plasma is collisionless and nearly dissipationless. The list of plasma parameters in the early plasma epoch ($z = 10^{11} - 10^3$) are summarized in Table I, in which our conclusions are also listed that are to be obtained in the following discussion.

We note that the past investigations of cosmological magnetic fields such as Harrison (1970, 1973) assumed primordial turbulence with nonzero vorticity and obtained magnetic fields of $\sim 10^{-18}$Gauss for galactic scales. More recently this idea has lost favor with most cosmologists, primarily because vortices decay during the cosmic expansion (Rees, 1987). Kajantie *et al.* (1986) discussed phase transition incurred fluctuations. In contrast to these works we resort to no assumption as for the primordial condition but for the thermal equilibrium in the following. Our treatment is based on the rigorous theory of fluctuation-dissipation theorem. The calculation of magnetic fields is undertaken in Sec. II. Section III covers the computational work, which support the theoretical finding. Some interesting physical properties are discussed in Sec. IV. In Sec. V we discuss cosmological implications of the present discovery.

II. The Fluctuation-Dissipation Theorem and Magnetic Fields

In or near thermal equilibrium the plasma has thermal fluctuations, whose level is related to the medium's dissipative characteristics and the temperature T, as formulated in the fluctuation-dissipation theorem (Kubo, 1957). We find an expression for the fluctuation spectrum of the magnetic field in an equilibrium plasma as a function of frequency. This is accomplished by deriving the magnetic fluctuations in wavenumber and frequency space $\langle B^2 \rangle_{\mathbf{k}\omega}/8\pi$

from the fluctuation-dissipation theory, then integrating over wavenumber. $\langle B^2 \rangle_\omega / 8\pi$ is nearly a black-body spectrum at high frequencies, but, when plasma collisionality is taken into account, it has a high, narrow peak at frequency $\omega = 0$.

The following derivation closely parallels the work of Geary *et al.* (1986). We look at waves in a homogeneous isotropic equilibrium plasma. However, since we are interested in spontaneous generation of magnetic fields, we consider a nonmagnetized plasma here in contrast to Geary *et al.* (1986). Furthermore, as we detail later, the work by Geary *et al.* (1986) neglected radiation, while we retain this in the following. To start with, we assume a wavevector $\mathbf{k} = k\hat{x}$. The strength of electric field fluctuations may be found in Sitenko (1967):

$$\frac{1}{8\pi} \langle E_i E_j \rangle_{\mathbf{k}\omega} = \frac{i}{2} \frac{\hbar}{e^{\hbar\omega/T} - 1} \left\{ \Lambda_{ij}^{-1} - \Lambda_{ij}^{-1*} \right\} , \tag{5}$$

where

$$\Lambda_{ij}(\omega, \mathbf{k}) = \frac{c^2 k^2}{\omega^2} \left(\frac{k_i k_j}{k^2} - \delta_{ij} \right) + \epsilon_{ij}(\omega, \mathbf{k}) , \tag{6}$$

where $\epsilon_{ij}(\omega, \mathbf{k})$ being the dielectric tensor of the plasma. Since Faraday's law is $\mathbf{B} = \frac{c\mathbf{k}}{\omega} \times \mathbf{E}$, and we have set $\mathbf{k} = k\hat{x}$, we find

$$\frac{\langle B_2^2 \rangle_{\mathbf{k}\omega}}{8\pi} = \frac{i}{2} \frac{\hbar}{e^{\hbar\omega/T} - 1} \frac{c^2 k^2}{\omega^2} \left\{ \Lambda_{33}^{-1} - \Lambda_{33}^{-1*} \right\} , \tag{7}$$

and

$$\frac{\langle B_3^2 \rangle_{\mathbf{k}\omega}}{8\pi} = \frac{i}{2} \frac{\hbar}{e^{\hbar\omega/T} - 1} \frac{c^2 k^2}{\omega^2} \left\{ \Lambda_{22}^{-1} - \Lambda_{22}^{-1*} \right\} , \tag{8}$$

where the subscript 1,2, and 3 refer to x, y, and z. We then have the total magnetic fluctuations as

$$\frac{\langle B_{\text{tot}}^2 \rangle_{\mathbf{k}\omega}}{8\pi} = \frac{i}{2} \frac{\hbar}{e^{\hbar\omega/T} - 1} \frac{c^2 k^2}{\omega^2} \left\{ \Lambda_{22}^{-1} + \Lambda_{33}^{-1} - \Lambda_{22}^{-1*} - \Lambda_{33}^{-1*} \right\} , \tag{9}$$

where c.c. refers to the complex conjugate.

In order to establish $\Lambda_{ij}^{(\omega,\mathbf{k})}$, from the equation of motion of a plasma, here we introduce a multi-fluid model of a plasma. With finite and constant collisionality:

$$m_\alpha \frac{d\mathbf{v}_\alpha}{dt} = e_\alpha \mathbf{E} - \eta_\alpha m_\alpha \mathbf{v}_\alpha \ , \tag{10}$$

where α is a particle species label and η_α is the collisional frequency but can include the viscosity effect. A more accurate description of electron dynamics such as kinetic treatments than Eq. (10) leads to better mathematical properties. However, for the sake of analytical tractability and physical transparency we take this simplified constant collision frequency model. We note that η should tend to zero for very short wavelength EM waves. Fourier transforming (10) gives

$$-i\omega m_\alpha \mathbf{v}_\alpha = e_\alpha \mathbf{E} - \eta_\alpha m_\alpha \mathbf{v}_\alpha \ , \tag{11}$$

which yields the current $\mathbf{j}_\alpha$

$$(-i\omega + \eta_\alpha)\,\mathbf{j}_\alpha = \frac{\omega_{p\alpha}^2}{4\pi}\mathbf{E} \ . \tag{12}$$

The susceptibility tensor $\chi_{\alpha ij}$ is defined to relate $\mathbf{j}_\alpha$ to $\mathbf{E}$ such that

$$j_{\alpha i} = -i\omega\, \chi_{\alpha ij}(\omega\mathbf{k}) E_j(\omega\mathbf{k}) \ . \tag{13}$$

The dielectric tensor $\epsilon_{ij}(\omega\mathbf{k})$ is given by

$$\epsilon_{ij}(\omega\mathbf{k}) = \delta_{ij} + 4\pi \sum_\alpha \chi_{\alpha ij} \ , \tag{14}$$

so

$$4\pi\, \chi_{\alpha ij}^{(\omega,\mathbf{k})} = \frac{\omega_{p\alpha}^2}{\omega(\omega + i\eta_\alpha)}\delta_{ij} \ , \tag{15}$$

and

$$\epsilon_{ij}(\omega,\mathbf{k}) = \delta_{ij} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega(\omega + i\eta_\alpha)}\delta_{ij} \ . \tag{16}$$

