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In the local approximation the electrostatic dispersion relation for the (ion or electron) temperature gradient driven modes is solved retaining the full guiding-center resonance condition $\omega = k_\parallel v_\parallel + \mathbf{k}\cdot\mathbf{v}_D(v_\perp^2, v_\parallel^2)$. Increase of the slab-to-toroidal ratio parameter $k_\parallel v_T/k_\perp v_D$ is shown to directly increase the threshold value of the temperature-to-density gradient parameter η_c. Therefore the local anomalous heat flux has a strong dependence on the safety factor q when $k_\parallel \sim 1/qR$.

In confinement stability theory the temperature gradient driven drift wave instability has been studied extensively in the limits of slab and toroidal geometry. In the slab geometry the resonant dynamics is from the one-dimensional ($d = 1$) $v_\parallel$ motion along the magnetic field $\mathbf{B}$, whereas in the toroidal geometry the resonant dynamics is from the ∇B-curvature drift $v_D(\frac{1}{2}v_\perp^2 + v_\parallel^2)/v_T^2$ in the three-dimensional ($d = 3$) velocity space. In fluid theory the threshold of the temperature gradient instability is given by the parameters $\eta = \partial_x \ln T/\partial_x \ln n$ and the ideal gas compressibility constant $\Gamma = (d+2)/d$. The critical or threshold temperature gradient value η_c is given by $\eta_c = \Gamma - 1 = 2, 1$, and $2/3$ for $d = 1, 2$, and 3, respectively, and arises from the balance of the thermal energy released by $\tilde{v}_E\, dp/dr$ and that taken up by plasma compression $\Gamma p \nabla\cdot\tilde{\mathbf{v}}$. Horton and Varma[1] give a derivation for $\eta_c = 2/3$ for $\Gamma = 5/3$ from the Braginskii equations.

In this letter we show that the kinetic dispersion relation gives a critical η_c that increases from $\eta_c = 2/3$ in the toroidal regime with $d = 3$, to the value $\eta_c = 2$ for the slab regime where the effective dimensionality is $d = 1$. This kinetic theory variation of η_c follows from the dimensionality of the guiding-center resonance in velocity space and the value of $k\rho$ due to the Bessel function weighting of the velocity resonance.

We show the details of the change in the unstable wavenumber domain $\gamma(k_y, k_\parallel)$ as the transition from the toroidal to the slab regime occurs. The parameter governing the transition is $x = k_\parallel v_T/\omega_D \simeq k_\parallel R/k_y \rho$ which, for fixed $k_y \rho$, is proportional to $1/q(r)$ when $2\pi/k_\parallel$ is taken as the length between the good and bad curvature regions qR. The increase of the threshold η with $1/q$ was found in Hong *et al.*[2] from a ballooning mode calculation, which gives the proper averaging over the poloidal angle of the local dispersion relation used here. Here we restrict attention to the local, electrostatic stability taking $k_\parallel$ as a parameter. From ballooning mode analysis we estimate that the relevant range of $k_\parallel$ is $-1/qR \le k_\parallel \le 1/qR$. A shear-dependent formula from the ballooning mode weighted average of $k_\parallel$ is given in Horton-Choi-Tang[3].

The electrostatic kinetic dispersion relation, $D_k(\omega)\Phi_k = 0$, in dimensionless units is

$$D = D_a - \int_0^\infty \int_{-\infty}^{+\infty} \frac{[\tau\omega - k_y(1+\eta(v^2/2 - 3/2))]\, J_0^2(k_\perp v_\perp/\tau^{1/2}) e^{-v^2/2}}{\tau\omega - k_y\,\varepsilon_n(\frac{1}{2}v_\perp^2 + v_\parallel^2) - k_\parallel v_\parallel \tau^{1/2}} \frac{v_\perp\, dv_\perp\, dv_\parallel}{(2\pi)^{1/2}} \tag{1}$$

with analytic continuation from $\mathrm{Im}(\omega) > 0$. Here D_a is the adiabatic part of the plasma response. Equation (1) applies to both the electron (η_e) and the ion (η_i) temperature gradient driven instabilities. For the η_e mode[4], $D_a = 1 + T_e/T_i$ and the quantities $k_\perp, k_\parallel$, and ω are normalized to $\rho_{ei} = v_{ei}/\omega_{ce}$, $r_n = L_n = -(\partial \ln n/\partial x)^{-1}$, and r_n/v_{ei} with $v_{ei} = (T_i/m_e)^{1/2}$. For the η_i mode, $D_a = 1 + T_i/T_e$ and $k_\perp, k_\parallel$, and ω are normalized to $\rho_s = c_s/\omega_{ci}$, r_n, and r_n/c_s with $c_s = (T_e/m_i)^{1/2}$. We note that for $\beta_e > 2m_e/m_i$ the η_e mode develops an electromagnetic component for $(m_e/m_i)^{1/2} \le k_\perp < (\beta_e/2)^{1/2}$.

