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An upper bound is determined for the turbulently generated axial electromotive force in reversed-field pinches, constrained solely by energy conservation in the approximation of incompressible magnetohydrodynamics. The resulting F–Θ curve is presented and comparisons are made with the "Taylor state".

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We compute a rigorous upper bound for the electromotive force (emf) due to steady-state turbulence in cylindrical pinch plasmas driven by an external electric field.

The reversed-field pinch (RFP) is a device that magnetically confines plasma.[1] In RFP experiments it is observed that toroidal-field reversal occurs and the reversed-field state is sustained, accompanied by fluctuations.[1] Self-reversal can be understood in general terms as a consequence of the tendency of the plasma to relax to a minimum-energy state while conserving total magnetic helicity, as proposed by Taylor.[2] The detailed dynamical mechanisms behind the relaxation have not yet been fully explained.

Following the success of Taylor's variational method, other authors have attempted to exploit variational principles based on one or another physical hypotheses[3,4]; however, none of those principles, including Taylor's, is rigorous. Furthermore, because of nonlinearity one needs many approximations in order to proceed analytically[5]; some are difficult or impossible to justify. Many numerical simulations have been done.[6,7] However, these require very long runs in order to achieve steady state.[7] Thus it is desirable to develop a model that requires less effort and time to solve, but still exhibits fundamental features such as field reversal; ideally, these would be predicted rigorously.

Rigorous analytic prediction of transport rates or mean profiles is probably too much to hope for. However, rigorous and suggestive bounds on certain transport fluxes can be obtained. We employ a rigorous, nonlinear variational principle formulated originally by Howard[8] (the so-called "optimum" theory[9]). The principle states "Maximize a preferred flux under (some) constraints obtained from the dynamical

equations." If the constraints are chosen judiciously, the bound can reasonably approximate the true flux. Furthermore, if there is a strong tendency in the real world toward a state of maximal flux, then the optimum theory offers model equations that include important features of the real physical system. It is difficult to quantify this tendency *a priori*. However, the utility of the optimum theory has been proven in both self-consistent fluid turbulence[9,10] and various models of passive advection.[11,12]

We consider a steady-state RFP parametrized by specified total axial current. For this case, the natural quantity to maximize is the spatially averaged turbulent axial emf $\overline{\varepsilon}$, which is positive-definite and can be interpreted as a generalized dynamo effect.[13] We use the so-called "basic" constraint[11] obtained from the global energy balance, which balances linear dissipation with the production of fluctuations due to the nonlinear interaction of the fluctuations with the mean fields. The role of, and difficulty with, additional helicity constraint, which plays an important role in RFP, are discussed in Ref. 14.

We use the resistive, viscous magnetohydrodynamic (MHD) equations in cylindrical geometry (r, θ, z). We assume for simplicity that the plasma is incompressible ($\vec{\nabla}\cdot\vec{u} = 0$, $\vec{u}$ being the fluid velocity), the electrical resistivity η is a constant, and the viscosity ν is isotropic and constant. This is the simplest possible model that can still exhibit field reversal.[6] As units of time, space, and magnetic field we use the resistive-diffusion time $\tau_{\mathrm{d}} \doteq 4\pi a^2/c^2\eta$, the radius a, and the toroidal field B_0. Two natural dimensionless numbers are the Hartmann number $\mathrm{H} \doteq \left(a^2 B_0^2/c^2 \rho_0 \eta \nu\right)^{1/2}$ and the "magnetic Prandtl"

number Pm $\doteq$ $(4\pi/c^2)(\nu/\eta)$. (The Lundquist number Lu can be expressed as Lu $=$ $\sqrt{\mathrm{Pm}}\,\mathrm{H}$.) Let $\vec{E}_\mathrm{t}$ be the total electric field, the sum of the external field $\vec{E}_\mathrm{ext} = \hat{z}\,E_0$ and the internal field $\vec{E}$ of the plasma. Then the MHD equations are Ohm's law $\vec{E}_\mathrm{t} + \vec{u}\times\vec{B} = \vec{j}$, or