In an electron-positron plasma neglecting ions, we have $\omega_{pe^+} = \omega_{pe^-}$ and $\eta_{e^+} = \eta_e = \eta$. So Eq. (16) becomes

$$\epsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} - \frac{\omega_p^2}{\omega(\omega + i\eta)}\,\delta_{ij}\ , \tag{17}$$

where $\omega_p^2 = \omega_{pe^+}^2 + \omega_{pe^-}^2$. We now obtain

$$\Lambda_{ij} = \begin{pmatrix} 1 - \dfrac{\omega_p^2}{\omega(\omega+i\eta)} & & \\ & 1 - \dfrac{c^2\,k^2}{\omega^2} - \dfrac{\omega_p^2}{\omega(\omega+i\eta)} & \\ & & 1 - \dfrac{c^2\,k^2}{\omega^2} - \dfrac{\omega^2}{\omega(\omega+i\eta)} \end{pmatrix} . \tag{18}$$

Combining Eqs. (9) and (18) after some algebra, we obtain

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi} = \frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\eta\omega_p^2\frac{k^2c^2}{\omega^2}\frac{1}{\left[\omega^2 - k^2c^2 - \omega_p^2\right]^2 + \eta^2\left[\omega - k^2c^2/\omega\right]^2}\ , \tag{19}$$

or

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi} = \frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\eta\omega_p^2\frac{k^2c^2}{(\omega^2+\eta^2)k^4c^4 + 2\omega^2(\omega_p^2 - \omega^2 - \eta^2)k^2c^2 + \left[(\omega^2-\omega_p^2)^2 + \eta^2\omega^2\right]\omega^2}\ . \tag{20}$$

The first form, with a pole being clearly offset from the electromagnetic plasma wave pole, might be more physically apprehensible, whereas the second form will make integration an easier task. Note that if relativistic effects are included, the only change is $\omega_p \to \omega_p/\sqrt{\gamma}$; this, in fact, has been done in the case of the early electron-positron plasma where $\gamma \sim 2.7$.

We stay with the electron-positron plasma for the time being. Our first goal is to find the wavenumber integration of $\langle B^2\rangle_{\mathbf{k}\omega}$ to get $\langle B^2\rangle_\omega$. Integrating Eq. (20) over wavenumber, we obtain

$$\frac{\langle B^2\rangle_\omega}{8\pi} = \frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\,\frac{2\eta}{2\pi^2\omega_{pe}^2}\left(\frac{\omega_{pe}}{c}\right)^3\int_0^\infty dx\frac{x^4}{(\omega'^2+\eta'^2)x^4+\cdots}\ , \tag{21}$$

where $x = k\,\frac{c}{\omega_{pe}}$ and the primed quantities are normalized by ω_{pe}. Note that if we let ω tend to zero with $\eta' = \eta/\omega_{pe}$ fixed, Eq. (21) diverges:

$$\frac{\langle B^2\rangle_\omega}{8\pi} = \frac{2}{\pi^2}\frac{T}{\omega_{pe}}\frac{1}{\eta'}\left(\frac{\omega_{pe}}{c}\right)^3 \int_0^\infty dx \,. \tag{22}$$

Before we arrive at a result, we have to solve the problem of divergence in these integrals. The fact that we get high wavenumber divergence is not surprising, since we have, up to this point, based our calculations on classical fluid equations of motion with a constant collision frequency η independent of k. Where the fluid picture breaks down, we need a new theory. This could be done by a kinetic theory which includes more exact collision effects, wave-particle interaction, etc. However, by doing so, we lose analytical tractability and thus the physical transparency of the present approach. So, for the moment we are content with this semi-phenomenological theory of collision effects. In order to overcome the large k divergence, we let η tend to zero first then we integrate over k to infinity. This will bring the high frequency expression. This procedure physically corresponds to the vanishing cross-section of collisions as $k \to \infty$. (To show this involves quantum kinetics, for which we have no space in the present paper.) We still need the correct plasma expression in the low frequency expression.

We start from our expression for $\langle B^2\rangle_{\mathbf{k}\omega}/8\pi$, Eq. (19). Note that at high frequency and high wavenumber ($\omega, ck \gg \omega_p$), this function has a substantial value only where $\omega^2 - c^2\,k^2 - \omega_p^2 \approx 0$. As we noted, a high-frequency, high wavenumber limit is obtained by letting $\eta \to 0$. We take this limit with the aid of a standard definition of the Dirac δ-function:

$$\delta(x) = \frac{1}{\pi}\lim_{\gamma\to 0}\frac{\gamma}{x^2+\gamma^2} \,. \tag{23}$$

Thus we obtain

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi} = \frac{2\hbar\omega}{e^{\hbar\omega/T}-1}\,\omega_p^2\,k^2\,c^2\,\pi\delta\left(\frac{\omega(\omega^2-c^2\,k^2-\omega_p^2)}{\omega^2-c^2\,k^2}\right)\frac{1}{(\omega^2-c^2\,k^2)^2} \,. \tag{24}$$

Integrating Eq. (24) over $4\pi\,k^2\,dk$ from 0 to ∞, we obtain

$$\frac{\langle B^2\rangle_\omega}{8\pi} = \frac{T}{2\pi}\,\delta(\omega)\int\frac{\omega_p^2}{\omega_p^2+c^2\,k^2}\,k^2\,dk + \frac{1}{2\pi\,c^3}\frac{\hbar}{e^{\hbar\omega/T}-1}(\omega^2-\omega_p^2)^{3/2} \,. \tag{25}$$

This expression (25) reduces to the familiar black-body radiation formula in the limit $\omega_p \to 0$. As the magnetic energy density is $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\langle B^2 \rangle}{8\pi}$ and half the energy in a black-body is stored in the electric field, we see that this is the standard black-body spectrum. The only difference of Eq. (25) from the conventional vacuum formulation is the presence of the plasma cutoff ω_p.