In the higher frequency domain $|\omega| \gg k_y\,\varepsilon_n, k_\parallel$, the resonant dominator is expanded to obtain the fluid theory response $D_{\text{fluid}} \simeq A + B/\omega + C/\omega^2$ which tends to limit $D(|\omega| \to \infty) = A = D_a - I_0(k_\perp^2)\exp(-k_\perp^2)$. At frequencies comparable to the guiding-center drift frequency, the small $k_\parallel$ limit is analyzed by Similon *et al.*[5] and shown to have a $\omega^{1/2}$ branch point on the $\omega > 0$ resonant side of the real ω axis and Im $D = 0$ for $\omega < 0$. For $k_y\,\varepsilon_n \ll k_\parallel$ the function reduces to the one-dimensional plasma $Z(\phi)$ function which is regular at $\omega = 0$ with Im $Z(\phi) = \pi^{1/2}\exp(-\phi^2)$.

The fraction of particles of species s resonant with the fluctuation $\phi_{\mathbf{k}\omega}$ is given by $R^s_{\mathbf{k}}(\omega) = \int \delta(\omega - \mathbf{k}\cdot\mathbf{v}_{gc}) f_s(\mathbf{v}) d\mathbf{v}$ with $\int_{-\infty}^{+\infty} d\omega\, R^s_{\mathbf{k}}(\omega) = n_s$. The resonant particles lie on the ellipse in $v_\perp - v_\parallel$ space with center at $v_\perp = 0$ and $v_\parallel = -k_\parallel/2\varepsilon_n\,k_y$. For each ω the resonant particles are in a band of width $\omega\tau_c$, where τ_c is the correlation time of the fluctuation, along the ellipse

$$\frac{\varepsilon_n}{2}\,v_\perp^2 + \varepsilon_n\left(v_\parallel + \frac{k_\parallel}{2\varepsilon_n\,k_y}\right)^2 = \frac{\omega}{k_y} + \frac{k_\parallel^2}{4\varepsilon_n\,k_y^2}\,. \tag{2}$$

For marginally stable toroidal η modes the frequency is $\omega_m \lesssim k_y\,2\varepsilon_n$ and $k_\parallel \ll 2\varepsilon_n\,k_y$ so the radius of the ellipse is of order the thermal velocity and it is centered at $v_\perp = v_\parallel = 0$. For slab modes the center of the ellipse moves off to a high parallel velocity $|v_\parallel|/v_T = k_\parallel/2\varepsilon_n\,k_y \gg 1$ and the resonant particles have $v_\parallel = \omega/k_\parallel$. The transition in velocity space is shown in Fig. 1. The toroidal resonance involves both $v_\perp$ and $v_\parallel$ and is thus a three-dimensional $d = 3$ resonance. The slab resonance is a one-dimensional $d = 1$ resonance for $k_\perp \ll 1$ where all $v_\perp$'s contribute. For $k_\perp \gg 1$, however, only the $v_\perp < 1/k_\perp$ particles contribute significantly to the resonance so the small pitch angle particles are more heavily weighted, changing the effective Γ.

In Fig. 2 with $\varepsilon_n = 0.1$ we show the transition of the critical temperature gradient η_c as a function of $x = k_\parallel R/2k_y\rho = k_\parallel\,r_n/2\varepsilon_n\,k_y$ for several values of $k_\perp = k_y$ from 0.1 to 1.0. The curves are obtained from the marginal stability analysis of $D(\omega, k) = 0$. There is a rapid transition from the toroidal value $\eta_c \simeq 2/3$ to the values $\eta_c \gtrsim 2$ to 3 as the slab-to-toroidal parameter $k_\parallel\,r_n/2\varepsilon_n\,k_y$ increases through one.

In Fig. 2 we re-express the results of the marginal stability analysis in terms of $x = k_{\|} r_n / 2\varepsilon_n k_y = k_{\|} R / 2 k_y \rho$ by means of two simple parameterizations of $\eta_c(x)$. We find it convenient to express the function as

$$\eta_c(x) = \frac{1 + x^4}{1 + x^4/3} \tag{3}$$

giving a rapid change from 1 to 3 as x increases from zero to $x \gg 1$. From Fig. 2 we see there is some residual k_y-dependence that is not contained in the simple formula of Eq. (3). An alternate parameterization $(1 + x^2)/(1 + x^2/5)$ is also shown for comparison.

