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Abstract

A unified theory of temperature gradient driven trapped ion modes and ballooning instabilities is developed using kinetic theory in banana regimes. All known results, such as electrostatic and purely magnetic trapped particle modes and ideal MHD ballooning modes (or shear Alfven waves) are readily derived from our single general dispersion relation. Several new results from ion-ion collision and trapped particle modification of ballooning modes are derived and discussed and the interrelationship between those modes is established.

I. Introduction

In this paper, we present a systematic analytical investigation of the low frequency and long wave-length modes, such as electrostatic and purely magnetic trapped particle modes and ballooning instabilities (or shear Alfven waves) in banana regimes. Our analytical method allows us to show interconnection among those modes proposed in each limiting case and provide a link between single particle motions and the collective plasma modes. In additon, the Lorentz collision operator is used to understand the effects of collisions on the low frequency modes.

In a toroidal plasma the existence of particles trapped in the magnetic field gives rise to a class of instabilities and kinetic modification of the ideal MHD ballooning modes.[1-8] The underlying physical mechanism which makes these electromagnetic modes so harmful to plasma confinement has been extensively discussed.[1-8] Therefore, we proceed directly to discuss interconnections among these modes and collision effects. The relation with previously derived models of anomalous transport and with experimentally observed favorable current scaling of the global energy confinement time τ_E from trapped particle modes is also discussed.

It is well known that the primary source of free energy available to drive the low frequency modes in the tokamak is the expansion energy associated with density and temperature gradients of confined plasmas due to the coupling motion between $\vec{E} \times \vec{B}$ drift and the magnetic (∇B and curvature) drift. In addition, unlike the circulating (untrapped) particles, which are free to sample many oscillations along a magnetic field line, the trapped particles in banana regimes are constrained to sample only one local poloidal magnetic well. The inherently different behavior of these particles not only causes charge separation, but also compresses and bends the magnetic field lines. This can lead to the breakdown of validity of the MHD equations and, hence, to the development of complicated electromagnetic modes. Perhaps the greatest theoretical progress lies in restricting oneself to each separate problem. Upon electrostatic approximation the sources of these expansion energy drive trapped particle modes,[1-8] while in the ideal MHD limit (where $\tilde{E}_{\|} = 0$), the expansion energy (of course mainly due to the circulating particles) drives ballooning instabilities.[9] By analyzing appropriate finite β (ratio of plasma to magnetic pressure) effects and considering the equilibrium condition $\nabla(4\pi P + B_0^2/2) = \vec{\kappa}\ B_0^2$ with $\vec{\kappa} = \vec{B}_0 \cdot \nabla\vec{B}_0/{B_0}^2$ for the curvature, Rosenbluth and Sloan[10] found that if β is increased from 0 to a critical value $\beta_c \sim$ a/R, the resulting favorable magnetic field gradient induced by the presure P of the plasma can nullify the destabilizing unfavorable field gradients associated with the vacuum magnetic field. However, whenever this happens, the reversal of the vacuum ∇B in the bad curvature regions by the diamagnetic plasma pressure P can lead to a magnetic instability having the polarization of compressional Alfven waves, i.e., with the perturbed magnetic field $\delta\vec{B} \| \vec{B}_0$ in tandem mirrors.[11] Unification of these matters in banana regimes of tokamak is a principal content of this paper. For general electromagnetic perturbations, a variational approach constructed from the gyro-kinetic equation is used to obtain a general analytic dispersion relation, which is radially local on each magnetic flux surface. It turns out to have three types of instabilities in the system: electrostatic trapped particle modes, ideal MHD ballooning instabilities and purely magnetic trapped particle modes. Usual approximations for each mode to be separated are very good because the couplings are small except for ideal ballooning modes. In the latter case, trapped particles make a stabilizing contribution in the low β and low shear limit. Collisions have a strong stabilizing impact on the purely magnetic trapped particle modes, a destabilizing effect on the electrostatic trapped particle modes, and a damping influence on the usual shear Alfven waves whenever ions are trapped. Threshold values are presented. One interesting result is that when the finite gyroradius term is retained, the trapped particles also drive dissipative drift ballooning instabilities near marginal ideal MHD ballooning stability.

We should point out that the comprehensive analysis conducted by Tang, Rewoldt, Cheng and Chance[12,13] for determining the stability properties of ballooning modes and trapped particle drift modes is different from that of our theory. Their work was numerically concerned primarily with ballooning modes and trapped electron modes, and frequencies higher than ion bounce frequency. The present paper, however, analytically investigates the long wave-length ballooning modes and trapped ion modes. This choice obviously enables a more natural understanding of anomalous ion loss from a number of recent experimental observations.[14-16] Although trapped ion modes have smaller growth rates by comparison, transport coefficients, at least in simple mixing-length estimate ($D \sim \gamma/k_\perp{}^2$, where γ is the linear growth rate and $k_\perp$ the wave number perpendicular to magnetic field) would not be necessarily smaller in magnitude since they support a large "convective cell" of width $\delta x = [k_\theta h(\theta)]^{-1}$ with $k_\perp^2 = k_\theta{}^2[1 + h^2(\theta)]$ (see Eq. (17)). Modes remain unstable for small k_θ untill the frequency falls below the ion collision rate.

The structure of the paper is as follows. In Sec. II., we discuss the variational form of the general electromagnetic modes in banana collisionality regimes. In Sec. III., we derive a general analytic dispersion relation by using a trial function of the modes and perturbed distribution function of ions. Several new results and several corrections to the well-known modes are obtained. It is shown that both trapped particle modes and ideal MHD ballooning instabilities can be derived from a single dispersion relation, which, in its simplest form, follows from the "disconnected-mode boundary condition".[21] The implication, of course, is that although trapped and untrapped particles play a different role in the perturbed electromagnetic fields, the structure of the modes is essentially the same. However, untrapped particles can extend along the entirety of the equilibrium magnetic field lines, and the "disconnected mode" approximation is not valid at the low shear limit in the magnetic field. In Sec. IV., we thus carefully investigate trapped particle modification of low shear and low β ballooning instabilities. The results are summarized and conclusions are drawn in Sec. V.

II. Variational Principle

We start with low-frequency gyro-kinetic equations derived in the high mode number limit using the eikonal representation. The appropriate set of field variables are: ϕ the electrostatic potential, $\delta B_{\parallel}$ the perturbed parallel magnetic field, and $A_{\parallel}$ the parallel component of the perturbed vector potential or, equivalently Λ, where

$$A_{\parallel} = \frac{c}{i\omega}\vec{b}\cdot\nabla\Lambda.$$

The perturbed distribution function $\tilde{f}$ is given by

$$\tilde{f} = f\exp[inS(r,\theta,\zeta) - i\omega t], \tag{1}$$

where

$$f = -\frac{q_j\phi}{T_j}F_{mj} + g_j J_0(k_\perp\rho_i)$$

with S being the usual eikonal ($\vec{k}_\perp = \nabla S$), r the minor radius, θ the poloidal angle and ζ the toroidal angle. F_{mj} is assumed to be Maxwellian for each species. In Eq. (1), g_j is the nonadiabatic portion of the perturbed distribution function and is the solution to the gyro-kinetic equation[17,18]

$$(\omega - \omega_{dj} + iv_{\parallel}\frac{\partial}{\partial l})g_j = (\omega - \omega_*^T)\frac{q_j}{T_j}F_{mj}\left[J_0\left(\phi - \frac{v_{\parallel}}{c}A_{\parallel}\right) - i\frac{v_\perp}{c}\hat{A}_\perp J_1\right] + iC(g_j), \tag{2}$$

where l is the arc length along a field line, c is the speed of light, $J_l = J_l(k_\perp\rho_j)$ is the usual Bessel function for $l = 0, 1$. $C(g_j)$ represents collisions which we approximate by the gyro-averaged pitch angle scattering operator appropriate if $k_\perp\rho_i < 1$,[2]

$$C(g_j) = \nu\frac{|v_{\parallel}|}{B}\frac{\partial}{\partial\mu}\left[|v_{\parallel}|\mu\frac{\partial g_j}{\partial\mu}\right], \tag{3}$$

