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ANALYSIS OF CHAOTIC, AREA-PRESERVING MAPS†

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Area-preserving maps have been extensively used as models of dynamical systems because they are computationally simple, yet they illustrate the range of behavior of dynamical systems described by differential equations. Many of these maps, like the standard map[1], have a nonlinearity parameter, ϵ. Where ϵ is small, the system is integrable, and orbits can be predicted with great accuracy for long periods. When ϵ is large, the integrals of the motion no longer exist, and orbit prediction, even with large computers, is difficult. Much work on the small-ϵ regime has been calculated with ϵ as the perturbation parameter. Here we discuss the large-ϵ, nearly totally stochastic system, for which $1/\epsilon$ is the perturbation parameter.

In this paper we consider periodic impulse maps[2], which have the form:

$$p_{n+1} = p_n + \epsilon f(x_n) \tag{1a}$$

$$x_{n+1} = x_n + p_{n+1} \tag{1b}$$

where $f(x)$ is periodic with period 2π. For the standard map, $f(x) = \sin(x)$. To analyze[3] this map, we will use the periodicity of f to write the Fourier series expansions,

$$f(x) = \sum_{\ell=-\infty}^{\infty} f_\ell e^{i\ell x} \tag{2}$$

and

$$e^{i\epsilon f(x)} = \sum_{\ell=-\infty}^{\infty} g_\ell(\epsilon) e^{i\ell x} \quad . \tag{3}$$

Since we are examining the large-ϵ, stochastic regime, it seems appropriate to consider averages of functions. For this purpose we introduce the characteristic functions of region R:

$$\chi_k^R(m_0, m_1, \ldots, m_k) \equiv \left[\int_R dx_0 dp_0\right]^{-1} \int_R dx_0 dp_0$$

$$\exp\left[i \sum_{j=0} m_j x_{n+j}(x_0, p_0)\right] \quad . \tag{4}$$

In Eq. (4), $x_{n+j}(x_0,p_0)$ gives the position after the $n+j^{th}$ mapping of an initial condition (x_0,p_0), and the region R is assumed to be an invariant region of the map, which implies the independence of χ_k^R on the parameter n.

The crucial point of this analysis is that the mapping equations (1) in the form,

$$x_{n+k} = 2x_{n+k-1} - x_{n+k-2} + \epsilon f(x_{n+k-1}) , \qquad (5)$$

allow one to express χ_{n+1} in terms of χ_n for $n \geqslant 1$ by use of Eq. (3). One finds

$$\chi_k^R(m_0,m_1,\ldots,m_k) =$$

$$\sum_{\ell} g_\ell(m_k\epsilon)\chi_{k-1}(m_0,m_1,\ldots,m_{k-3},m_{k-2} - m_k,m_{k-1} + 2m_k + \ell) . \qquad (6)$$

Moreover, Eq. (4) gives χ_1 in the form

$$\chi_k^R(m_0,m_1) =$$

$$\left[\int_R dx_0 dp_0\right]^{-1} \int_R dx_0 dp_0 \exp\left[i(m_0 + m_1)x_0 - im_0p_0\right] \qquad (7a)$$

$$\approx \delta_{m_0,0}\,\delta_{m_1,0} . \qquad (7b)$$

(The approximate equality [Eq. (7b)] follows from the near ergodicity of the mapping in the highly chaotic regime.) Thus, continued use of Eqs. (6) and (7) gives all of the characteristic functions.

It is important to note that this method is complete. The characteristic function determines all of the statistical properties. For example, the diffusion coefficient is given by

$$D^R/\epsilon^2 = \frac{1}{2}\left[\int_R dx_0 dp_0\right]^{-1} \int_R dx_0\, dp_0 f^2(x_0) + \sum_{j=1}^{\infty} \sum_{m,n} f_m f_n \chi_j^R (m,0,\ldots,0,n) \quad , \qquad (8)$$

~~in which the first term is the quasilinear term.~~

This method has been used to analyze the standard map and the sawtooth map. Application to the standard map yields a series representation for the diffusion coefficient of Rechester and White[4] and many other statistical properties. One must use this series with care, however, since it is asymptotic and divergent. Application of this method to the sawtooth map, which is rigorously ergodic, yields exact results for the diffusion coefficient.

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