In the toroidal limit $x \to 0$, Nyquist analysis gives the marginal frequency

$$\omega_m = 2\varepsilon_n k_y \left(\frac{3}{2}\eta - 1 \right) \Big/ (\eta - 2\varepsilon_n) , \tag{4}$$

which must be positive for a resonance to occur in Eq. (1) at $k_{\|} = 0$. Thus, the condition $\eta > 2/3$ is necessary for instability. For small ε_n, the condition $\eta > 2/3$ is also sufficient to have instability since Re $D(\omega_m) = D_a - c_1 \eta / \varepsilon_n < 0$. For finite aspect ratio ε_n, the compression of the plasma in the nonuniform toroidal magnetic field leads to the additional condition from Re $D(\omega_m) < 0$ that

$$\varepsilon_T = \varepsilon_n / \eta < \frac{c_1}{D_a} \simeq \frac{0.7}{D_a} , \tag{5}$$

as obtained earlier by Horton-Hong-Tang[4] and Dominguez and Waltz[6].

In the slab limit $x^2 \gg 1$, the condition of marginal stability[7,8] gives

$$\omega^2 - 2 \frac{k_{\|}^2 r_n^2}{k_y \eta} \omega + k_{\|}^2 r_n^2 \left[\frac{2}{\eta} - 1 - 2k_y^2 \left(1 - \frac{I_1}{I_0} \right) \right] = 0 . \tag{6}$$

For small k_y and $\eta > 2$ there are two real roots and the system is unstable at small $k_{\|} r_n$. The marginal frequency is then

$$\omega_m = \pm k_{\|} r_n \left(1 - \frac{2}{\eta} \right)^{1/2} . \tag{7}$$

The condition Re $D(\omega_m) < 0$ from the Nyquist analysis then gives

$$k_\parallel r_T = \frac{k_\parallel r_n}{\eta} < \frac{k_y I_0 e^{-k_y^2}}{D_a} \leq \frac{0.4}{D_a} \tag{8}$$

where the upper limit $0.4/D_a$ occurs for $k_y \gtrsim 1$ where the slab η_c drops to one[7,8].

Above the critical $\eta > \eta_c(x)$ the change in the unstable wavenumber domain is shown by the growth rate contours $\gamma(k_\perp, k_\parallel)$ in Fig. 3. Figure 3(a) shows the toroidal system with $\varepsilon_n = 0.1$, and Fig. 3(b) shows the slab system with $\varepsilon_n = 0$, both for $\eta = 3$. Both systems have a critical $k_\parallel(k_y)$ curve below which $|k_\parallel| < k_\parallel(k_y)$ the modes are unstable and above which $\gamma_{\mathbf{k}} < 0$. Both systems have growth through all k_y. The slab system has no growth for $k_\parallel = 0$. The toroidal system, however, has a large maximum growth rate $\gamma_{\text{tor}} \simeq 0.23 \simeq 2\gamma_{\text{slab}}$ along $k_\parallel = 0$. The toroidal system has a local maximum growth at $k_y \lesssim 1/(1+\eta)^{1/2}$ and a secondary maximum growth rate at $k_y \simeq 1.5$. Clearly, the faster growing, long wavelength modes of the toroidal regime make the transport in the toroidal system larger than in the slab system. An example of this dominance of the transport in the toroidal regime for the heat flux is shown in Fig. 4 of Hong and Horton[9].

The marginal stability frequency is important in controlling both the $\mathbf{E} \times \mathbf{B}$ transport in profiles near marginal stability and also the division in the ion energy spectrum between those particles giving energy to the mode and those receiving energy from the mode. In the local electrostatic limit the transfer of energy from the fluctuation $\phi_{\mathbf{k}\omega}$ to the particles given by $\langle \mathbf{j} \cdot \mathbf{E} \rangle$ reduces to

$$\delta \langle \mathbf{j} \cdot \mathbf{E} \rangle / |\phi_{\mathbf{k}\omega}|^2 = \frac{2}{\pi^{1/2}} \int_0^\infty d\epsilon e^{-\epsilon}\, h_{\mathbf{k}\omega}(\epsilon) \Delta_{\mathbf{k}\omega}(\epsilon) \ , \tag{9}$$

where

$$h_{\mathbf{k}\omega}(\epsilon) = \omega \left[\omega - k_y(1 + \eta(\epsilon - 3/2))\right] \ ,$$

and $\Delta_{\mathbf{k}\omega}(\epsilon)$ is the pitch-angle averaged resonance function

$$\Delta_{\mathbf{k}\omega}(\epsilon) = \int_{-1}^{+1} \delta(\omega - k_\parallel v_\parallel - \omega_D) \frac{d\mu}{2} = \frac{\pi}{2} \left[\left(\frac{k_\parallel^2}{2} + \varepsilon_n k_y \omega \right) \epsilon - \varepsilon_n^2 k_y^2 \epsilon^2 \right]^{-1/2} > 0 \tag{10}$$

for energies in the range $\epsilon_{\min} \leq \epsilon \leq \epsilon_{\max}$, with $\Delta_{\mathbf{k},\omega}(\epsilon) = 0$ outside this range. The lowest resonant energy is

$$\epsilon_{\min} = \frac{\omega^2}{k_{\|}^2 + 2\varepsilon_n\, k_y\omega + k_{\|}(k_{\|}^2 + 4\varepsilon_n\, k_y\omega)^{1/2}} \tag{11}$$