Here we list below the definitions of the terms which appear in Eqs. (1)-(3).

$$\omega_{*j}^T = \frac{cT_j\vec{k}\times\vec{b}\cdot\nabla F_{mj}}{q_jBF_{mj}} = \omega_{*j}\left[1 + \eta_j(\frac{E}{T_j} - \frac{3}{2})\right],$$

$$\omega_{dj} = \omega_{kj} - \frac{c\mu}{q_jB^2}\sum_\alpha \vec{k}\times\vec{b}\cdot\nabla P_\alpha,$$

$$\omega_{kj} = \hat{\omega}_{kj}(2E - \mu B)/T_j, \hat{\omega}_{kj} = \frac{cT_j}{q_jB}\left(\vec{\kappa}\times\vec{b}\cdot\vec{k}\right),$$

$$
\begin{aligned}
\vec{\kappa} &= (\vec{b}\cdot\nabla\vec{b}),\\
\mu &= \frac{m_j v_\perp^2}{2B^2},\\
E &= \frac{m_j v^2}{2},\\
\nu_{ei} &= 2\left(\frac{T_e}{E}\right)^{3/2}\frac{\omega_{pe}^2 e^2 m_e^{1/2}}{(2T_e)^{3/2}}\ln\Lambda\left\{1+H\left[\left(\frac{E}{T_e}\right)^{1/2}\right]\right\},\\
\nu_{ii} &= 2\left(\frac{T_i}{E}\right)^{3/2}\frac{\omega_{pi}^2 e^2 m_i^{1/2}}{(2T_i)^{3/2}}\ln\Lambda H\left[\left(\frac{E}{T_i}\right)^{1/2}\right],\\
H(z) &= \frac{1}{\pi^{1/2}}\frac{1}{z}e^{-z^2}+\left(1-\frac{1}{2z^2}\right)\frac{2}{\pi^{1/2}}\int_0^z dte^{-t^2}
\end{aligned}
\tag{4}
$$

with P_j the pressure of the jth species. $\omega_{*j} = -k_\theta\rho_j v_{thj}/(2L_{nj})$ is the diamagnetic drift frequency, $\rho_j = v_{thj}/\Omega_{cj}$ the Larmor radius, $v_{thj} = \sqrt{(2T_j/m_j)}$ the thermal speed, $\Omega_{cj} = q_jB/(m_jc)$ the cyclotron frequency, $L_{nj} = -(d\ln n_j/dr)^{-1}$ the density gradient scale length, $k_\theta = -m/r$ the poloidal wave vector, and $\eta_j = L_{nj}/L_{Tj}$ with L_{Tj} the temperature gradient scale length.

Eq. (2) may be rewritten by noting $C(F_{mj}) = 0$ and $C(\omega_{*j}^T F_{mj}) = 0$ and defining

$$
g_j = \left(1-\frac{\omega_{*j}^T}{\omega}\right)\frac{q_jF_{mj}}{T_j}J_0\Lambda + h_j, \tag{5}
$$

and

$$
\psi = \phi - \Lambda
$$

to obtain

$$
(\omega-\omega_{dj}+iv_\parallel\frac{\partial}{\partial l})h_j = \left(\omega-\omega_{*j}^T\right)\frac{q_j}{T_j}F_{mj}\left[J_0\left(\psi-\frac{\omega_d}{\omega}\Lambda\right)+\frac{v_\perp^2}{2\Omega_{cj}c}\delta B_\parallel\right]+iC(h_j), \tag{6}
$$

where we have neglected a small term proportional to $C(J_0)$. Since we are interested in $k_\perp\rho_i < 1$, we have taken the lowest order expansion for $J_1 = (k_\perp\rho_i/2)(v_\perp/v_{thi})$ and used $J_0 = 1-((k_\perp\rho_i)^2/4)(v_\perp/v_{thi})^2$.

In this form, the gyro-kinetic equation (6) is solved for h_j and the result is used in the quasineutrality condition

$$
-\frac{n_0e^2}{T_i}(1+\tau^{-1})\psi - \frac{n_0e^2}{T_i}\frac{k_\perp^2\rho_i^2}{2}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\Lambda + \sum_j q_j\int d^3vh_jJ_0 = 0, \tag{7}
$$

with $\tau = T_e/T_i$ and in the component of Ampere's law in the $\hat{e}_\perp$ direction as follows:

$$\delta B_\parallel = -ik_\perp A_\perp = \sum_j \frac{4\pi n_j q_j}{B} \frac{\omega_{*j}}{\omega}(1+\eta_j)\Lambda - B_1. \tag{8a}$$

Here B_1 is defined as

$$B_1 = 4\pi \sum_j q_j \int d^3v \frac{v_\perp^2}{2\Omega_{cj}c} h_j. \tag{8b}$$

Note that B_1 has the polarization of compressional Alfven modes.

By taking the $\sum_j q_j \int d^3v$ moment of the gyro-kinetic equation (6), and by making use of the quasineutrality condition, the number conservation property of the collision operator, and the parallel component of Ampere's law, this moment equation becomes

$$\begin{aligned} \frac{c^2}{4\pi\omega^2}\frac{\partial}{\partial l}\left(k_\perp^2 \frac{\partial \Lambda}{\partial l}\right) = &- \frac{n_0 e^2}{T_i}\frac{k_\perp^2\rho_i^2}{2}\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right](\psi + \Lambda) \\ &- \sum_j \frac{n_j q_j^2}{T_j}\frac{\omega_{*j}\hat{\omega}_{dj}}{\omega^2}(1+\eta_j)\Lambda - \sum_j \frac{n_j q_j}{B}\frac{\omega_{*j}}{\omega}(1+\eta_j)\delta B_\parallel \\ &+ \sum_j q_j \int d^3v \frac{\omega_{dj}}{\omega} J_0 h_j. \end{aligned} \tag{9}$$

Substituting Eq. (8a) into (6) and (9), one easily finds that this only leads to the cancellation of the diamagnetic pressure terms from ω_d. Thus these two equations have simple forms

$$\begin{aligned} \frac{c^2}{4\pi\omega^2}\frac{\partial}{\partial l}\left(k_\perp^2 \frac{\partial \Lambda}{\partial l}\right) = &- \frac{n_i e^2}{T_i}\frac{k_\perp^2\rho_i^2}{2}\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right](\psi + \Lambda) \\ &- \sum_j \frac{n_j q_j^2}{T_j}\frac{\omega_{*j}\hat{\omega}_{kj}}{\omega^2}(1+\eta_j)\Lambda + \sum_j q_j \int d^3v \frac{\omega_{kj}}{\omega} J_0 h_j, \end{aligned} \tag{10}$$

and

$$(\omega - \omega_{dj} + iv_\parallel \frac{\partial}{\partial l})h_j = \left(\omega - \omega_{*j}^T\right)\frac{q_j}{T_j}F_{mj}\left[J_0\left(\psi + \frac{\omega_k}{\omega}\Lambda\right) - \frac{v_\perp^2}{2\Omega_{cj}c}B_1\right] + iC(h_j). \tag{11}$$

If we consider the evolution in Eq. (11) to occur on a long time scale associated with the drifting motion of particles across field line and collision, we have the following expansion procedure. To the lowest order, Eq. (11) requires

$$v_\parallel \frac{\partial h_j}{\partial l} = 0$$

that is, h_j is a constant along a field line.