$$\partial\vec{B}/\partial t = \vec{\nabla}\times\left(\vec{u}\times\vec{B} - \vec{j}\right), \tag{1}$$

the equation of motion

$$\mathrm{Pm}^{-1}\frac{d\vec{u}}{dt} = -\vec{\nabla}p + \mathrm{H}^2\vec{j}\times\vec{B} + \nabla^2\vec{u}, \tag{2}$$

where $d/dt \doteq \partial/\partial t + \vec{u}\cdot\vec{\nabla}$, and Ampère's law $\vec{\nabla}\times\vec{B} = \vec{j}$. At the wall it is assumed that $\hat{n}\cdot\vec{B} = \hat{n}\times\vec{j} = 0$ and $\vec{u} = 0$, where $\hat{n}$ is normal to the wall. Thus, we assume a fairly ideal situation for supplying the magnetic flux such that E_0 is uniform and finite inside the wall but vanishes at the wall.[7,4]

With our assumption $\vec{\nabla}\cdot\vec{u} = 0$, the role of the pressure p is to maintain the incompressibility. It can be expressed as an instantaneous functional of $\vec{u}$ and $\vec{B}$ by solving the Poisson equation obtained by taking the divergence of Eq. (2):

$$\nabla^2 p = -\mathrm{Pm}^{-1}\vec{\nabla}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}) + \mathrm{H}^2\vec{\nabla}\cdot(\vec{j}\times\vec{B}).$$

However, we will not need to solve this equation in the present work.

We denote the average over the directions θ and z (assumed to be homogeneous) by angular brackets and fluctuations by tildes: $\tilde{f}(r,\theta,z) \doteq f(r,\theta,z) - \langle f\rangle(r)$. A bar denotes

the radial average $\overline{f} \doteq 2\int_0^1 dr\, r f$. Steady states are parametrized by $\mathcal{J} \doteq \overline{\langle j_z \rangle}$, a measure of the total axial current. Then the goal is to maximize the functional

$$\mathcal{A}\{\tilde{\vec{u}}, \tilde{\vec{B}}\} \doteq \overline{\varepsilon}(\mathcal{J}) + \lambda_{\rm E}\mathcal{C}_{\rm E} + \overline{\left\langle \lambda_{\tilde{\vec{u}}}(\vec{x}) \left(\vec{\nabla}\cdot\tilde{\vec{u}}\right)\right\rangle} + \overline{\left\langle \lambda_{\tilde{\vec{B}}}(\vec{x}) \left(\vec{\nabla}\cdot\tilde{\vec{B}}\right)\right\rangle}.$$

Here $\varepsilon \doteq Q_z$ ($\vec{Q} \doteq \langle \tilde{\vec{B}} \times \tilde{\vec{u}} \rangle$), $\mathcal{C}_{\rm E}$ is the energy constraint, and the λ's are Lagrange multipliers.

The fundamental nonlinearity in the theory is the self-consistent adjustment of the mean profiles to the fluctuations. To obtain $\mathcal{C}_{\rm E}$ it is useful to express the means in terms of the fluctuations. We begin by averaging Ohm's law. Barring the z component of the result leads to $E_0 = \mathcal{J} + \overline{\varepsilon}$, so one may interpret the nonlinear effects as providing a generalized nonlinear resistivity. By using the mean Ohm's law and Ampère's law, $\langle \vec{B} \rangle$ can be found. The $\langle \vec{u} \rangle$ can be obtained from Eq. (2). Details can be found in Ref. 14.