which reduces to $\epsilon_{\min} = \omega^2/2\, k_{\|}^2$ for $\varepsilon_n < k_{\|}^2/k_y\omega$ and $\epsilon_{\min} = \omega/2\varepsilon_n\, k_y$ for $\varepsilon_n \gg k_{\|}^2/k_y\omega$. The maximum resonant energy moves in pitch angle from $\mu = 0$ as $k_{\|} \to 0$ to $\mu < 0$ for $\varepsilon_n \to 0$, and is given approximately by

$$\varepsilon_{\max} \cong \begin{cases} \dfrac{k_{\|}^2}{2\varepsilon_n^2\, k_y^2} & \text{for } x \gg 1 \\[2ex] \dfrac{\omega}{\varepsilon_n\, k_y} & \text{for } x \ll 1\ . \end{cases} \tag{12}$$

At marginal stability there is no power transfer, $\langle \mathbf{j} \cdot \mathbf{E} \rangle = 0$, due to the balance of higher energy particles emitting into the wave and lower energy particles absorbing energy from the wave. The critical relative energy ϵ_{crit}, above which the particles emit energy into the mode, is given by

$$\epsilon_{\text{crit}} = \frac{mv_{\text{crit}}^2}{2T} = \frac{3}{2} + \left(\frac{\omega_m}{k_y} - 1\right) \Big/ \eta\ . \tag{13}$$

Below this energy the particles absorb energy. At marginal stability the rate of absorption and emission are in balance when the energy integral over the energy spectrum is performed. In the toroidal limit the marginal frequency vanishes. From Eq. (4) at $\eta = 2/3$, $\omega_m = 0$ and only the lowest energy particles are resonant as shown in Fig. 1 and Eq. (13). For $\eta > 2/3$, the division between absorbing and emitting particles shifts to higher energies according to Eq. (13). As $k_{\|}$ increases, the marginal frequency increases, first in the negative direction, and then reverses at $x = 1$ to 1.5 and becomes positive, reaching a few tenths of $k_{\|}\, v_T$ at $x = 3$. When the mode is rotating in the negative direction, only a few particles with negative $k_{\|}\, v_{\|}$ are resonant.

In Fig. 4 we show the effect of varying the toroidicity ε_n on the growth rate when $\eta \gg \eta_c(x)$. For finite $k_{\|}$, Fig. 4(a) clearly shows how the transition from the slab mode to the toroidal mode occurs as the toroidicity ε_n increases. The transition point in toroidicity is given by $\varepsilon_n \simeq k_{||}/2k_y$, which is the point where the nature of the velocity resonance changes from slab to toroidal. In Fig. 4(a), for $2k_y = 1$ the change in character is particularly strong along the $k_{||} = 0.2$ growth rate curve. The small toroidal resonance contributes to the stabilization for the slab ($k_{||} = 0.2$) mode. Similarly, the slab Landau resonance contributes a stabilizing effect on the toroidal mode, which is also shown in Fig. 3(a) by the decrease of γ with increasing $k_{||}$.

The toroidal mode is restabilized as the toroidicity exceeds a critical value. In previous studies which used the fluid approach[4,6], this restabilization was shown to be due to the compression effect, and the critical value of the toroidicity was given to be $(\varepsilon_n)_{\rm crit} = 0.35\eta$ or $(\varepsilon_T)_{\rm crit} = 0.35$ for $T_i = T_e$. Figure 4(b) shows that the value of $(\varepsilon_n)_{\rm crit}$ also significantly depends on the wavelength $1/k_y$, and a more general formula is

$$(\varepsilon_n)_{\rm crit} = 0.12\eta/k_y$$

or $\varepsilon_T = \varepsilon_n/\eta = r_T/R < 0.12/k_y \leq 0.4 \sim 0.5$ for instability at fixed $k_{||}\, r_n \doteq 0.1$. As seen from Fig. 4(b), the flat density profiles with $\varepsilon_n > 1$ require that ε_T be less than a critical value for instability as shown in Horton *et al.*[4] (1988) and Dominguez and Waltz *et al.*[6] The longer wavelength modes are the first to become unstable as ε_T is decreased from large values.

The fluid turbulence formulas for the thermal diffusivity χ are proportional to $\gamma'_m(\eta-\eta_c)^m$. In the toroidal limit Ref. 3 gives $m = 0.5$, and in the slab limit Hamaguchi-Horton[10] give $m = 1.0$. Here we calculate $\max\gamma^\ell(\mathrm{k})$ and find that $m = 0.7$ with $\gamma'_m = 0.13$ for $\varepsilon_n = 0.1$ and $k_{||} = 0$ (toroidal modes), but $m \simeq 1.0$ with $\gamma'_m = 0.06$ for $\varepsilon_n = 0.0$ (slab modes). We suggest that the fluid turbulence formulas[3,10] are good measures of the thermal diffusivities χ provided that the kinetic values of $\eta_{\rm crit}$ are used in place of the corresponding fluid values.