The next order Eq. (11) when averaged over the bounce motion of the particle yields an evolution equation for h_j

$$(\omega - \overline{\omega}_{dj})h_j = \left(\omega - \omega_*^T\right) \frac{q_j}{T_j} F_{mj} \left[J_0 \left(\overline{\psi + \frac{\omega_k}{\omega}\Lambda} \right) - \overline{\frac{v_\perp^2}{2\Omega_{cj} c} B_1} \right] + i\overline{C(h_j)}, \qquad (12)$$

with $\overline{G} = (\int dlG/|v_\parallel|)/(\int dl/|v_\parallel|)$ being averaged over a trapped-particle trajectory. Note that parallel compression (by extension, acoustic wave) has been removed from the analysis by bounce-averaging in restricting the width of the modes. Thus, some modes of interest, e.g., slab-like η_i, modes are not considered.

We now consider the boundary conditions on h_j. Since the distribution function is continuous at the boundary between trapped and untrapped particles, and the nonadiabatic circulating particle response is negligible, i.e., $h_j \sim O(\frac{\omega}{\omega_{tj}}) \ll 1$ (this removes a potential dissipation mechanism, namely, Landau-damping on the non-adiabatically responding circulating ions which must be taken into account in the radially nonlocal analysis), we may impose a boundary condition on h_j

$$h_j(E, \mu)|_{\mu = \frac{E}{B_{max}}} = 0. \qquad (13)$$

For deeply trapped particles ($v_\parallel = 0$), we choose

$$\frac{\partial h_j(E, \mu)}{\partial \mu}|_{\mu = \frac{E}{B_{min}}} < \infty. \qquad (14)$$

The bounce-averaged drift kinetic equation (12) together with the boundary conditions (Eqs. (13) and (14)), the quasineutrality condition Eq. (7), and moment equation (10) complete the formal specification of the problem.

We now construct the variational form. Multiplying Eqs. (7), (8*b*), (10) and (12) by ψ/B_0, B_1/B, Λ/B_0 and $h_j/[F_{mj}(\omega - \omega_{*j}^T)]$, integrating along a field line and over velocity space, and adding them together, we obtain the following variational principle:

$$Q(h_j, \psi, \Lambda, B_1) = \frac{c^2}{4\pi\omega^2} \int \frac{dl}{B} \left(k_\perp \frac{\partial \Lambda}{\partial l} \right)^2 - \frac{n_i e^2}{T_i} \left[1 - \frac{\omega_{*i}}{\omega}(1 + \eta_i) \right] \int \frac{dl}{B} \frac{k_\perp^2 \rho_i^2}{2} (2\psi\Lambda + \Lambda^2)$$
$$- \sum_j \frac{n_j q_j^2}{T_j} \frac{\omega_{*j}}{\omega^2}(1 + \eta_j) \int \frac{dl}{B} \hat{\omega}_{kj} \Lambda^2 - \frac{n_0 e^2}{T_i}(1 + \tau^{-1}) \int \frac{dl}{B} \psi^2$$

$$
\begin{aligned}
&-2\pi\sum_j \frac{T_j}{m_j^2}\int_{Tr}\frac{dl}{|v_\parallel|}dEd\mu\left\{\left[\frac{h_j^2(\omega-\overline{\omega_{dj}})}{F_{mj}(\omega-\omega_{*j}^T)}\right]-i\frac{h_j\overline{C}(h_j)}{F_{mj}(\omega-\omega_{*j}^T)}\right\}\\
&+\frac{1}{4\pi}\int\frac{dl}{B}B_1^2\\
&+2\sum_j q_j\int\frac{dl}{B}\int_{Tr}d^3vh_j\left[J_0\left(\psi+\frac{\omega_{kj}}{\omega}\Lambda\right)-\frac{v_\perp^2}{2\Omega_{cj}c}B_1\right].
\end{aligned}
\tag{15}
$$

The velocity space integration in Eq. (15) is understood to be performed on trapped populations. We observe that by demanding $Q(h_j,\psi,\Lambda,B_1)$ to be stationary with respect to h_j, ψ, Λ and B_1, we reproduce Eqs. (12), (7), (10) and (8*b*), respectively. It further follows from the kinetic equation, the quasineutrality condition, and the first moment equation that, for the functions in which Q is stationary, the value of Q is zero.

We emphasize that h_j in the variational form (15) was constrained to satisfy the kinetic equation (12). We then use Eq. (12) to rewrite the final term in Eq. (15) and obtain

$$
\begin{aligned}
Q(\psi,\Lambda,B_1)=&\frac{c^2}{4\pi\omega^2}\int\frac{dl}{B}\left(k_\perp\frac{\partial\Lambda}{\partial l}\right)^2+\frac{1}{4\pi}\int\frac{dl}{B}B_1^2\\
&-\sum_j\frac{n_jq_j^2}{T_j}\frac{\omega_{*j}}{\omega^2}(1+\eta_j)\int\frac{dl}{B}\hat{\omega}_{kj}\Lambda^2-\frac{n_0e^2}{T_i}(1+\tau^{-1})\int\frac{dl}{B}\psi^2\\
&+2\pi\sum_j\frac{T_j}{m_j^2}\int_{Tr}\frac{dl}{|v_\parallel|}dEd\mu\left\{\left[\frac{h_j^2(\omega-\overline{\omega}_{dj})}{F_{mj}(\omega-\omega_{*j}^T)}\right]-i\frac{h_j\overline{C}(h_j)}{F_{mj}(\omega-\omega_{*j}^T)}\right\}\\
&-\frac{n_ie^2}{T_i}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\int\frac{dl}{B}\frac{k_\perp^2\rho_i^2}{2}(2\psi\Lambda+\Lambda^2).
\end{aligned}
\tag{16}
$$

The terms in Eq. (16) have the following simple physical interpretation. The first term represents the energy required to bend magnetic field lines, the second term is the work done in compressing the magnetic field and plasma, and the third term drives the ballooning and interchange modes. The fourth term is the energy needed to produce charge separation due to the longitudinal electric field and the fifth group of terms which represent the energy required to compress the plasma in the MHD model, are the interactions among various electromagnetic perturbations through the plasma. The last term is polarization and the finite Larmor radius term of ions. If we take the limit $\overline{C}(h_j)=0$, Eq. (16) is the collisionless and the low-frequency kinetic energy principle for isotropic equilibrium pressure in high toroidal mode number limit($n \to \infty$).[19,20] It is worth to point out that the similar variational form was also constructed in the electrostatic limit in the tandem mirror.[24]

We note that if we substitute into Eq. (16) three test functions Λ, ψ and B_1, with a first-order error, the eigenvalue ω will have an error of second order.