We can now construct $\mathcal{C}_{\rm E}$ by forming the evolution equation for the energy of the fluctuations. We add the scalar products of Eq. (1) and Eq. (2) with $\tilde{\vec{B}}$ and $\tilde{\vec{u}}$, respectively. The result contains third-order moments; however, these can be eliminated completely and rigorously by applying the barring operation and employing the boundary conditions. This annihilation is the essence of the method; it replaces the uncertainty conventionally introduced by statistical closure. In steady state, we find

$$0 = \mathcal{C}_{\rm E} \doteq \overline{\varepsilon}\mathcal{J} - \left[\left(\overline{\left\langle \tilde{j}^2 \right\rangle} + \mathrm{H}^{-2}\overline{\langle \tilde{\omega}^2 \rangle}\right) + \left(\overline{\Delta\varepsilon^2} + \overline{Q_\theta^2}\right) + \mathrm{H}^2\overline{|\vec{M}|^2}\right], \qquad (3)$$

where $\widetilde{\vec{\omega}} \doteq \vec{\nabla}\times\widetilde{\vec{u}}$, $\Delta\varepsilon(r) \doteq \varepsilon(r) - \overline{\varepsilon}$, and

$$\vec{M} \doteq \hat{r}\times\left[\left\langle\widetilde{\vec{B}}\times(\hat{r}\times\widetilde{\vec{B}})\right\rangle - \mathrm{Lu}^{-2}\left\langle\widetilde{\vec{u}}\times(\hat{r}\times\widetilde{\vec{u}})\right\rangle\right].$$

The first parenthesized term in Eq. (3) represents dissipation; the other terms are related to production. Clearly $\overline{\varepsilon}$ is positive. One can show that Eq. (3) is nothing but Poynting's theorem: The volume-averaged dissipation of the total energy (mean plus fluctuation) balances the inward Poynting flux.

By introducing the quantity $\zeta \doteq 1 + \lambda_{\mathrm{E}}^{-1}$ one can write the resulting Euler-Lagrange equations in the form

$$0 = \vec{\nabla}\times\widetilde{\vec{j}} - \widetilde{\vec{u}}\times\vec{L} + \widetilde{\vec{\Omega}}_{\vec{B}} + \vec{\nabla}\lambda'_{\vec{B}}, \tag{4a}$$

$$0 = \mathrm{H}^{-2}\vec{\nabla}\times\widetilde{\vec{\omega}} + \widetilde{\vec{B}}\times\vec{L} + \widetilde{\vec{\Omega}}_{\vec{u}} + \vec{\nabla}\lambda'_{\vec{u}}, \tag{4b}$$

where

$$\vec{L} \doteq -\hat{\theta}\,Q_\theta + \hat{z}\left(\tfrac{1}{2}\zeta\mathcal{J} - \Delta\varepsilon\right),$$

$$\widetilde{\vec{\Omega}}_{\vec{B}} \doteq \mathrm{H}^2\mathbf{P}\cdot\left(\vec{M}\times\widetilde{\vec{B}}\right),$$

$$\widetilde{\vec{\Omega}}_{\vec{u}} \doteq \mathrm{Pm}^{-1}\mathbf{P}\cdot\left(\widetilde{\vec{u}}\times\vec{M}\right),$$

$\mathbf{P} \doteq \hat{z}\hat{z} - (\hat{x}\hat{x} + \hat{y}\hat{y})$, and $\lambda' \propto \lambda$. The nonlinearities associated with $\langle\vec{B}\rangle$ are assembled inside $\vec{L}$; those due to $\langle\vec{u}\rangle$ are represented by the $\widetilde{\vec{\Omega}}$'s. To determine λ_{E} we take scalar products of Eqs. (4a,b) with $\widetilde{\vec{u}}$ and $\widetilde{\vec{B}}$, respectively, bar the resulting relations, and use the constraints to find

$$\zeta = 2 - \left(\overline{\left\langle\widetilde{j}^2\right\rangle} + \mathrm{H}^{-2}\overline{\langle\widetilde{\omega}^2\rangle}\right)/(\overline{\varepsilon}\mathcal{J}).$$