We now break up the integral of Eq. (21) into two intervals: one from zero to $k_{\rm cut}$ and the other from $k_{\rm cut}$ (or $x_{\rm cut} \equiv k_{\rm cut}\, c/\omega_{pe}$) to ∞. In the first integral we keep η finite since the range of this integral includes $\omega \sim \mathcal{O}(\eta)$, while in the second integral we let $\eta \to 0$:

$$\frac{\langle B^2 \rangle_\omega}{8\pi} = \frac{1}{\pi^2} \frac{\hbar\omega'}{e^{(\hbar\omega_{pe}/T)\omega'} - 1} 2\eta' \left(\frac{\omega_{pe}}{c}\right)^3 \int_0^{x_{\rm cut}} dx \, \frac{x^4}{(\omega'^2 + \eta'^2)x^4 + \cdots}$$
$$+ \frac{\hbar(\omega'^2 - \omega_p'^2)^{3/2}}{2\pi(e^{(\hbar\omega_{pe}/T)\omega'} - 1)} \left(\frac{\omega_{pe}}{c}\right)^3 \Theta\left(\omega - \sqrt{c^2 k_{\rm cut}^2 + \omega_p^2}\right) , \qquad (26)$$

where Θ is the Heaviside step function. The second term is what we have obtained in the above. Note that the divergence of $\omega \to 0$ is removed:

$$\lim_{\omega\to 0} \frac{\langle B^2 \rangle_\omega}{8\pi} = \frac{\hbar\omega'}{\pi^2(e^{\hbar\omega_{pe}\omega'/T} - 1)} 2 \left(\frac{\omega_{pe}}{c}\right)^3 \frac{1}{\eta'} \int_0^{x_{\rm cut}} dx = \frac{2T}{\pi^2\, \eta'\, \omega_{pe}} \left(\frac{\omega_{pe}}{c}\right)^3 x_{\rm cut} , \qquad (27)$$

where the second equality holds for the classical limit $T \gg \hbar\omega_{pe}$. Also note that at higher temperatures $\omega_p'^2 = 2$ is replaced by $2/\gamma$. We chose $k_{\rm cut}$ (or $x_{\rm cut}$) so as to make the frequency spectrum (26) be smooth at the joint between the low-frequency spectrum and the black-body spectrum. The size of $k_{\rm cut}$ thus becomes approximately $k_{\rm cut} \sim \omega_{pe}/c (x_{\rm cut} \sim 1)$. This determines the spectrum intensity at $\omega \sim 0$. This intensity does not vary sensitively with $k_{\rm cut}$. A value of $k_{\rm cut} = \sqrt{3/\pi}\, \omega_{pe}/c$ is derived below (see Eqs. (26) and (27) and following comments). The above integral can be evaluated analytically. We show three cases of plots of the magnetic fluctuation frequency spectrum of Eq. (26) for the early (~ 1 sec), middle ($\sim 10^8$ sec), and late (10^{12} sec) phase of the plasma epoch in Figs. 1, 2, and 3, respectively. Note that the rise of the zero frequency peak is so sharp that the curve and the graph margin are very difficult to distinguish (a break indicates where the zero frequency peak ends). The frames (b) in Figs. 1–3 show the log-log blowup of the low frequency behavior, indicating the ω^{-2} frequency spectrum near $\omega \sim 0$.

In an electron ion plasma the dielectric function is

$$\epsilon_{ij}(\omega, \mathrm{k}) = \delta_{ij} - \frac{\omega_{pe}^2}{\omega(\omega + i\eta_e)}\,\delta_{ij} - \frac{\omega_{pi}^2}{\omega(\omega + i\eta_i)}\,\delta_{ij}\ , \tag{28}$$

from which we find in the limit of $\omega \longrightarrow 0$:

$$\frac{\langle B^2\rangle_\omega}{8\pi} = 2T\int_0^{k_{\rm cut}} \frac{dk}{(2T_i)^3}\, 4\pi\, k^2\, \frac{k^2\, c^2}{k^4\, c^4} \left(\frac{\omega_{pe}^2}{\eta_e} + \frac{\omega_{pi}^2}{\eta_i}\right) = \frac{T}{\pi^2\, c^2} \left(\frac{\omega_{pe}^2}{\eta_e} + \frac{\omega_{pi}^2}{\eta_i}\right) k_{\rm cut}\ . \tag{29}$$

In an equilibrium hydrogen plasma, $\omega_{pe}^2 \sim 2000\, \omega_{pi}^2$. Also, $\eta_e = 2.91\times 10^{-6}\, n_e \ell n\, \Lambda\, T^{-3/2}\,\mathrm{sec}^{-1}$ and $\eta_i = 4.78 \times 10^{-18}\, n_e \Lambda\, T^{-3/2}\,\mathrm{sec}^{-1}$. So $\frac{\omega_{pe}^2/\eta_e}{\omega_{pi}^2/\eta_i} \approx 16.4$. Therefore, ion motion raises the $\omega = 0$ peak by about 6% of the value it would have if the ions were frozen.

We now discuss the wavenumber spectrum. In the limit $\eta \longrightarrow 0$ the integral of Eq. (20) over ω gives

$$\begin{aligned}\frac{\langle B^2\rangle_{\rm k}}{8\pi} &= \int_{-\infty}^{\infty} d\omega\, \frac{2\hbar\omega}{e^{\hbar\omega/T} - 1}\, \omega_p^2\, k^2\, c^2 \pi\, \delta\left(\frac{\omega(\omega^2 - c^2\, k^2 - \omega_p^2)}{\omega^2 - c^2\, k^2}\right) \frac{1}{(\omega^2 - c^2\, k^2)^2} \\ &= \int_{-\infty}^{\infty} d\omega\, \frac{\hbar\omega}{e^{\hbar\omega/T} - 1}\, \omega_p^2\, k^2\, c^2 \left[\delta(\omega) + \delta\left(\omega - \sqrt{c^2\, k^2 + \omega_p^2}\right) + \delta\left(\omega + \sqrt{c^2\, k^2 + \omega_p^2}\right)\right] \\ &\times \frac{1}{(\omega^2 - c^2\, k^2)(3\omega^2 - c^2\, k^2 - \omega_p^2) - 2\omega^2(\omega^2 - c^2\, k^2 - \omega_p^2)}\ .\end{aligned} \tag{30}$$

This leads to the expression

$$\frac{\langle B^2\rangle_{\rm k}}{8\pi} = \frac{\hbar\, k^2\, c^2}{\left(e^{\hbar/T\,(\omega_p^2 + k^2\, c^2)^{1/2}} - 1\right)} \frac{1}{(\omega_p^2 + c^2\, k^2)^{1/2}} + T\,\frac{\omega_p^2}{\omega_p^2 + c^2\, k^2}\ . \tag{31}$$

First, note that, once again, $\omega_p \longrightarrow 0$ gives the standard black-body result arising from the first term. (Remember: $\langle B^2\rangle/8\pi = \int_0^\infty \frac{dk}{(2\pi)^3}\, 4\pi\, k^2 \left\langle B^2\right\rangle_{\rm k}/8\pi$.) Second, the second term is that given for $\langle B^2\rangle_{\rm k}/8\pi$ by Geary *et al.* (1986). They obtain this expression via the Darwin approximations, that is to say, without radiation. Our result satisfies both radiative and non-radiative limits.