In what follows, we consider for simplicity shifted circular magnetic surface equilibria, and the equilibrium magnetic field strength is $B = B_0/\overline{h}(\theta)$, where $\overline{h}(\theta) = 1 + \varepsilon\cos\theta$, $\varepsilon = r/R_0$, and R_0 is the major radius at the center of the selected magnetic surface. For the model MHD equilibrium,[9] we then have

$$\begin{aligned} \vec{k_\perp} &= n\nabla S = -k_\theta[\vec{e}_\theta + h(\theta)\vec{e}_r], \\ \vec{b}\times\vec{\kappa}\cdot\vec{k} &= \frac{k_\theta}{R_0}[\cos\theta + h(\theta)\sin\theta], \\ h(\theta) &= \hat{s}(\theta-\theta_k) - \alpha(\sin\theta - \sin\theta_k), \\ \frac{\partial}{\partial l} &= (\vec{b}\cdot\nabla\theta)\frac{\partial}{\partial\theta} = \frac{1}{R_0 q}\frac{\partial}{\partial\theta}, \end{aligned} \tag{17}$$

where $\hat{s} \equiv d\ln q/d\ln r$ measures the average shear, and $\alpha = -q^2R_0 d\beta/dr$ is a measure of the local Shafranov shift, with $\beta \equiv 8\pi\sum_j n_jT_j/B_0^2$. For the latter calculation, we will take $\theta_k = 0$, in the lowest order in the ballooning hierarchy, corresponding to the fastest growing modes. Also, $v_\parallel(\theta) = \sigma v[1-\mu B_0/\overline{h}(\theta)E]^{1/2}$, and $v_\perp(\theta) = v[\mu B_0/\overline{h}(\theta)E]^{1/2}$, where $\sigma = \pm 1$. Circulating particles correspond to $0 \le \mu B_0/E < 1-\varepsilon$, and trapped particles to $1-\varepsilon < \mu B_0/E \le \overline{h}(\theta)$, at a given θ.

III. Dispersion Relation

This section is divided into four parts. In the first part, the quadratic form of Eq. (16) is solved to obtain a general dispersion relation. The dissipative drift ballooning instabilities and the damping shear Afven waves driven by trapped particle collisions are discussed in the second part, and the third part is devoted to the study of the effect of ion-ion collisions on temperature gradient driven trapped particle modes. Finally the purely magnetic trapped particle modes are derived.

A. General Dispersion Relation

To proceed, we will consider the frequency ranges of

$$\omega_{be}, \omega_{te} \gg \nu_{ei,eff} \gg \omega, \omega_{de},$$

$$\omega_{bi}, \omega_{ti} \gg \omega, \omega_{di}, \nu_{ii,eff}. \tag{18}$$

Due to collisions the electron response is adiabatic, that is, $h_e = 0$ and trapped electron drift modes are dropped at this point. In order to find out the ion nonadiabatic response, h_i, we construct a trial function and seek for the qualitative form of the true solution by recalling the difference in response between trapped and untrapped ions. The collision operator acts mainly through pitch angle scattering to enforce that h_i vanishes at the trapping boundary. Thus, we write the simplest trial function in following form

$$h = h_0(\omega, E, \psi_0, \Lambda_0, B_{10})(\mu - \mu_t), \tag{19}$$

with the variational parameter $h_0(\omega, E, \psi_0, \Lambda_0, B_{10})$. Here $\mu_t = (E/B_0)(1-\varepsilon)$ is the trapping boundary and ψ_0, Λ_0 and B_{10} are three constants along the magnetic field lines.

By looking at the quasineutrality condition Eq. (7) and B_1 in Eq. (8), we can make the following assumptions:

$$\begin{aligned} \psi &= \psi_0(1+cos\theta), \\ B_1 &= B_{10}(1+cos\theta), \\ \Lambda &= \Lambda_0(1+cos\theta). \end{aligned} \tag{20}$$

These functions are even in θ and satisfy the disconnected-mode boundary conditions whose eigenfunctions vanish at the inside of the torus. Note that they give reasonable values for the marginal stability points where $\omega = 0$ only for high shear ballooning modes. The physical basis is that the destabilizing effects associated with trapped particles and unfavorable curvature are known to be strongest at the outside of the torus.

By substituting the trial functions Eqs. (19) and (20) into the variational form Eq. (15), performing relevant θ-integrals and pitch angle integrals, and making a variation of quantity Q with respect to variable h_0, we obtain

$$h_0 = 1.29\frac{eF_m}{T_i}(\omega - \omega_{*i}^T)\frac{(1 - b_\perp x^2)\left(\psi_0 + \frac{b\omega_{k0}}{\omega}x^2\Lambda_0\right) - \frac{T_i}{eB_0}x^2 B_{10}}{\omega - a_1\omega_{k0}x^2 + 0.4i\nu_{ii,eff}}. \tag{21}$$

Here

$$\begin{aligned}
\omega_{k0} &= \frac{k_\theta \rho_i v_T}{R_0}, \\
a_1 &= 0.36 + 0.21\hat{s} - 0.21(1 + \frac{7}{6q^2})\alpha, \\
b &= 0.38 - \alpha + 0.22\hat{s}, \\
b_\perp &= \frac{(k_\perp \rho_i)^2}{2}, \\
x &= \frac{v}{v_{Th}}.
\end{aligned}$$

From Eq. (21), we can see that except for the numerical difference, the energy-dependent Krook collision operator is a good approximation. The fact that the magnetic drift frequency ($a_1\omega_{k0}$, $b\omega_{k0}$) depends on α represents a measure of the diamagnetic plasma pressure.

We now substitute trial functions Eqs. (19) with Eq. (21) and (20) back into the quadratic form (15), and construct the variations of quantity Q with respect to variables B_{10} and Λ_0, respectively. This manipulation then produces the following dispersion relation:

$$\begin{aligned}
&\left[-3(1+\tau^{-1}) + \delta_1 - \frac{\delta_2^2}{6/\beta_i + \delta_3}\right] \\
&\times \left\{I_1 - \alpha I_2 + \delta_3 \beta q^2 b^2 - \frac{\delta_3^2}{6/\beta_i + \delta_3}\beta q^2 b^2 - \frac{3}{2} I_3 \Omega^2 \left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\right\} \\
&- (k_\theta \rho_i)^2 \left\{-\frac{3}{2} I_3 \Omega^2 \left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right] + \frac{\delta_2 \sqrt{\beta} b q}{k_\theta \rho_i} - \frac{\delta_2 \delta_3}{6/\beta_i + \delta_3}\frac{\sqrt{\beta} b q}{k_\theta \rho_i}\right\}^2 = 0.
\end{aligned} \tag{22}$$