The multipliers $\lambda_{\vec{B}}$ and $\lambda_{\vec{u}}$ can be eliminated by applying the operators $\widehat{r}\cdot(\vec{\nabla}\times)^n$ ($n = 1,2$) to Eqs. (4). To satisfy the solenoidal constraints, we express the fields in terms of potentials; for example,

$$\widetilde{\vec{B}} \doteq \vec{\nabla}\times(\widehat{r}\psi_B) + \vec{\nabla}\times\vec{\nabla}\times(\widehat{r}\chi_B)\,.$$

After Fourier transformation in θ and z and extremely tedious algebra,[14] we arrive at a 12th-order nonlinear system of o.d.e.'s. Since the equations have a $1/r$ singularity at $r = 0$, we must require that the variables be regular there. We determine appropriate regularity conditions by performing an eigenvector analysis near $r = 0$, following Lentini and Keller.[15]

We solve this system numerically. First, we compute the largest possible current $\mathcal{J}_c$ that does not drive turbulence [*i.e.*, $\overline{\varepsilon}(\mathcal{J}_c) = 0$]. This (linear) eigenvalue calculation serves two purposes. First, it is identical to the energy stability problem.[10] Second, in solving the nonlinear problem we employ Pereyra's algorithm[16]; since iteration is involved it is important to have a good initial guess in order to guarantee convergence.

It is found that the critical maximizing modes are $m = \pm 1$, $n = \pm 2$ when the aspect ratio A is unity. (To date, we have considered only $A = 1$, as an interesting and exemplary case.) The critical current is $\mathcal{J}_c \approx 40\mathrm{H}^{-1}$. Since H may be very large, $\mathcal{J}_c$ is far below the actual value observed in the RFP experiments, as one would expect. The dominance of $m = 1$ modes is in agreement with the RFP research, both numerical simulations and experiments. Also, it fits well with the speculation of Caramana[17] that $|n| \sim 2A$.

Now, we consider the bounding curve $\mathcal{J}(\bar{\varepsilon}; \mathrm{H}, \mathrm{Pm})$ for non-zero $\bar{\varepsilon}$ when $A = 1$. For a preliminary analysis that substantially simplifies the numerical work we shall neglect the effects of $\langle \vec{u} \rangle$ because the essential nonlinearity for the study of field reversal is believed to be the $\vec{u} \times \vec{B}$ term in Ohm's law. Then H is the only parameter in the problem. (Pm appears only in the terms associated with $\langle \vec{u} \rangle$.) The inclusion of the mean flow should modify the answer by a relative contribution of at most $\mathcal{O}(1)$. We intend to include these terms in the future; though complicated, they pose no problem of principle.

We retain only the mode $(m, n) = (1, -2)$. The $(1, 2)$ mode is ignored because modes with $m/n < 0$ (for which the mode-rational surface falls inside the plasma) are believed to be more important. We believe that this single-mode calculation is correct for $\mathcal{J}$ sufficiently close to $\mathcal{J}_c$. For sufficiently large $\mathcal{J}$, presumably bifurcations occur[10] such that the maximizing solutions consist of multiple modes. This scenario has been verified in a variety of fluid applications.[9] It is extremely difficult to compute the onset and properties of the bifurcated states, and we have not attempted to do so.

The single-mode bounding curve up to $\mathrm{H}\mathcal{J} \approx 2 \times 10^3$ is shown in Fig. 1. For $\mathcal{J}$ close to $\mathcal{J}_c$, $\bar{\varepsilon} \propto \mathcal{J} - \mathcal{J}_c$, which can be proven analytically by perturbation analysis. For $\mathrm{H}\mathcal{J} > 5 \times 10^2, \bar{\varepsilon} \approx 0.4\mathrm{H}^{0.1}\mathcal{J}^{1.1}$. For higher $\mathrm{H}\mathcal{J}$, the formation of boundary layers makes the numerical computations rather difficult; however, there is no reason to believe that the form of the single-mode bounding curve should change dramatically for larger H. For $\mathrm{H} = 400$, the F–Θ curve is shown in Fig. 2, where $F \doteq \langle B_z \rangle(\mathrm{wall})/\overline{\langle B_z \rangle}$ and $\Theta \doteq \langle B_\theta \rangle(\mathrm{wall})/\overline{\langle B_z \rangle}$. Unlike the prediction from Taylor's hypothesis, the curve