Note that when $\hbar(\omega_p^2 + c^2 k^2)^{1/2} \ll T$, two terms together yield

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} \to T \,, \tag{32}$$

giving the equipartition law of classical statistical mechanics. We can show that this limit is obtained also by using the Kramers-Kronig theorem.

It is worth noting here that the $\omega = 0$ peak shows up in our calculation of $\langle B^2 \rangle_{\mathbf{k}} / 8\pi$ by way of the $\delta(\omega)$ in the integral $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\langle B^2 \right\rangle_{\mathbf{k}\omega} / 8\pi$ as seen in Eq. (30). This $\delta(\omega)$ gives rise to the second term in Eq. (30) in the final result. The earlier note by Tajima and Shibata (1989) on the magnetic fluctuations $\langle B^2 \rangle^0 / 8\pi$ is to carry out the integral of this second term $\langle B^2 \rangle_k / 8\pi \sim T\omega_p^2/(\omega^2 + k^2 c^2)$ in Eq. (31) over the interval of 0 to ω_{pe}/c. This integral gives rise to the expression of the magnetic energy density near the zero frequency (i.e. non-radiative modes) as

$$\frac{\langle B^2 \rangle^0}{8\pi} = \int_0^{x_{\rm cut}\,\omega_p/c} \frac{dk\, 4\pi k^2}{(2\pi)^3} \frac{\langle B^2 \rangle_{\mathbf{k}}^0}{8\pi} \cong \frac{1}{6\pi^2} T \left(\frac{\omega_p}{c}\right)^3 x_{\rm cut}^3 \,, \tag{33}$$

where the approximate expression of Eq. (33) is obtained with $\omega_p^2/(\omega_p^2 + k^2 c^2) \approx 1$ in the integrand. On the other hand, the integration of the frequency spectrum $\langle B^2 \rangle_\omega / 8\pi$ for the zero frequency modes in Eq. (27) over frequency interval of 0 to η yields the expression of the magnetic energy density near the zero frequency as

$$\frac{\langle B^2 \rangle^0}{8\pi} = \int_0^{\eta} \frac{d\omega}{2\pi} \frac{\langle B^2 \rangle_\omega^0}{8\pi} = \frac{1}{2\pi^3} T \left(\frac{\omega_p}{c}\right)^3 x_{\rm cut} \,, \tag{34}$$

Since the expressions (33) and (34) should agree, we obtain $x_{\rm cut} = k_{\rm cut}\omega_p = (3/\pi)^{1/2} = 0.977$, close to unity. Considering the involved approximations in evaluating integrals in Eqs. (33) and (34), this agreement should be quite satisfactory. The agreement of Eqs. (33) and (34) may be termed as a version of Perseval's theorem. The ordinary version of the integrals of the spectral intensity in different spaces such as $\mathbf{k}\omega$ space and $\mathbf{x}t$ space give rise to the same answer. Here we have not only the entire electromagnetic energy satisfying such a theorem, but also a portion (i.e. the zero frequency fluctuations) of magnetic energy spectrum satisfying a similar relationship. In short, the photon energy is in the first term of Eq. (31) and the "zero-frequency" magnetic energy is in the second term. A schematic figure is shown in Fig. 4.

Another observation of interest may be made. The energy under the $\omega = 0$ peak is approximately equal to the energy "deficit" around the plasma cutoff $\omega = \omega_p$. The energy under the peak is evaluated from Eq. (27) as $\sim T(\omega_p/c)^3\,4/\pi^2$. The energy deficit, using the Rayleigh-Jeans formula, is approximately

$$\int_{-\omega_p}^{\omega_p} \frac{d\omega}{2\pi}\,\frac{T}{2\pi}\,\frac{\omega^2}{c^3} = \frac{8}{6\pi^2}\,T\left(\frac{\omega_p}{c}\right)^3 .$$

This is pictorially shown in Fig. 4. This result may be figuratively stated that the electromagnetic energy cutoff by the plasma $\omega < \omega_{pe}$ is squeezed toward the zero frequency fluctuations when compared with the black-body radiation without plasma.

We compare the size of the $\omega = 0$ peak of $\langle B^2\rangle_\omega/8\pi$ relative to the size of the black-body peak. The black-body spectrum has its maximum at $\omega \approx 2.81\,T/\hbar$. So, the ratio of the zero frequency peak fluctuations to the black-body radiation in an electron-positron plasma is

$$\frac{\langle B^2\rangle_{\omega=0}}{\langle B^2\rangle_{b-b}} = \alpha_1\,\hbar^2\,\frac{\omega_{pe}^2\,ck_{\rm cut}}{\eta T^2} = (2\pi)^3\alpha_1\delta\left(\frac{\lambda_B}{\lambda_{D_e}}\right)^3\frac{T}{\hbar\eta}\,, \tag{35}$$

where λ_B is the thermal deBroglie wavelength $2\pi\hbar/\sqrt{m\,T}$ and $\delta \sim 0.81$. For the electron-ion case α_1 in Eq. (35) becomes 0.47 for electron-ion plasma and 0.89 for $e^- - e^+$. Note, for example, that this ratio Eq. (46) can be as great as $0.1 - 1$ at $t = 10^{-2}$ sec (see Table I). This is an extremely impressive value.

We also compare $\langle B^2\rangle_{\omega=0}/8\pi$ to the pressure produced by $\omega = 0$ fluctuations in the ion density. Consider the longitudinal **E**-field fluctuations:

$$\frac{\langle E_x^2\rangle}{8\pi} = \frac{i}{2}\,\frac{\hbar}{e^{\hbar\omega/T}-1}\left(\Lambda_{11}^{-1} - \Lambda_{11}^{-1*}\right) . \tag{36}$$

(Once again, $\mathbf{k} = k\hat{x}$.) The definition of Λ_{ij} shows

$$\Lambda_{11} = \epsilon_{11}(\omega, \mathbf{k}) \ , \ \Lambda_{11}^{-1} = \frac{1}{\epsilon_{11}(\omega, \mathbf{k})} \ . \tag{37}$$

By taking

$$\epsilon_{11}(\omega, \mathbf{k}) = 1 + \frac{k_D^2}{k^2} - \frac{\omega_{pi}^2}{\omega(\omega + i\eta_i)} \ , \tag{38}$$

we find

$$\frac{\langle E_x^2 \rangle_{k\omega}}{8\pi} = \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} \frac{\eta \omega_{pi}^2}{\left[\omega^2(1 + k_D^2/k^2) - \omega_{pi}^2\right]^2 + \eta_i^2 \, \omega^2 (1 + k_D^2/k^2)^2} \ , \tag{39}$$

where k_D is the Debye wavenumber. Notice that while $\langle B^2 \rangle_\omega$ is inversely proportional to η at $\omega = 0$, $\langle E_x^2 \rangle_\omega$ is directly proportional to η at $\omega = 0$. Mathematically, this stems from the term $\frac{c^2 \, k^2}{\omega^2}$ which is present in Λ_{22} and Λ_{33} (transverse fluctuations) but absent from Λ_{11} (longitudinal).