Here

$$\begin{aligned}
\Omega &= \frac{\omega}{\omega_A}, \\
I_1 &= \frac{3}{4}\alpha^2 - \frac{8}{3}\hat{s}\alpha + \hat{s}^2(\frac{\pi^2}{3} - \frac{1}{2}) + 1, \\
I_2 &= 2 + \frac{5}{3}\hat{s} - \frac{5}{4}\alpha, \\
I_3 &= \frac{5}{12}\alpha^2 - \frac{10}{9}\hat{s}\alpha + \hat{s}^2(\frac{\pi^2}{3} - \frac{15}{16}) + 1, \\
\delta_l &= \frac{29.64\sqrt{2}\varepsilon}{\pi^{\frac{3}{2}}} \int_0^\infty dx x^{2l} e^{-x^2}(1 - b_\perp x^2)\frac{[\omega - \omega_{*i}(1 - \frac{3}{2}\eta_i)] - \omega_{*i}\eta_i x^2}{\omega - a_1\omega_{k0}x^2 + 0.4i\nu_{ii,eff}}.
\end{aligned} \tag{23}$$

with $\omega_A = v_A/qR_0$ the transit frequency of a shear Alfven wave between the regions of good and bad curvature and with $v_A = \sqrt{B^2/4\pi n_i m_i}$ the Alfven velocity. We only keep finite Larmor radius corrections in δ_1 which is crucial to reverse propagation from the electron to the ion direction in usual electrostatic trapped ion ∇P-driven modes (see Eq. (31)),[6] instead for all δ_l.

The ideal MHD ballooning instabilities (or shear Alfven waves) and trapped particle modes (electrostatic and magnetic) are three roots of the equation (22) and are coupled via the finite-ion-gyroradius term and the diamagnetic pressure evolution by compression across the magnetic field due to the last group of terms. Because of the large differences in the various mode frequencies, which are given by $\omega^2 \gg \omega_* \omega_{k0}$, $\omega^2 \sim \omega_* \omega_{d0}$ and $\omega^2 \sim \sqrt{\varepsilon}\beta\omega_*\omega_{d0}$, respectively, it is impossible for resonances to occur among these modes, except near marginal MHD stability.

B. Dissipative Drift Ballooning Instabilities and Damped Shear Alfven Waves

When $\omega^2 \gg \omega_* \omega_{k0}$, we obtain ballooning instabilities from Eq. (22)

$$\frac{3}{2} I_3 \Omega^2 \left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right] = I_1 - \alpha I_2 + \delta_3 \beta q^2 b^2, \tag{24}$$

where we have neglected the compressional Alfven wave terms, charge separation due to the longitudinal electric field, and coupling terms, which are at least order $\varepsilon^{1/2}$ smaller than the trapped particle term in Eq. (24). If we ignore the finite Larmor radius and trapped particle effects, Eq. (24) is the usual finite β ideal MHD ballooning modes dispersion relation,[21] which clearly shows the two marginal stability boundary points of α for fixed $\hat{s}$

$$2\alpha^2 - \frac{13}{3}\hat{s}\alpha - 2\alpha + \hat{s}^2\left(\frac{\pi^2}{3} - \frac{1}{2}\right) + 1 = 0. \tag{25}$$

The trapped particle terms without collisions tend to contribute stabilizing effects on ideal MHD ballooning modes but are relatively weak since they are order $\varepsilon^{3/2}$ smaller than the pressure gradient driving term. However, collisional effects introduced by trapped particles near marginal stability are worthy of being discussed in detail. Eq. (24) can be rewritten as

$$(\omega - \omega_1)(\omega - \omega_2) = \frac{2}{3I_3}\delta_3 \beta q^2 b^2 \omega_A^2, \tag{26a}$$

where

$$\omega_{1,2} = \frac{1}{2}\left[\omega_{*i}(1+\eta_i) \pm \sqrt{[\omega_{*i}(1+\eta_i)]^2 + 4I_4^2\omega_A^2}\right],$$

and

$$I_4 = \frac{2}{3}\frac{I_1 - \alpha I_2}{I_3}.$$

At marginal stability, we have

$$\omega_{*i}(1+\eta_i) \gg I_4\omega_A.$$

By assuming $\omega = \omega_{*i}(1+\eta_i) + i\gamma$ with $\omega_{*i}(1+\eta_i) \gg \gamma$, from Eq. (26$a$) we obtain

$$\gamma = -\frac{b^2}{I_3}\frac{\omega_s^2 q^2}{\omega_{*i}^2}\frac{\eta_i}{(1+\eta_i)^3}\frac{19.76\sqrt{2\varepsilon}}{\pi^{\frac{3}{2}}}\int_0^\infty dx\nu_{ii,eff}\frac{\left(\frac{5}{2}-x^2\right)x^6 e^{-x^2}}{1+[0.4\frac{\nu_{ii,eff}}{\omega_{*i}(1+\eta_i)}]^2}. \qquad (26b)$$

where $\omega_s = \sqrt{\beta}\omega_A = v_{thi}/(qR_0)$ is the sound transit frequency. The threshold for the dissipative drift ballooning instabilities is given by

$$\eta_i > 0$$

and

$$\nu_{ii,eff}|_{cr} \geq 1.3\omega_{*i}(1+\eta_i). \qquad (26c)$$

Since the frequency ordering of Eq. (18) was assumed in arriving at the dissipative drift ballooning modes, and if we estimate $\omega_{ti} \simeq v_{th}/(2qR_0)$ for typical tokamak parameters, the results thus apply only to perturbations with sufficiently low toroidal mode numbers $n < 10 - 20$.[41]

On the other hand, if α is further below the marginal stability threshold and satisfies the following conditions

$$\omega_{bi} > I_4\omega_A \gg \omega_{*i}(1+\eta_i), \qquad (27a)$$

the usual shear Alfven waves are excited. By substituting $\omega = \pm I_4\omega_A + i\gamma$ with $I_4\omega_A > \gamma$ and $\omega_{1,2} = \pm I_4\omega_A$ into Eq. (26a), we obtain

$$\gamma = -\frac{\beta q^2 b^2}{I_3 I_4^2}\frac{7.81\sqrt{2\varepsilon}}{\pi^{\frac{3}{2}}}\int_0^\infty dx\nu_{ii,eff}\frac{x^6 e^{-x^2}}{1+[0.4\frac{\nu_{ii,eff}}{\omega_A I_4}]^2}. \qquad (27b)$$

Therefore, we find that the introduction of trapped particle ion-ion collisions results in the damping of shear Alfven waves.

C. Effect of Collisions on Ion Temperature Gradient Drive Trapped Particle Modes

If we assume that $\omega^2 \sim \omega_* \omega_{d0}$, we obtain normal electrostatic trapped particle modes

$$0 = -3(1+\tau^{-1}) + \frac{29.64\sqrt{2\varepsilon}}{\pi^{\frac{3}{2}}} \times$$
$$\times \int_0^\infty dx x^2 e^{-x^2}(1-b_\perp x^2)\frac{[\omega - \omega_{*i}(1-\frac{3}{2}\eta_i)] - \omega_{*i}\eta_i x^2}{\omega - a_1\omega_{k0}x^2 + 0.4i\nu_{ii,eff}}. \tag{28}$$

Here we have gotten rid of the coupling terms which are quite small. We now examine the dispersion relation (28). The first term in Eq. (28) is the adiabatic electron and ion response. The second set of terms represents the non-adiabatic trapped ion response. These terms survive because the trapped ions respond differently than circulating ions and electrons, resulting in incomplete charge cancellation, and hence trapped particle modes.