In conclusion, both the toroidal regime and the slab regime are important sources of anomalous heat flux. The toroidal regime, however, has both a lower threshold in η for onset and a larger small $k_\perp$ growth rate, which lead to a considerably stronger anomalous heat flux. Therefore, if there is either a small q and/or strong magnetic shear to force the system into the slab-like regime, the anomalous heat flux can be strongly reduced. Clearly, this variation with $x = k_\parallel R/k_y\rho \sim \overline{q}/q(r)$ of the marginal stability condition $\eta_c(x)$ and the shift of the maximum growth rate in $k_y, k_\parallel$ space give a strong q dependence to the anomalous heat flux. Database studies of χ_i and χ_e do show a strong improvement at low q (high toroidal current); however previous theoretical studies[3,11] have typically given only a rather weak q dependence for the transport.

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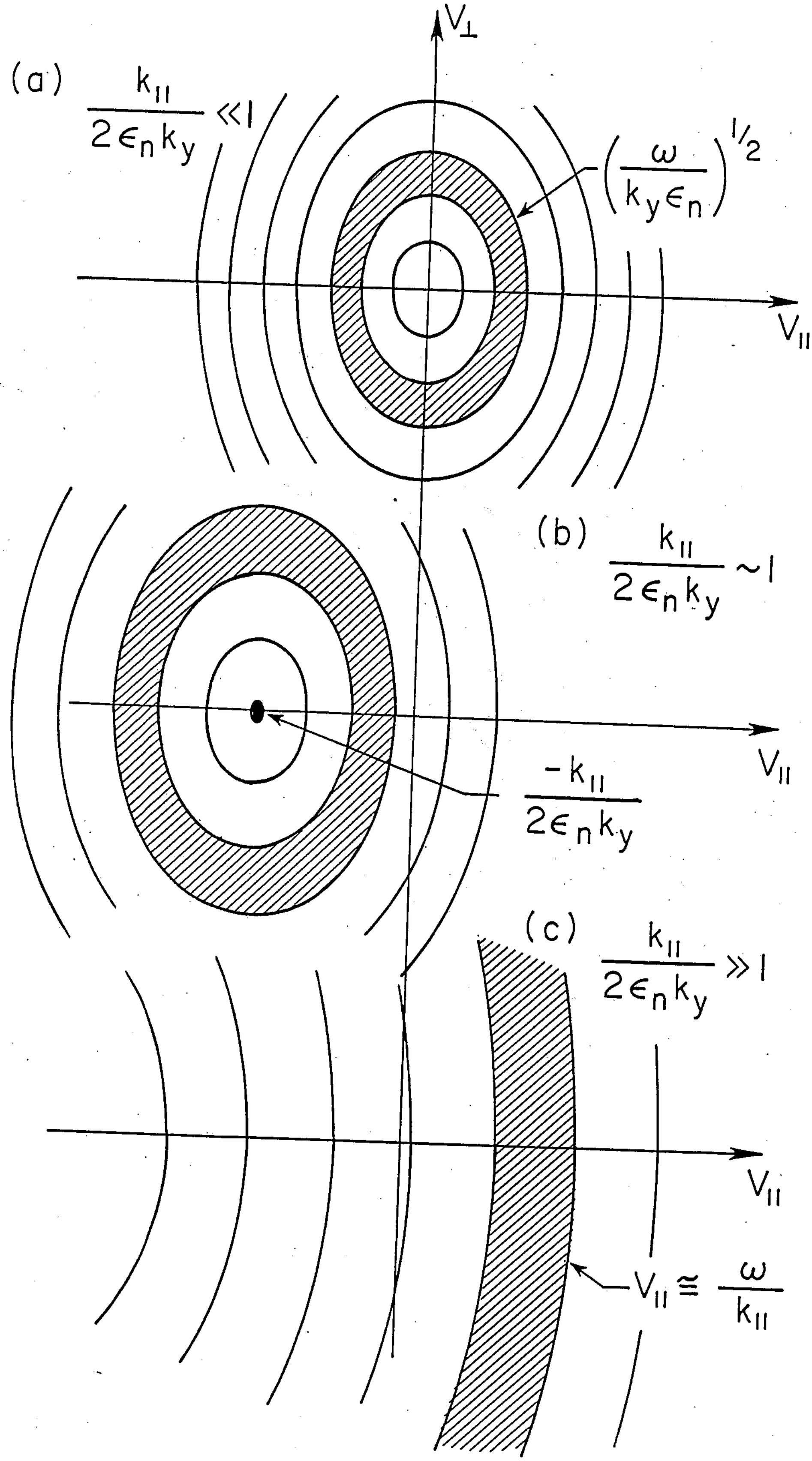
Figure Captions

1. Resonant particle velocity regions (shaded) as a function of kω. (a) The toroidal regime. (b) The mixed slab-toroidal regime. (c) The slab limit.