Using the Poisson equation, we obtain the charge density fluctuations

$$\left\langle \delta\rho^2 \right\rangle_{\mathbf{k}\omega} = \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} \frac{k^2 \, \eta \omega_{pi}^2 / 2\pi}{\left(\omega^2 + \omega^2 \, k_D^2/k^2 - \omega_{pi}^2\right)^2 + \eta^2 \, \omega^2 (1 + k_D^2/k^2)^2} \ . \tag{40}$$

Now, this is total charge fluctuation. If we are interested in the ion density fluctuations, we use

$$\delta n_i = \epsilon_e \, \delta\rho_e / e \ , \qquad \text{where } \epsilon_e = 1 + \frac{k_D^2}{k^2} \ , \tag{41}$$

to obtain

$$\left\langle \delta n_i^2 \right\rangle_\omega = \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} \frac{\eta \omega_{pi}^2}{2\pi \, e^2} \int_0^{k_D} \frac{dk}{(2\pi)^3} 4\pi k^2 \left(1 + \frac{k_D^2}{k^2}\right)^2 \frac{k^2}{\left[\omega^2 + \omega^2 \frac{k_D^2}{k^2} - \omega_{pi}^2\right]^2 + \eta^2 \omega^2 \left(1 + \frac{k_D^2}{k^2}\right)^2} \ . \tag{42}$$

In the zero frequency limit $\omega \to 0$ we have

$$\left\langle \delta n_i^2 \right\rangle_\omega = T \, \frac{\eta}{\omega_{pi}^2} \, \frac{1}{4\pi^3} \, \frac{28}{15} \, k_D^5 / e^2 \ , \tag{43}$$

where k_D is chosen as the cutoff since density fluctuations will be correlated very weakly for wavenumbers higher than k_D. This limit may be an overestimate, but, as will be seen, the calculated value of the pressure effects $\langle \delta n_i^2 \rangle_{\omega=0}$ is insignificant compared to $\langle B^2 \rangle / 8\pi$. The ratio of the plasma pressure fluctuations to the magnetic fluctuation is

$$\left(\frac{T \langle \delta n_i^2 \rangle / n_{i0}\, e^2}{\langle B^2 \rangle / 8\pi} \right)_{\omega \to 0} = 4\pi\alpha \left(\frac{M}{m} \right) g^2 \left(\frac{k_D}{k_E} \right)^2 , \tag{44}$$

where $\alpha \approx 0.56$, g, the plasma parameter, is introduced to evaluate the ratio of the collision frequency to the plasma frequency disregarding the numerical coefficient of the order of unity (Ichimaru, 1973) $g = \eta_e / \omega_{pe}$, and $k_E = c/\omega_{pe}$. Equation (44) is much less than unity in our typical cases. Thus, magnetic fluctuations play a much more important role than density fluctuations in shaping the primordial Universe.

From Eq. (34) the plasma beta due to the magnetic field energy density associated with the zero frequency is evaluated to be

$$\beta = \frac{nT}{\langle B^2 \rangle^0 / 8\pi} = 2\pi^3 n \left(\frac{c}{\omega_p} \right)^3 , \tag{45}$$

where x_{cut} is equated to unity for simplicity. If we assume that at each instance of cosmic time the level of magnetic fluctuations is determined by the fluctuation-dissipation theorem, the plasma beta scales as $\beta \propto n^{-1/2} \propto a^{3/2}$ (see Table I). The beta at $t = 10^{-2}$ sec is as small as 1-10. Once again this is an impressive value. On the other hand, if the primordial magnetic fields are created at $t = t_d$ (at this moment we do not have sufficient knowledge to determine t_d) according to the fluctuation-dissipation theorem and the magnetic field evolution is detached from that of the plasma temperature or the photon temperature as the Universe continues to cool, the plasma beta may scale as $\beta \propto a^0$ (invariant), because $B \propto a^{-2}$ (the flux conservation). One, however, notes that most likely at some point of time the dynamo effect comes into play, which tends to amplify magnetic fields in competition of the cosmic expansion. The complete consideration of all these processes is beyond the scope of the present paper.

III. Simulation

We carry out particle simulation of a thermal equilibrium plasma and measure the level of magnetic fields and their frequency spectrum to corroborate with the theory described above. The code employed is the standard electromagnetic particle code in 1D and 2D (see, for example, Tajima, 1989). The 1-D simulations have been employed to control runs and parameters were: $L_x = 256$ cells, particles/cell=$10e^- + 10e^+$, $\Delta t = 0.1/\omega_{pe}$, the number of time steps of runs $N_t = 2048$, and $c = 5\Delta\omega_{pe}$, where Δ is the grid spacing. Simulations were run for three different temperatures: the relativistic factor due to thermal velocities $\gamma_{\text{therm}} = 1.05$, 1.22, and 34.7 (i.e. $T \sim 3 \times 10^{8\circ}$K, $1.3 \times 10^{9\circ}$K, and $2 \times 10^{11\circ}$K). To test the performance of these codes, we examined the dispersion relations produced for B_z and compared them with the standard result $\omega^2 = c^2 k^2 + \frac{\omega_{p0}^2}{\gamma}$ where $\omega_{p0}^2 = \dfrac{4\pi n e^2}{m_e} + \dfrac{4\pi n e^2}{m_i}$. We made an additional test on the 1-D code by running a nonrelativistic simulation for $N_t = 4096$. Again, the $\omega = 0$ peak appears and its width does not change from the width it had in the $N_t = 2048$ simulation. We also examined B_z fluctuation strengths as functions of wavevector. The dispersion relation comparisons were excellent, the fluctuation strengths compared fairly well. In each of these three cases, a strong B_z fluctuation peak is seen at $\omega = 0$. We tested cases with temperatures of $2 \times 10^{11\circ}$ K, $10^{9\circ}$ K, and $3 - \times 10^{8\circ}$ K. The results of 1D cases are not shown here.

In the 2-D simulation, parameters were 32×32 cells, $9e^- + 9e^+$ per cell, $\Delta t = 0.1/\omega_{pe}$, $N_t = 2048$, and $\gamma_{\text{therm}} = 1.05$. Again, a strong B_z fluctuation peak is seen at $\omega = 0$. Figure 5 shows the 2D results with the corresponding 1D results.