In low collision frequency regimes $\omega_{*i} > \omega > \omega_{k0}, \nu_{ii,eff}$, an analytical solution can be found as

$$\frac{1.27(1+\tau^{-1})}{\sqrt{2\varepsilon}} + \frac{\omega_{*i}}{\omega}[1 - 3b_\perp(1+\eta_i)] + \frac{3a_1}{2}\frac{\omega_{*i}\omega_{k0}}{\omega^2}(1+\eta_i) -$$
$$- \frac{0.16i}{3\sqrt{\pi}}\frac{\omega_{*i}\nu_{ii,eff}}{\omega^2}(1-\frac{3}{2}\eta_i) = 0. \tag{29}$$

If $\eta_i > 2/3$ but $\eta_i \sim 1$, we obtain

$$\omega = -\frac{\sqrt{2\varepsilon}}{1.27(1+\tau^{-1})}[1-3b_\perp(1+\eta_i)]\omega_{*i} - i\frac{0.28(1+\tau^{-1})}{\sqrt{2\varepsilon}}(1-\frac{3}{2}\eta_i)\nu_{ii,eff}. \tag{30}$$

Eq. (30) is the well known dissipative trapped ion modes, which propagate in the electron diamagnetic drift direction.[22]

If $\eta_i > 1/\sqrt{2\varepsilon}$, both ion curvature drift and ion-ion collision drive the instability:

$$\omega = -\frac{\sqrt{2\varepsilon}}{2.54(1+\tau^{-1})}[1-3b_\perp(1+\eta_i)]\omega_{*i} \pm \frac{\sqrt{2\varepsilon}}{2.54(1+\tau^{-1})}\times$$
$$\left\{[1-3b_\perp(1+\eta_i)]^2\omega_{*i}^2 - \frac{5.08(1+\tau^{-1})}{\sqrt{2\varepsilon}}[\frac{3a_1}{2}\omega_{k0} + \frac{0.8i}{\sqrt{\pi}}\nu_{ii,eff}]\omega_{*i}\eta_i\right\}^{\frac{1}{2}}. \tag{31}$$

Now we analyze this dispersion relation in two limits: the collisionless and collisional regimes.

(i). ∇P_i-Driven Trapped Particle Modes

If $a_1\omega_{k0} > \nu_{ii,eff}$, we obtain the collisionless trapped ion ∇P_i-driven modes with growth rate[6]

$$\gamma \sim \varepsilon^{\frac{1}{4}}\sqrt{a_1\omega_{k0}\omega_{*T_i}}. \tag{32}$$

where $\omega_{*T_i} = \omega_{*i}\eta_i$. When $3b_\perp(1+\eta_i) > 1$, these modes may propagate in the ion diamagnetic drift direction. However with increased plasma diamagnetism, the magnetic drift becomes negative and the system is stable. We may determine α_{cr} by noting that a_1=0 at drift reversal. The stability threshold is

$$\alpha_{cr} > \frac{1.71+\hat{s}}{1+\frac{7}{6q^2}}, \tag{33}$$

which is slightly bigger than the first ballooning instability limit, given by Eq. (25). For $\hat{s}=1$, $q=1$, we obtain $\alpha_{balloon,cr1} = 0.81$ and $\alpha_{balloon,cr2} = 2.37$ while $\alpha_{trap,cr} = 1.25$.

Remember that our frequency range is

$$\omega_{bi}, \omega_{ti} \gg \omega \gg \omega_{di} \gg \nu_{ii,eff}$$

which thus corresponds to the range for the growing wavelengths,

$$\frac{\nu_{ii,eff}R_0}{v_{thi}} \ll k_\theta \rho_i \ll \frac{\varepsilon^{\frac{3}{4}}}{q}. \tag{34}$$

The resulting anomalous ion thermal diffusivity from a mixing-length estimate is thus given by

$$\chi_i^{\nabla P} \sim \frac{\epsilon^{\frac{5}{4}}}{[1+h^2(\theta)]\nu_{ii,eff}}\frac{\rho_i^2 v_{thi}^2 a_1^{\frac{3}{2}}}{\sqrt{L_T R_0^3}}. \tag{36}$$

which is similar to the result given by Biglari, Diamond and Rosenbluth.[6] But our model explicitly indicates the θ and α dependence of χ_i. As the diamagnetic plasma pressure gradient α approaches the threshold α_{cr}, the ion thermal diffusivity greatly reduced and eventually vanishes at the threshold.

(ii). Enhanced Dissipative Trapped Ion Modes

For $\nu_{ii,eff} \gg a_1\omega_{k0}$, we find that the enhanced dissipative trapped ion modes become dominant with complex frequency,

$$\omega \sim i^{\frac{1}{2}}\epsilon^{1/4}(\nu_{ii,eff}\omega_{*T_e})^{1/2}. \tag{37}$$

which always propagates in the electron diamagnetic drift direction.

From our frequency ordering, the maximum growing wavelength is determined by

$$k_\theta\rho_i \gg \frac{\nu_{ii,eff}L_T}{v_{thi}}\varepsilon^{-\frac{1}{2}}. \tag{38}$$

Using Eqs. (37) and (38), we may provide an estimate of the ion thermal diffusivity

$$\chi_i^{\nu_{ii}} \sim \frac{\epsilon^2}{[1+h^2(\theta)]\nu_{ii,eff}}\frac{\rho_i^2 v_{thi}^2}{L_T^2}. \tag{39}$$

Comparison of χ_i in Eqs. (36) and (39) shows a large difference in magnitude but similar scalings and there is no explicitly α-dependence in (39). Both models show very unfavorable temperature scaling ($\chi_i \sim T^{7/2}$), experimentally observed favorable current scaling, and characteristics of ballooning modes ($\chi_i \sim [1+(\hat{s}\theta)^2]^{-1}$ for high shear $\hat{s} > \alpha$ in magnetic field).

Eq. (28) can be solved numerically to obtain the instability threshold η_{ic} for $\eta_i \gg 1$ and $\omega, \nu_{ii,eff} \gg a_1\omega_{k0}$. By setting $\omega = \omega_r$, we multiply the denominator in Eq. (28) by their complex conjugates, and separate the real and imaginary parts of the resulting expressions. These equations are then simutaneously solved numerically to give the threshold for instability:

$$\eta_i > \eta_{ic} = 0.65\frac{\nu_{ii,eff}}{\omega_{*i}}\frac{(1+\tau^{-1})}{\sqrt{2\varepsilon}}. \tag{40}$$

Eq. (40) also can be interpreted as a threshold for ion-ion collisions for fixed η_i, and shows that at high collision frequency regimes, there are no unstable η_i-type modes.

D. Purely Magnetic Trapped Modes

We now consider the magnetic trapped particle modes with $\omega^2 \sim \sqrt{\varepsilon}\beta\omega_{*i}\omega_{d0}$. The crude picture of the instability has been primarily given by Rosenbluth.[11] Trapped particles (with small $v_{\parallel}$) tend to move on a constant-B surface. Thus the energy of these particles is nearly a function of $|\vec{B}|$. Supposed that $|\vec{B}|$ is perturbed by an amount B_1, then the energy change of single particle is μB_1. Therefore any purely magnetic perturbation can cause energy transfer from the magnetic field to the transverse motion of trapped particles. If and only if the diamagnetic pressure due to the particles trapped in unfavorable curvature is just sufficient to reverse the vacuum gradient of B, then the region where flux lines become sparse can be positive particle energy regions and the flux will become still more sparse, i.e., the system will be unstable.