2. The transition of the critical temperature gradient from toroidal to the slab regime as a function of the parameter $x = k_{\|} r_n/\varepsilon_n k_y \rho = k_{\|} v_T/\omega_D$ for several $k_y \rho$ compared with the fitting function in Eq. (3).

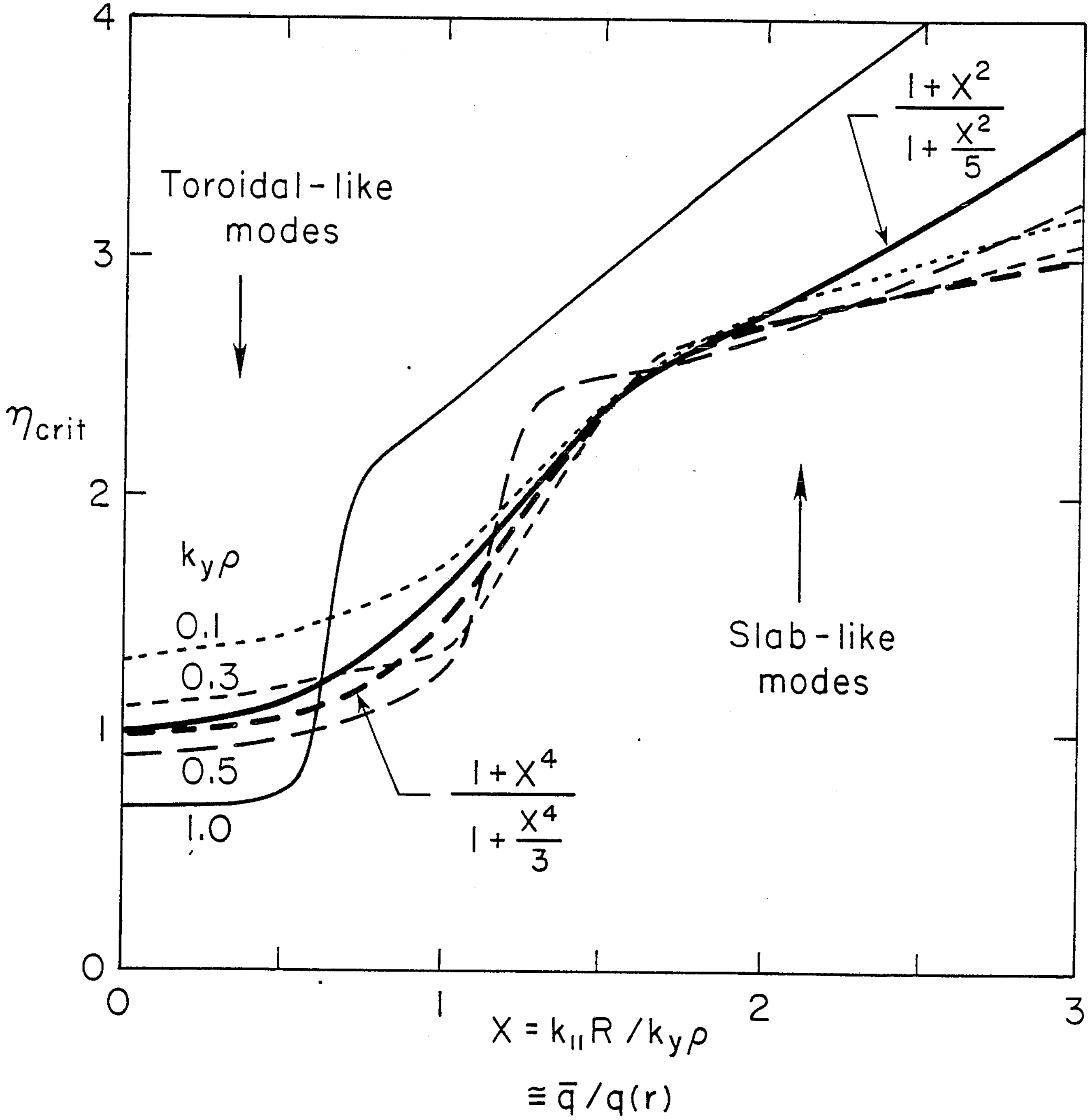
3. The growth rate in the wavenumber domain when $\eta = 3$ (a) in the toroidal regime with $\varepsilon_n = 0.1$ (b) in the slab limit with $\varepsilon_n = 0.0$.

4. The toroidicity dependence of the growth rate when $\eta = 3$ (a) for $k_{\|} r_n = 0, 0.1, 0.2$ with $k_y \rho = 0.5$ (b) for $k_y \rho = 0.5, 1.0$, and 2.0 at $k_{\|} r_n = 0.1$.