Our simulation results for $S(\mathrm{k})$ follow the standard expression $(1/(\omega_p^2 + c^2 k^2)$ the second term in Eq. (31)) more closely than our low-wavenumber expansion (Eq. (32)). See frames (b) and (d) of Fig. 5. This is explained by the conditions of the simulation. First of all, the grid nature of the simulation puts a cap on the maximum k at π/Δ. Second, as can be seen from our derivation of $S(\mathbf{k})$, the first term in Eq. (31) comes from the energy contained in the radiation. The results shown were obtained by summing $S(\mathbf{k}, \omega)$ over frequencies ranging from 0 to $+5\omega_p \sim 1/\Delta t$. When the wave frequency of a given mode is higher than this range, the high-frequency energy of the radiation mode will not enter into the sum. These damping factors of the radiation branch, plus sharing of energy between modes, account for the faster decay of $S(\mathrm{k})$ than theory Eq. (32), favoring the expression with only the second term in Eq. (31) in this situation. This simulation is therefore in satisfactory agreement with our present theory, indicating in fact the presence of a zero-frequency peak in the magnetic field spectrum.

IV. Collisional Effects

In this section we examine the essential results of Sec. II that was based on the rigorous theoretical analysis of the fluctuation-dissipation theorem and attach our physical interpretation to it. The basic physical picture of what the fluctuation-dissipation theorem says is: an individual mode (or field) decays by a certain dissipation, giving up energy to particles or other modes, while particles (or other modes) excite new modes and repeat the process and the amount of fluctuations is related to the dissipation. In this context our discovery of the zero frequency peak in fluctuation spectra may be a little surprising. Therefore, it is instructive to provide more intuitive ground to our theory. We find that the physical basis of the zero frequency peak is due to collisions (or other kinetic dissipation) or more precisely collision-induced quasi-modes. In the following we relate physical consequences to collision-induced quasi-modes.

From Maxwell's equations with all the terms on the right-hand side except the source term (the third term) written in terms of $\mathbf{E}$, we obtain the dispersion relation of the quasi-modes:

$$\omega^2 - k^2 c^2 - \frac{\omega_p^2}{1 + i\eta/\omega} = 0 \ . \tag{46}$$

In the low frequency limit Eq. (46) yields the dispersion relation

$$\omega = i \, \frac{k^2 c^2}{\omega_p^2} \, \eta \ . \tag{47}$$

Or, equivalently, the spatial size λ of magnetic field fluctuations for a given lifetime $\tau_\ell (\equiv \omega^{-1})$ is

$$\lambda(\tau_\ell) = 2\pi \, \frac{c}{\omega_p} \, (\eta \, \tau_\ell)^{1/2} \ . \tag{48}$$

Equation (47) states that the lifetime τ_ℓ of magnetic fluctuations (or maybe called "magnetic bubble") of size λ is proportional to the size squared ($\tau_\ell \propto \lambda^2$); the larger the size of the bubble, the longer it lasts.

This entails an important ramification. Suppose two magnetic bubbles touch or collide with each other and coalesce into one. The time for coalescence of magnetic bubbles involves reconnection of magnetic field lines. It is generally known (Parker 1957; Bhattacharjee *et al.* 1983; Tajima *et al.* 1987) that this process (or related ones) is much faster than the diffusive time related to Eq. (47). Thus the coalescence time is much shorter than the individual life time. Therefore, before bubbles die away, they can form a coalesced bubble when they collide with each other, as long as they collide frequently enough. Once a larger coalesced bubble is formed, its life time is substantially longer, as Eq. (47) shows the life time is proportional to the square of the size of the bubble. It is possible to imagine that once larger bubbles are formed, they become even more longer-lived and may be able to encounter more opportunities to collide with other bubbles. In this way a preferential formation of larger bubbles may become possible. This process is not far different from that of polymerization. A detailed mathematical analysis of the magnetic bubble size growth is beyond the scope of the present paper and will be left for a future publication (Tajime and Isichenko 1991).

V. Cosmological Implications

We have discovered in the previous sections that electromagnetic waves in the primordial plasma fall into two categories: one with large wavelengths ($k \lesssim \omega_{pe}c$) and nearly zero frequency ($\omega \ll \omega_{pe}$) and one with small wavelengths ($k \gg \omega_{pe}/c$) and frequency greater than ω_{pe}. Those modes $k > \omega_p/c$ are not significantly modified by the presence of the plasma ('hard photon'), while those with $k < \omega_p/c$ significantly modified ('soft or plastic photon'). It is those 'plastic photons' or their magnetic fields that we are interested in, as they can have more 'magnetic' fields in nature and can leave possible structural imprints on the primordial plasma.

In Table I we summarize our results. Including the physical quantities we already discussed, we survey physical quantities of importance that characterize the radiation epoch (or the plasma epoch). The density scales as $n \propto a^{-3}$ where a is the cosmic scale factor, which increases as $a \propto t^{1/2}$ during the radiation epoch. The wavelength of photons also scales as $\lambda \propto a$ and thus the temperature of photons $T \propto a^{-1}$, as the frequency of the maximum intensity of the black-body radiation $\omega_{\max} = 2.81\, T/\hbar$. It follows that the plasma frequency scales as $\omega_{pe} \propto n^{1/2} \propto a^{-3/2}$. The electron collision frequency goes like $\eta_e \propto n\, T^{-3/2} \propto a^{-3/2}$. The plasma parameter (and the collisionality) is therefore $g = (n\lambda_{De}^3)^{-1} \cong \eta_e/\omega_{pe} \propto a^0$ (independent of a) and thus invariant during the epoch in which the numbers of constituent particles are conserved; e.g. during $t = 10^{-2} - 10^0\,\mathrm{sec}$ $g \sim 10^{-3}$ (invariant) and it changes around $t = 10^0\,\mathrm{sec}$ as positrons annihilate with electrons to $g \sim 10^{-7}$ and stays invariant till the recombination. On the other hand, the collision frequency between electrons and photons may be given from the Thompson cross-section or in relativistic cases from the Klein-Nishina cross-section to be $\nu_{\mathrm{TH}} \propto n\, T^{1/2} \propto a^{-7/2}$ and $\nu_{\mathrm{KN}} \propto n\, T^{-1} \propto a^{-2}$, respectively. The Reynolds number R_e is Lv/μ, where L, v, and μ are the typical sizes of the length, velocity, and viscosity. By taking v the thermal velocity, Lv/μ scales as $L\, a^{-1}$ and if we take L as the horizon size ct, $R_e \propto t^{1/2}$ in the radiation epoch.