From Eq. (22), the dispersion relation of purely magnetic trapped particle modes is given by

$$\frac{6}{\beta_i} + \delta_3 = 0. \tag{41}$$

Eq. (41) resembles Eq. (28) but with a different sign, which exactly represents different energy sources to drive these two ion instabilities. For trapped ion responses, these two instabilities are essentially similar, but the former instabilities are caused by magnetic expansion energy whereas the second are driven by ion-electron electrostatic energy which is negative. This fact shows that whenever the usual trapped particle modes in Eq. (28) are stabilized by diamagnetic pressure, the purely magnetic trapped particle modes ensue. We use a similar method to discuss the dispersion relation in the two regimes: collisionless and collisional.

If we assume that $\omega_{*i} > \omega \gg \omega_{k0} \gg \nu_{ii,eff}$, Eq. (41) becomes

$$\frac{1.12}{\beta_i\sqrt{2\varepsilon}} - \frac{15\sqrt{\pi}}{16}\frac{\omega_{*i}}{\omega}(1+2\eta_i) - \frac{105\sqrt{\pi}}{32}\frac{a_1\omega_{k0}\omega_{*i}}{\omega^2}(1+3\eta_i) = 0. \tag{42}$$

We find that if and only if the magnetic drift is reversed ($a_1 < 0$), the system is surely unstable. The instability threshold is given by Eq. (33) and the growth rate is

$$\gamma \sim [\beta(1+3\eta_i)]^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\sqrt{a_1\omega_{k0}\omega_{*i}}. \tag{43}$$

In the collisional regimes, $\omega_{*i} > \omega, \nu_{ii,eff} \gg a_1\omega_{k0}$, we may separate the real and imaginary parts of Eq. (41) as we did in the preceding subsection. The resulting imaginary part is given by

$$D_I = \frac{29.64\sqrt{2}\varepsilon}{\pi^{\frac{3}{2}}} \int_0^\infty dx x^6 e^{-x^2} \frac{\omega_{*i}(1-\frac{3}{2}\eta_i) - \omega_{*i}\eta_i x^2}{\omega_r^2 + (\gamma + 0.4\nu_{ii,eff})^2}(\gamma + 0.4\nu_{ii,eff})$$

D_I will not change sign for any value of $\nu_{ii,eff}$. By Nyquist analysis, we thus find that in the collisional case, the purely magnetic trapped particle modes are stable.

In order to find the minimum collision rate needed to stabilize the magnetic trapped particle modes, we may assume the collision frequency is not energy dependent, which at most makes a numerical difference. For $\omega_{*i} > \omega \gg \omega_{k0}, \nu_{ii,eff}$, we obtain the critical value of collisions to be

$$\nu_{ii,eff}|_{cr} = 5.7[\beta(1+3\eta_i)]^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\sqrt{a_1\omega_{k0}\omega_{*i}}. \tag{44}$$

The corresponding anomalous ion thermal diffusivity from a mixing-length estimate is

$$\chi_i^B \sim a_1 \frac{\beta\epsilon^{\frac{1}{2}}}{[1+h^2(\theta)]\nu_{ii,eff}} \frac{\rho_i^2 v_{thi}^2}{R_0 L_n}(1+3\eta_i). \tag{45}$$

We thus conclude that even if magnetic trapped particle modes are quite weak instabilities, on the basis of the simplest estimates of anomalous transport, they possibly cause anomalous ion thermal diffusivity similar in magnitude and scaling to the usual trapped particle modes.

IV. Trapped Particle Modification of Low Shear Ballooning Modes

In Sec. III., we discussed the general dispersion relation based on the disconnected mode boundary conditions. Ballooning instabilities are magnetohydrodynamic modes, usually solved in the infinite series ballooning representation with the boundary conditions ϕ, $A_\|$ and $\delta B_\| \to 0$ as $|\theta| \to \infty$.[9] As Coppi. *et al.* pointed out, the most unstable ($\hat{s} \sim 1$) solutions do not show much difference between the two approximations.[21] However, for low values of shear $\hat{s}$ near the magnetic axis where the disconnected-mode approximation tends to break down completely, the infinite-series ballooning representation must be used in order to correctly obtain the trapped particle modification of ideal MHD ballooning instabilities.

We start with our three basic equations (7), (10) and (12). For a low β plasma, we may drop the compressional Alfven wave terms. Restricting ourselves to the relevant frequency ordering

$$\omega_{be}, \omega_{te} \gg \nu_{ei,eff} \gg \omega, \omega_{de},$$

$$\omega_{bi}, \omega_{ti} \gg \omega \gg \omega_{di}, \nu_{ii,eff}, \tag{46}$$

we now ignore the trapped particle drift resonances and collision completely. The perturbed distribution function can be solved from the bounce-averaged gyro-kinetic equation (12) as

$$h_j = \left(1 - \frac{\omega_*^T}{\omega}\right) \frac{q_j}{T_j} F_{mj} J_0 \left(\overline{\psi} + \frac{\overline{\omega_k}}{\omega}\Lambda\right). \tag{47}$$

When this equation is substituted into the quasineutrality condition Eq. (7), taking note of fact that the trapped particle response to the longitudinal electric field ψ is $\sqrt{\varepsilon}$ smaller than the adiabatic response of the particles, we find

$$\psi = -\frac{k_\perp^2 \rho_i^2}{2} \frac{1}{(1+\tau^{-1})} \left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right] \Lambda$$
$$+ \frac{1}{(1+\tau^{-1})} \int_{Tr} d^3v J_0^2 \frac{F_{mj}}{n_0} \left(1 - \frac{\omega_*^T}{\omega}\right) \frac{\overline{\omega_k}}{\omega} \Lambda. \tag{48}$$

Subsequently inserting the Eqs. (47) and (48) into (10) yields a ballooning mode equation which retains trapped particle modification.

$$\frac{\partial}{\partial\theta}\left\{[1+h^2(\theta)]\frac{\partial\Lambda}{\partial\theta}\right\} + \frac{\omega^2}{\omega_A^2}[1+h^2(\theta)]\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\Lambda + \alpha[\cos\theta + h(\theta)\sin\theta]\Lambda$$

$$
\begin{aligned}
&-\beta q^2 \int_{Tr} d^3v[\cos\theta + h(\theta)\sin\theta]\frac{F_{mj}}{n_0}\left(1-\frac{\omega_*^T}{\omega}\right)\left(\frac{E}{2T}\right)^2\left(1+\frac{v_\parallel^2}{v^2}\right)\\
&\times \overline{\left(1+\frac{v_\parallel^2}{v^2}\right)[\cos\theta + h(\theta)\sin\theta]\Lambda}\\
&+\frac{1}{(1+\tau^{-1})}\frac{\omega\omega_{k0}}{\omega_A^2}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\int_{Tr} d^3v[\cos\theta + h(\theta)\sin\theta][1+h^2(\theta)]\Lambda\\
&\times \frac{F_{mj}}{n_0}\left(1-\frac{\omega_*^T}{\omega}\right)\left(\frac{E}{2T}\right)\left(1+\frac{v_\parallel^2}{v^2}\right)\\
&+\frac{1}{(1+\tau^{-1})}\frac{\omega\omega_{k0}}{\omega_A^2}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right][1+h^2(\theta)]\\
&\times \int_{Tr} d^3v\frac{F_{mj}}{n_0}\left(1-\frac{\omega_*^T}{\omega}\right)\left(\frac{E}{2T}\right)\overline{\left(1+\frac{v_\parallel^2}{v^2}\right)[\cos\theta + h(\theta)\sin\theta]\Lambda} = 0. \qquad (49)
\end{aligned}
$$