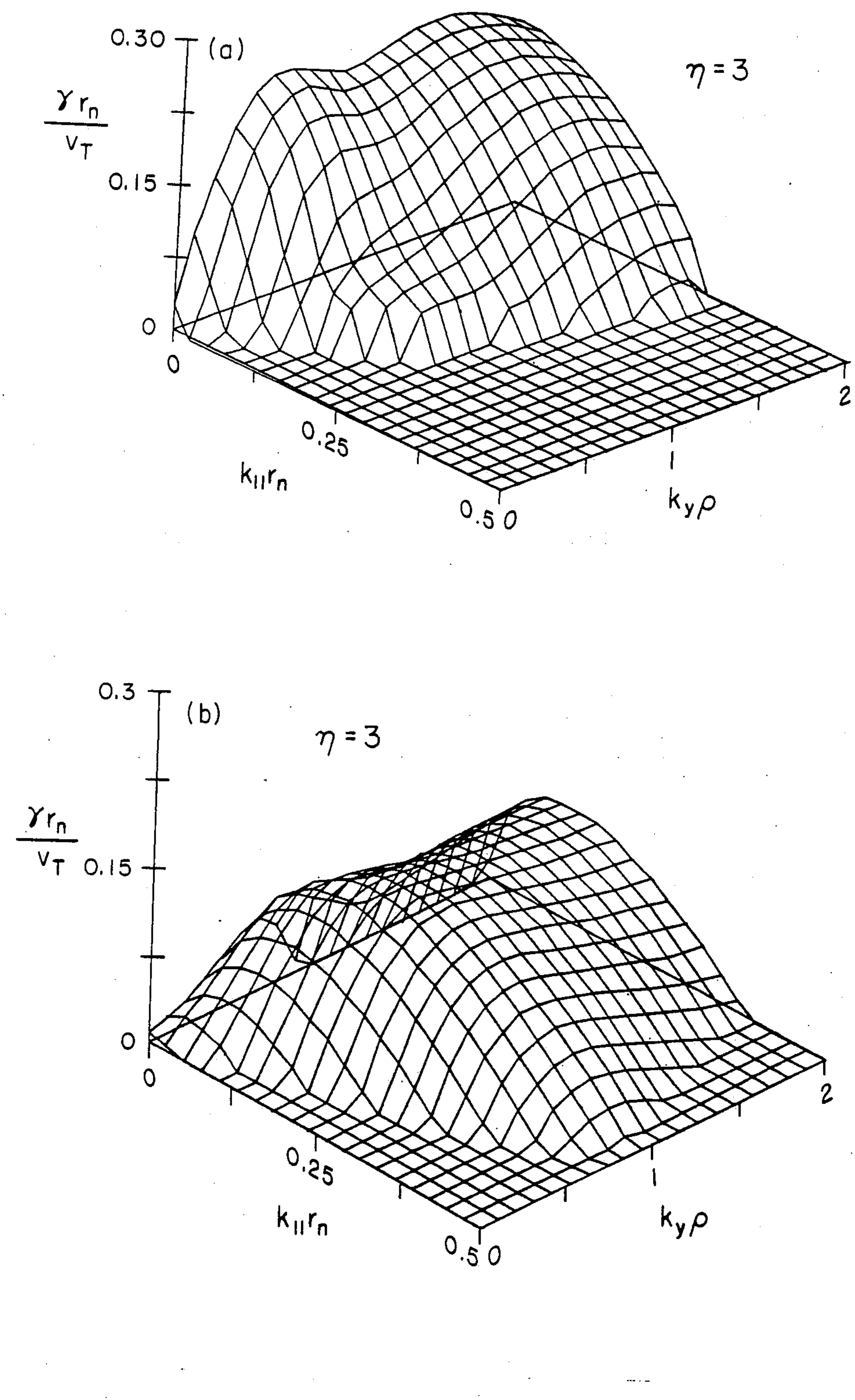
$V_\perp$
(a) $\frac{k_\parallel}{2\epsilon_n k_y} \ll 1$
$\left(\frac{\omega}{k_y \epsilon_n}\right)^{1/2}$
$V_\parallel$
(b) $\frac{k_\parallel}{2\epsilon_n k_y} \sim 1$
$\frac{-k_\parallel}{2\epsilon_n k_y}$
$V_\parallel$
(c) $\frac{k_\parallel}{2\epsilon_n k_y} \gg 1$
$V_\parallel$
$V_\parallel \cong \frac{\omega}{k_\parallel}$

Fig. 1



4
3
2
1
0
0
1
2
3
η crit
Toroidal-like modes
Slab-like modes
k y ρ
0.1
0.3
0.5
1.0
1 + X²
1 + X²/5
1 + X⁴
1 + X⁴/3
X = k∥ R / k y ρ
≅ q̄ / q(r)