The electron magnetic energy $\langle B^2 \rangle_\omega^{bb}$ contained in the black-body radiation is proportional to ω^2 and $\langle B^2 \rangle_\omega^{bb} \propto T^3 \propto a^{-3}$. On the other hand, the zero frequency magnetic fluctuation energy $\langle B^2 \rangle_{\omega \to 0} \propto T\omega_p^3/\mu \propto a^{-4}$. Thus the ratio of the zero frequency fluctuations to the black-body energy is proportional to a^{-1}. If we assume here that the level of magnetic fields is determined to be the level by the formula (34) at each instance of time after ω integration, the plasma beta scales as $\beta = \dfrac{nT}{\langle B^2 \rangle^0 / 8\pi} \propto n(c/\omega_p)^3 \propto a^{3/2}$. This conclusion is based on the instantaneous adjustment of the magnetic fields to the level of thermal energy of the Universe. It should be noted that nevertheless this result of $\beta \propto a^{3/2}$ differs from the earlier discussion of the magnetic field scaling when the magnetic flux conservation was invoked. For example, it may be possible to imagine a scenario in the absence of dynamo that $\beta \propto a^{3/2}$ until a certain time t_d, when magnetic fields detach from plasma and thereon $\beta \propto a^0$.

The significance of the presence of static (or nearly zero frequency) magnetic fields in the cosmological plasma may be appreciated in the following. Two main scenarios (Rees 1987) have been considered for primordial fluctuations, adiabatic fluctuations and isothermal fluctuations. The adiabatic (or isentropic) fluctuations are like those accompanied by ordinary sound waves and a cartoon illustration of this situation is displayed in Fig. 6(a). In such fluctuations the density of matter (electrons, positrons, and protons (and helium ions) for the case of the early radiation spoch) is accompanied by that of photons, as indicated in Fig. 6(a). Therefore, after electrons and positrons annihilate around $t = 1\,\text{sec}$ or after electrons and ions recombine around $t = 10^{13}\,\text{sec}$, the imprint of matter fluctuations would remain in photon fluctuations as a fossil of the primordial plasma structure. Thus the background microwave spectra would show a certain fluctuation or anisotropy/inhomogeneity on top of the black-body spectra. This would be a contradiction to the latest observations by COBE etc. (Mather *et al.* 1990; Gush *et al.* 1990).

On the other hand, imagine that as we have shown, there exist static magnetic fields action and the effects of noise re-entry to the horizon, the magnetic field at $t = 10^{13}$ sec (the recombination time) is as large as 10^{-12} Gauss. For the moment we exclude any possibility of such and our estimate of magnetic field strength is solely based on the instantaneous conditions of the thermal plasma at each instance. Based on Eq. (31), we can define the strength of magnetic fields whose wavelength is larger than a certain size λ as $\langle B^2 \rangle_\lambda\ 8\pi$. This becomes $\langle B^2 \rangle_\lambda /8\pi = (T/2)(4\pi/3)\lambda^{-3}$. For $\lambda_p = 2\pi c/\omega_p$,

$$\sqrt{\langle B^2 \rangle_{\lambda_p}} = 1.4 \times 10^{-12}(n/10^4\,\mathrm{cc})^{3/4}(T/10^4\mathrm{K})^{1/2}\,\mathrm{Gauss}\,. \tag{49}$$

We might be interested in global magnetic fields whose wavelengths at $t = 1$ sec are 10^{14} cm or longer, which may correspond to the length of the present galaxies. In this case $\sqrt{\langle B^2 \rangle_\lambda} = 4 \times 10^{-24}$ Gauss in the absence of dynamo actions, horizon re-entry, or coalescence of magnetic structures. It is important therefore to recall that this value is the theoretical minimum for a large scale magnetic field. This number together with 10^{16} G for small scales gives us several important implications. The primordial plasma *spontaneously* generates rather strong magnetic fields at small scales. However, the strength of magnetic fields of such a plasma at large scales is much smaller, implying the cosmology is not just a series of frozen snapshots of thermal equilibrium plasmas. This is hardly a surprise. In fact it would be rather surprising if such large magnetic fields at small scales did not act as seed fields for late developments.

Since the main objective of the present paper is to demonstrate through the rigorous theory of fluctuation-dissipation theorem that the (even) thermal primordial plasma can sustain substantial zero frequency magnetic fluctuations, extensive treatments of their later evolution and impact on cosmological consequences such as galaxy formation need to wait for subsequent papers. Without going into details, however, we discuss how our finding can give rise to some cosmological correspondences. First is the possibility of successive coalescence (Bhattacharjee *et al.* 1983) of magnetic structures and their subsequent stabilization: Our large magnetic fields with small scales tend to fuse to polymerized larger structures. We have already mentioned this physical basis at the end of Sec. IV. In this mechanism the long epoch of radiation dominance ($t = 1 - 10^{13}$ sec) once thought of as an epoch of no significant happening can be considered as a (perhaps quiet but) incubating period of large structures of magnetic fields.

Second is the possibility of dynamo amplification of the primordial magnetic fields. In addition to the possibility of the dynamo amplification of magnetic fields after the recombination ($t = 10^{13}$ sec) due to the gravitationally triggered rapid relaxational motion, it is possible to consider the dynamo action due to the fluctuations inherent to the thermal plasma. On top of this it should also be remembered that the sizes that were larger than the horizon could not have been in a thermal equilibrium and thus velocity fluctuations with sizes larger than the horizon of the older epoch now come within the horizon at a certain time during the plasma epoch. For example, velocity fluctuations (say $\lambda \sim 10^{14}$cm at $t = 1$ sec) can come within the horizon before $t = 10^9$ sec. Velocity fluctuations with lesser sizes enter within the horizon correspondingly earlier. Thus these velocity fluctuations coupled with magnetic fluctuations can drive dynamo action before the recombination time.

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Table I — **Primordial Magnetic Fluctuations with $\omega \sim 0$:** The zero frequency magnetic fluctuations in early Universe. ($t = 10^{-2}1$, and 10^{13} sec after the big bang). The temperature T, density of the plasma electrons, the horizon size $L_{\rm hor}$, the zero frequency magnetic fluctuations B, the ratio of the zero frequency magnetic fields $\left\langle B^2\right\rangle^0 /8\pi$ to the blackbody component $\left\langle B^2\right\rangle_{\rm bb} /8\pi$, the plasma beta β, and the (maximum) Reynolds number Re are tabulated. B and $\left\langle B^2\right\rangle^0 / \left\langle B^2\right\rangle_{\rm bb}$ are from Eq. (45) and β from Eq. (56). When kinetic effects are included, some of the numbers change.