is almost linear near the reversal $F = 0$ and the reversal takes place at $\Theta_0 \approx 3.0$, which is somewhat higher than the usual experimental and simulation value ≈ 1.5. However, there is a tendency for Θ_0 to become smaller as H increases. (For H $= 100$, $\Theta_0 \approx 3.7$; for H $= 250$, $\Theta_0 \approx 3.2$.) If we extrapolate these three data points, then $\Theta_0 \approx 1.3$ for H $= 10^5$. Of course, this value must not be taken too seriously. However, it is quite plausible that the optimum theory can predict field reversal fairly close to the experimental observations. Space constraints preclude including a graph of the bounding mean magnetic fields. However, they display no unusual features, except that for currents near field reversal $\langle B_z \rangle$ has a shallow maximum near $r/a \approx 0.4$ which may not be physical.

For H $= 400$ the relative fluctuation level $\widetilde{B}/|B|$ is about 10%, which agrees with the HBTXI experiment,[1] where H can be estimated to be of this order if we assume $\nu \approx v_i^2 \tau_i$, where v_i and τ_i are the ion thermal velocity and the ion collision time, respectively. The magnetic-field fluctuations are larger than the velocity fluctuations; the energy of the former is about ten times higher than that of the latter. The fluctuation level decreases as H increases. Boundary layers with width $\mathcal{O}(\mathrm{H}^{-1/2})$ are observed. (This can be explained by balancing the linear terms with the nonlinear terms in Eq. (3) and introducing the thickness δx of a boundary layer; for field reversal $\widetilde{\vec{B}} \approx \mathrm{H}^{-1/2}$, $\widetilde{\vec{u}} \approx \mathrm{H}^{1/2}$, and $\delta x \approx \mathrm{H}^{-1/2}$.) Between $r = 0.3$ and $r = 0.7$ the current and the magnetic field are aligned. They are not aligned in the core and near the boundary. On the axis $\langle j_z \rangle(0) = E_0 - \varepsilon(0) < 0$, thus $\langle \vec{j} \rangle$ and $\langle \vec{B} \rangle$ are anti-aligned. The maximal-emf state is neither force-free nor a Taylor state.

Although various facets of this work deserve to be refined, we conclude that the utility of the optimum theory has been demonstrated for this self-consistent problem of physical interest, inasmuch as it predicts field reversal close to (of the same order as) the observed values and makes a specific prediction for the turbulent emf. Future efforts are desirable in the following areas: (1) Include more modes, pursue the bounding curve to higher H, and include the mean velocity; (2) consider the role of the helicity constraint; (3) use a nonuniform driving electric field in order to model the experiments more closely; (4) include two-point constraints[11,12] in order to include the dynamical effects of finite correlation time and length.

After the present work was completed, A. Bhattacharjee and E. Hameiri brought to our attention Ref. 18 in which a variational principle described as "minimum entropy production" is proposed that is formally very similar to the principle considered in the present work. Those authors concluded that the resulting optimum state is, in fact, a locally attracting relaxed state, a result stronger than we have been able to deduce from our strict application of the theory of bounds. At present, the implications and the rigor of Ref. 18 are not fully understood. However, the results are intriguing, and further research is desirable.

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FIGURE CAPTIONS

Figure 1. The bounding curve up to $H\mathcal{J} \approx 2 \times 10^3$.

Figure 2. The F–Θ curve for $H = 400$.