We note that Eq. (49) basically corresponds to Eq. (22) by looking at the frequency ordering given by Eq. (46). The first three terms are the usual one-dimensional (along the magnetic field line) ballooning mode equation. The fourth term is ion anisotropic pressure caused by trapped particles. The last two terms are a trapped particle modification of polarization drift via the finite longitudinal electric field ψ. We have thrown away two small terms, one corresponding to a higher order Larmor radius correction and one to a trapped particle compression term, both from finite ψ. In order to be able to solve the eigenvalue equation (49) in the low shear and low β limit, we carry out a two spatial scale analysis, using χ and $z = \hat{s}\chi$. We then write

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\chi} + \hat{s}\frac{\partial}{\partial z}$$

and introduce the following subsidiary orderings

$$\Omega^2 \sim \alpha^4 \sim \sqrt{\varepsilon}\beta \sim |\varepsilon(1-q^2)| \sim \hat{s}^2. \qquad (50)$$

Following the procedures developed by Weiland and Chen[23] based on the systematic perturbation expansion for small $\hat{s}$ and α up to $0(\alpha^4)$, we obtain the solvability equation

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left[(1+z^2)\frac{\partial\Lambda_0}{\partial z}\right] + (u_c - u_{Tr})\Lambda_0 = \qquad (51)\\
&-(1+z^2)\left(\frac{\Omega}{\hat{s}}\right)^2\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\left\{1+\frac{2}{1+\tau^{-1}}\frac{\omega_{k0}}{\omega}\frac{2\sqrt{\varepsilon}}{\pi}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\right\}\Lambda_0.
\end{aligned}
$$

Here

$$u_c = \frac{\alpha}{2\hat{s}^2}\left[2\frac{\hat{s}\alpha - (7/32)\alpha^3}{1+z^2} + \frac{1}{8}\alpha^3\frac{1-z^2}{1+z^3} - \varepsilon(1-q^{-2})\right]$$

and

$$u_{Tr} = \beta q^2 \frac{1.05\sqrt{\varepsilon}}{\pi\hat{s}^2}\left[1 - \frac{\omega_{*i}}{\omega}(1+2\eta_i)\right]$$

Eq. (51) is solved again as in Ref. 23 and a analytic dispersion relation is obtained:

$$\Omega\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right]^{\frac{1}{2}}\left\{1 + \frac{2}{1+\tau^{-1}}\frac{\omega_{k0}}{\omega}\frac{2\sqrt{\varepsilon}}{\pi}\left[1 - \frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\right\} \tag{52}$$
$$= -i\left\{\beta q^2\frac{1.05\sqrt{\varepsilon}}{2\hat{s}}\left[1 - \frac{\omega_{*i}}{\omega}(1+2\eta_i)\right] - \frac{\pi}{4}\left[\alpha^2\left(1 - \frac{7}{32}\frac{\alpha^2}{\hat{s}}\right) - \hat{s} - 2\varepsilon\frac{\alpha}{\hat{s}}(1-q^{-2})\right]\right\},$$

where $\sqrt{\varepsilon}$ is the fraction of trapped particles. The finite longitudinal electric field ψ correction via trapped particles to the ion polarization drift is clearly seen to be small due to the frequency ordering given in Eq. (46). However, trapped particles contribute a significant stabilizing effect due to ion pressure anisotropy, which lead then to a threshold for instability:

$$\beta > \beta_{cr} = 0.71\frac{\varepsilon^{\frac{5}{2}}}{\hat{s}q^2}\left[1 + \sqrt{1 + 2.23\frac{\hat{s}^3}{\varepsilon^3}}\right]. \tag{53}$$

Eq. (53) shows two stabilizing effects: shear stabilization and trapped particle dynamics. At very low shear, trapped particle terms which corresponds to 1 under square root in Eq. (53) are dominant over the line bending.

V. Summary and Conclusions

We have seen that plasma instabilities separate into different α regimes in which different sources of free energy are available, and different perturbed electromagnetic fields are operative. In increasing order of α, we find the following:

(A.) Trapped ion ∇P-driven modes are unstable in the low α limit and eventually become stabilized by diamagnetic pressure at a sufficiently high value of α_{cr} given by Eq. (33) which is slightly bigger than the first ideal MHD ballooning instability limit. However, dissipative trapped ion modes remain unstable. In this range of α, the basic electromagnetic perturbations are caused by charge separation in the presence of the trapped particles in tokamak.

(B.) At marginal ideal MHD ballooning instability thresholds, the trapped particle ion-ion collisions coupling with the diamagnetic drift contribute an expansion free energy related only to ion temperature gradients and lead to dissipative drift ballooning instabilities which propagate in the ion diamagnetic direction. Because they require a high ion-ion collisionality, these modes thus may become important in the edge plasma.

However, if α is further from the marginal stability thresholds and satisfies the conditions of Eq. (27a), then trapped particles introduce a collisional damping effect on the usual shear Alfven waves with the rate given by Eq. (27b).

(C.) In the range $\alpha_{bc1} < \alpha < \alpha_{bc2}$ obtained in Eq. (25) for fixed $\hat{s}$, we encounter ballooning instabilities, which are a competition between the stabilizing influence of magnetic tension and the destabilizing impact of the expansion-free energy associated with the unfavorable curvature of the magnetic field. The dual role (destabilizing effect and shortening of connection length) that α plays results in the appearance of a second stable region in the $\alpha - \hat{s}$ plane.

(D.) Starting from $\alpha_{bc1} < \alpha_{cr} < \alpha_{bc2}$ discribed in (A) and (C), we have shown that trapped particles induce purely magnetic modes for large α. These purely magnetic trapped particle modes are excited via resonances between the modes and the trapped particle processional drift. The instability mechanism is the energy transfer from the compressional magnetic field into the perpendicular motion of trapped particles when the vacuum magnetic drift is reversed by diamagnetic pressure, which possibly persists in the second ballooning stable region. Fortunately, these modes are easily stabilized by trapped ion-ion collisions.

The basic trend discussed above is that long wavelength toroidal instabilities are likely to be present over a wide range of α, with the electrostatic branch dominant at low α, the shear Alfven branch dominant at moderate α, and compressional Alfven branch dominant at high α. A model Lorentz operator is used to introduce collisional dissipation into the unified theory. The collisional dissipation was found to produce dissipative trapped particle modes and dissipative drift ballooning modes, to damp shear Alfven waves, and to stabilize the electrostatic trapped ion at long wavelength limit and to stabilize the purely magnetic trapped particle modes. These residual dissipative instabilities appear to persist up to the larger α values ($\alpha > \alpha_{bc2}$) associated with the "second stability" ballooning regime.

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