	$t = 10^{-2}$	$t = 1$	$t = 10^{13}$	$t = 3 \times 10^{17}$ sec
T eV	10^7	10^6	0.4	$T_\gamma = 0.0003$
n cm^{-3}	5×10^{34}	4×10^{31}	10^3	10^{-6}
$L_{\rm hor}$ cm	10^8	10^{10}	10^{23}	10^{28}
B Gauss	10^{16}	10^{13}	10^{-12}*	
$\dfrac{\left\langle B^2\right\rangle^0}{\left\langle B^2\right\rangle^{bb}}$	0.1-1	10^{-2}	10^{-25}*	
β	1	$10 - 10^2$	10^{15}*	
R_e	10^{17}	10^{18}	10^{15}	
*: no dynamo nor trigger mechanisms included				

References

Bhattacharjee, A., F. Brunel, and T. Tajima, 1983, Phys. Fluids **26**, 3332.

Braginskii, S.I., 1965, in *Reviews of Plasma Physics* ed. M.A. Leontovich (Consultant Bureau, New York) p. 205.

Geary, J., T. Tajima, J.-N. Leboeuf, E.G. Zaidman, and J.-H. Han, 1986, Comp. Phys. Comm. **42**, 313.

Gush, H.P., M. Halpern, and E.H. Wishnow, 1990, Phys. Rev. Lett. **65**, 537.

Harrison, E.R., 1970, Phys. Rev. **D1**, 2726.

Harrison, E.R., 1973, Phys. Rev. Lett. **30**, 18.

Ichimaru, S., 1973, "Basic Principles of Plasma Physics" (Addison-Wesley, Redwood City).

Kajantie, K., and H. Kurki-Suonio, 1986, Phys. Rev. **D34**, 1719.

Kubo, R, 1957, J. Phys. Soc. Jpn. **12**, 570.

Mather, J.C. *et al.*, 1990, Ap. J. Lett. **354**, L37.

Misner, C.W., K.S. Thorne, and J.A. Wheeler, 1970, *Gravitation* (W.H. Freeman, San Francisco, CA) p. 764.

Parker, E.N., 1957, J. Geophys. Res. **62**, 509.

Rees, M., 1987, in *The Very Early Universe*, ed. by G.W. Gibbons, S.W. Hawking, and S.T.C. Siklos (Cambridge Univ. Press, Cambridge) p. 29.

Sitenko, A.G., 1967, *Electromagnetic Fluctuations in Plasma* (Academic Press, New York).

Tajima, T., *et al.*, 1987, Ap. J. **321**, 1031.

Tajima, T., 1989, *Computational Plasma Physics* (Addison-Wesley, Redwood City).

Tajima, T., and M. Isichenko, 1991, to be submitted to Ap. J.

Weinberg, S., 1972, *Gravitation and Cosmology* (John Wiley and Sons, New York).

Weinberg, S., 1977, *The First Three Minutes* (Basic Books, New York).

Figure Captions

1. The spectral intensity $S(\omega) = \langle B^2 \rangle_\omega / 8\pi$ for plasma 1 sec after big bang. $T = 10^{10\circ}$K; $n_e = 4.8 \times 10^{30}$/cc.

   a) $\ell n(S(\omega)/S_0)$ plotted linearly in ω. Zero-frequency peak is at the top of the graph, where S_0 is the normalization.

   b) $\ell n(S(\omega)/S_0)$ plotted linearly in ω. Zero-frequency peak is at the top of the graph, is seen to be higher than black-body peak.

   c) $\ell n(S(\omega)/S_0)$ plotted logarithmically in ω. Low-frequency line has slope around -2. Rises to peak at $\omega = 0$.

   d) $\ell n(S(\omega)/S_0)$ plotted logarithmically in ω. Note suppression of black-body radiation around $\omega = \omega_{pe}$.

2. Spectral intensity $S(\omega) = \langle B^2 \rangle_\omega / 8\pi$ for plasma 10^8 sec after big bang. $T = 10^{6\circ}$ K; $n_e = 6.5 \times 10^9$/cc.

   a) $\ell n(S(\omega)/S_0)$ plotted linearly in ω. Zero-frequency peak is at the top of the graph.

   b) $\ell n(S(\omega)/S_0)$ plotted logarithmically in ω. Slope of low-frequency line is ~ -2. Rises to peak at $\omega = 0$.

3. a) Spectral intensity $S(\omega) = \langle B^2 \rangle_\omega / 8\pi$ for plasma 10^{12} sec after big bang. $T = 10^{4\circ}$ K; $n_e = 6.5 \times 10^3$/cc.

   b) $\ell n S(\omega)/S_0$ plotted logarithmically in ω. Slope of low-frequency line is around -2. Continues to rise until peaking at $\omega = 0$.

4. (a) Schematic plot of the spectrum of magnetic fluctuations $\langle B^2 \rangle_\omega / 8\pi$ in a thermal plasma with temperature T, plasma frequency ω_p. The zero frequency peak has the height inversely proportional to the dissipation (such as collision frequency) and the width proportional to itself. The black-body profile is hardly modified. Only the low frequency ($\omega < \omega_p$) is severely modified by the plasma effects. (b) Schematic plot of the spectrum $\langle B^2 \rangle_k / 8\pi$. The shaded area corresponds to the second term in Eq. (31) and to the shaded area in Fig. 4(a).

5. Spectral intensities: a) $\ell n S(\omega)/\overline{S}$ from 1D simulation of $e^+ - e^-$ plasma. $\gamma_{\text{therm}} = 1.05 (T = 3 \times 10^{8\circ}$ K). Simulation was run for 4096 timesteps. Width of central peak is the same as from simulation run for 2048 timesteps.

   b) $\ell n S(k)/T_0$ from 1D simulation of $e^+ - e^-$ plasma. $\gamma_{\text{therm}} = 1.05$. 4096 timesteps. Line is from simulation results. Dots represent theoretical value $-1.7 - \ell n(1 + k^2 \frac{\gamma_c^2}{\omega_p^2} e^{k^2 a^2})$ where -1.7 was obtained from least squares fitting.

   c) $\ell n S(\omega)/\overline{S}$ from 2D simulation of $e^+ - e^-$ plasma. $\gamma_{\text{therm}} = 1.05$ ($T = 3 \times 10^{8\circ}$ K). Zero-frequency peak still present in 2-D.

d) $\ell n S(k)/S_0$ from 2-D simulation of e^-, e^+ plasma. $\gamma_{\mathrm{therm}} = 1.05$. Line is from simulation results. Dots represent $-2.6 - \ell n(1 + c^2 \frac{\gamma}{\omega_p^2} k^2 e^{k^2 a^2})$ where -2.6 was obtained from least squares fitting.

6. Cartoon illustration of zero frequency magnetic fields and their influence on the plasma.

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