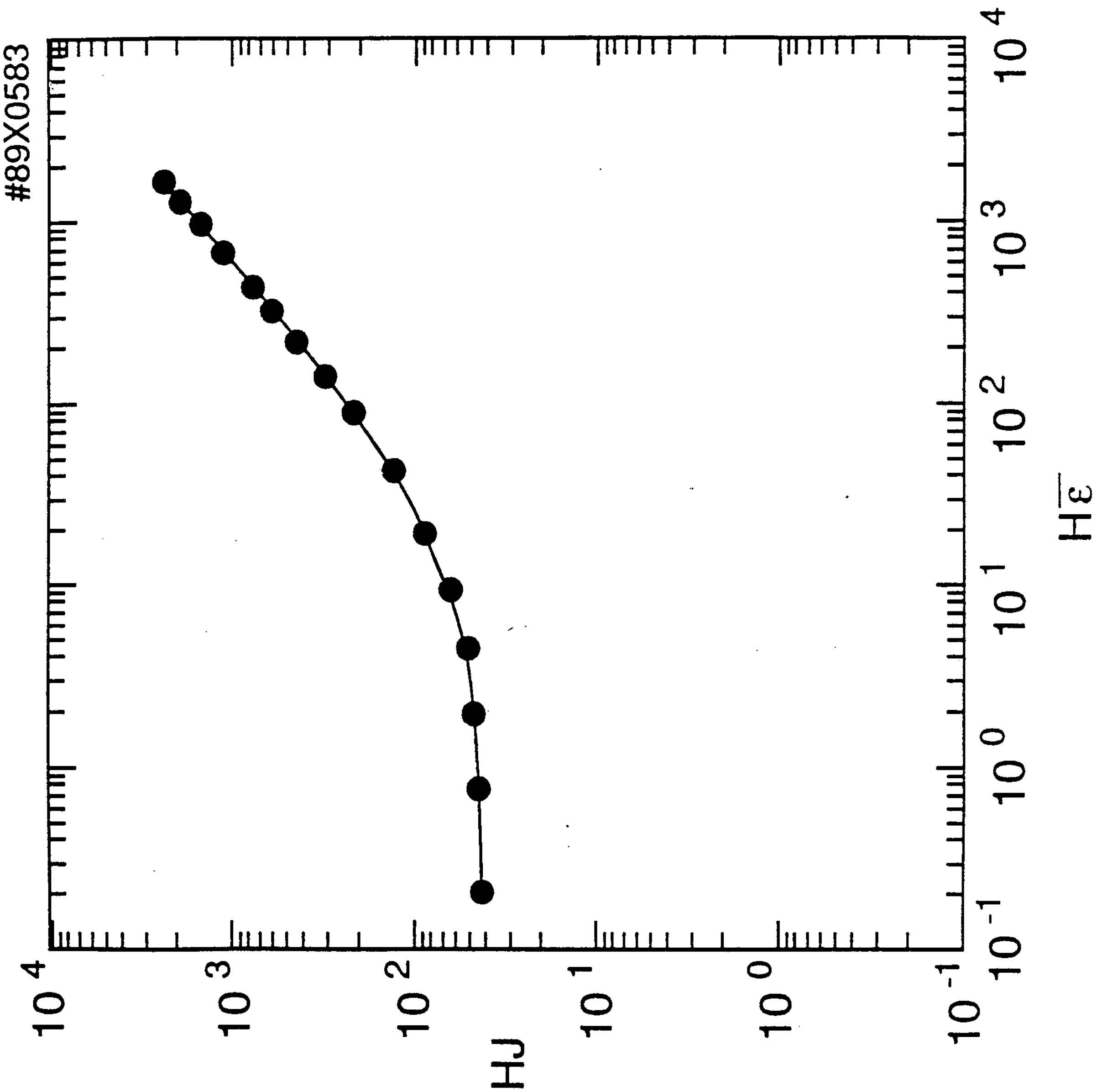

Figure 1