Fig. 2



0.30
(a)
γr_n / v_T
0.15
0
η = 3
0
0.25
k_∥r_n
0.5
0
1
2
k_yρ
0.3
(b)
η = 3
γr_n / v_T
0.15
0
0
0.25
k_∥r_n
0.5
0
1
2
k_yρ

Fig. 3

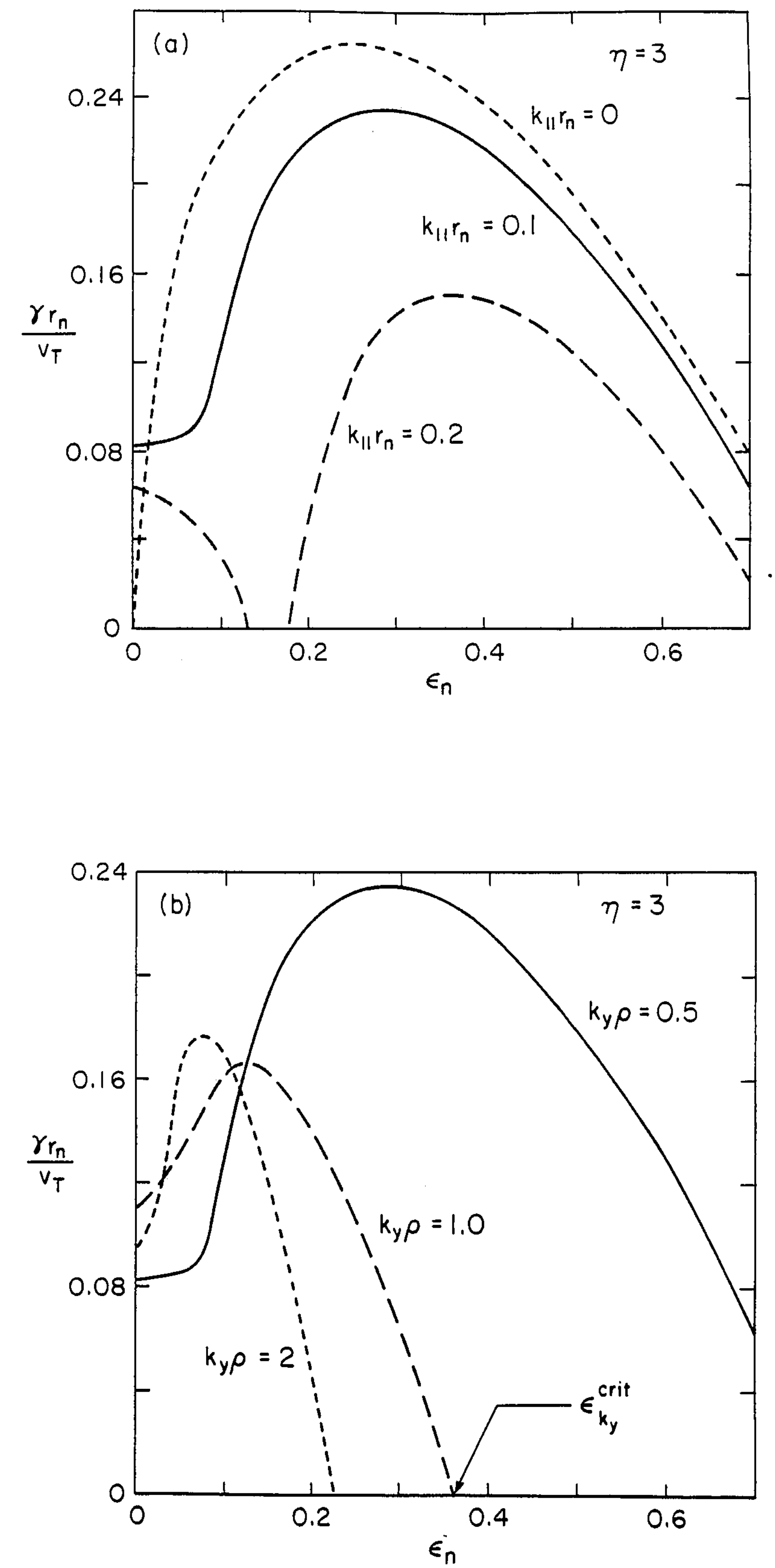

(a)
η = 3
0.24
0.16
0.08
0
$\frac{\gamma r_n}{v_T}$
$k_{\parallel} r_n = 0$
$k_{\parallel} r_n = 0.1$
$k_{\parallel} r_n = 0.2$
0
0.2
0.4
0.6
ϵ_n
(b)
η = 3
0.24
0.16
0.08
0
$\frac{\gamma r_n}{v_T}$
$k_y \rho = 0.5$
$k_y \rho = 1.0$
$k_y \rho = 2$
$\epsilon_{k_y}^{crit}$
0
0.2
0.4
0.6
ϵ_n

Fig. 4