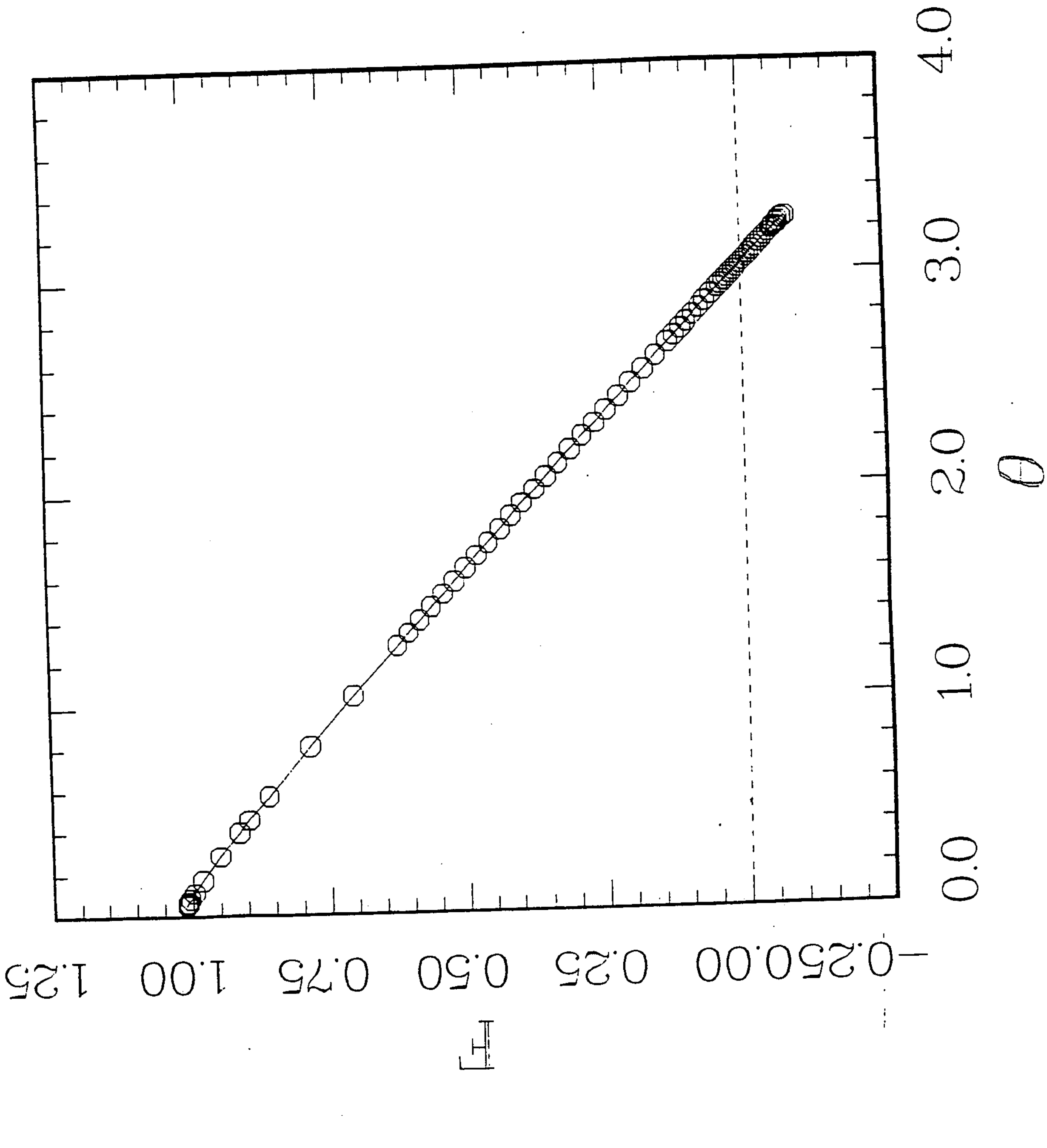

Figure 2