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J. W. Connor and *R. J. Hastie*
Culham Laboratory
(UKAEA/Euratom Fusion Association)
Abingdon, Oxfordshire OX14 3DB, England

J. B. Taylor
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712
USA

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U.S.A.

Abstract

This is part II of a study of resonant perturbations, such as resistive tearing and ballooning modes, in a torus. These are described by marginal ideal mhd equations in the regions between resonant surfaces; matching across these surfaces provides the dispersion relation. In part I we described how all the necessary information from the ideal mhd calculations could be represented by a so-called E-matrix. We also described the calculation of this E-matrix for tearing modes (even parity in perturbed magnetic field) in a large aspect ratio torus. There the toroidal modes comprise coupled cylinder tearing modes and the E-matrix is a generalisation of the familiar Δ' quantity in a cylinder.

In the present paper we discuss resistive ballooning, or twisting-modes, which have odd-parity in perturbed magnetic field. We show that, unlike the tearing modes, these odd-parity modes are **intrinsically** toroidal and are not directly related to the odd-parity modes in a cylinder. This is evident from the analysis of the high-n limit in ballooning-space, where a transition from a stable Δ' to an unstable Δ' occurs for the twisting mode when the ballooning effect exceeds the interchange effect, which can occur even at large aspect ratio (as in a tokamak). Analysis of the high-n limit in coordinate space, rather than ballooning space, clarifies this singular behaviour and indicates how one may define a twisting-mode Δ'. It also yields a prescription for treating low-n twisting modes and a method for calculating an E-matrix for resistive ballooning modes in a large aspect ratio tokamak. The elements of this matrix are given in terms of cylindrical **tearing** mode solutions.

1 Introduction

In part I of this work[1] we investigated toroidally coupled tearing modes. These are examples of 'resonant' perturbations described by the marginal ideal mhd equations outside critical layers at $nq = m$, where there are discontinuities Δ_m in the small component. These discontinuities are matched to corresponding $\Delta_m(\omega)$ calculated from the layer equations (which contain additional physical processes such as resistivity). In a cylinder, where harmonics of different m are uncoupled, the ideal mhd equations determine each Δ_m, but in a torus they determine only a single relation between all the Δ_m – called the "E-matrix" in I.

The tearing modes are distinguished by the fact that they have even-parity (ie symmetric in perturbed normal magnetic field ψ) in the critical layer. In a large aspect ratio torus different even-parity harmonics are weakly coupled and the toroidal mode is closely related to the cylinder tearing-mode.

In the present paper we consider toroidal **twisting** modes, ie modes with odd-parity in the layers. We will show that these exist only by virtue of toroidal coupling and are not related to the cylinder twisting mode.

The intrinsic toroidal nature of the twisting modes is already apparent in their behaviour in the high-n limit. This is usually described using the "ballooning-transformation" [2,3,4] when a quantity Δ_B, determined from the asymptotic $|\eta| \to \infty$ behaviour of the ideal mhd solution in transform space, replaces Δ. In a large aspect-ratio tokamak, with zero average curvature, Δ_B for the tearing mode approaches Δ for a cylinder as pressure, and therefore toroidal coupling, tends to zero. However, although the twisting and tearing modes have the same (negative) Δ in a cylinder (the solution for one parity is obtained from the other by inverting the function on one side of the resonant surface – leaving Δ unchanged), Δ_B for the twisting parity toroidal mode remains positive as pressure tends to zero!

In section (2) we show how the cylinder limit of high-n toroidal twisting modes is resolved by the introduction of non-zero average curvature, and in section (3) we show how these modes can be calculated in coordinate space. This leads to the main result of the present paper – the calculation of **low-n** toroidal twisting modes in section (4).

2 High-n modes in Ballooning space

Perturbations with high toroidal mode number n and many resonant surfaces $nq = m$, are best described using the "ballooning-transformation".[2–4] This exploits the fact that, at large n, harmonics centered on different rational surfaces are equivalent to one another. Formally the transformation is written

$$\psi(r,\theta) = \sum \psi_m(r) e^{-im\theta} \to \sum e^{-im\theta} \int e^{imn} \hat{\psi}(\eta, r) d\eta \tag{1}$$

where r, θ are polar coordinates in the poloidal plane and

$$\hat{\psi}(\eta, r) = e^{-inq(r)\eta} F(\eta, r). \tag{2}$$

Then at large n F varies slowly with r and to a first approximation $F \simeq F(\eta)$. Clearly $F(\eta)$ can be regarded as the fourier-transform, with respect to $nq(r)$, of $\psi_m(r)$ which, in accordance with the equivalence of different harmonics, $\sim \psi\,(m - nq(r))$. The singularity at $(m - nq(r)) = 0$ is reflected in the asymptotic behaviour of $F(\eta)$ through Δ_B, the ratio of the small to the large component at large $|\eta|$. This must be matched to the solution of a more complete plasma model,which vanishes as $|\eta| \to \infty$.

Strauss [5] introduced a model for a large aspect ratio, low shear ($s = rq'/q \ll 1$) tokamak for which he calculated Δ_B. The low shear permits an averaging of the ideal mhd equations over a connection length, so that the electrostatic potential for a marginal ideal mode is given by

$$\left[\frac{d}{dz}(1+z^2)\frac{d}{dz} + \frac{\lambda^2}{1+z^2} + \delta\right]\phi = 0 \tag{3}$$

where $z = s\eta, \lambda^2 = \alpha^2/s$ and $\alpha = -2Rp'q^2/B^2$. The term $\delta = \alpha\epsilon(1-q^2)/q^2s^2$ represents the effect of average curvature (interchange energy).

Strauss considered $\delta = 0$. Then the twisting and tearing parity solutions are

$$\phi^{TW} = \cos(\lambda \tan^{-1} z) \underset{|z| \to \infty}{\longrightarrow} \cos(\frac{\lambda\pi}{2}) \left[1 + \frac{\lambda \tan(\lambda\pi/2)}{|z|}\right] \tag{4a}$$

$$\phi^{TE} = \sin(\lambda \tan^{-1} z) \underset{|z| \to \infty}{\longrightarrow} \sin\left(\frac{\lambda\pi}{2}\right) \left[1 - \frac{\lambda \cot(\lambda\pi/2)}{|z|}\right] sgn(z) \tag{4b}$$

[N.B. ϕ has the opposite parity to the magnetic field perturbation ψ] so that

$$\Delta_B^{TW} = \frac{\lambda}{s}\tan\left(\frac{\lambda\pi}{2}\right) \tag{5a}$$

$$\Delta_B^{TE} = -\frac{\lambda}{s}\cot\left(\frac{\lambda\pi}{2}\right). \tag{5b}$$

As $\lambda \to 0$, $\Delta_B^{TE} \to -2/\pi s$, corresponding to $\Delta' = -2nq/r$ for high-n modes in a cylinder[6], and the expression (4b) is equivalent to the form

$$\phi^{TE}(x) \underset{x \to 0}{\longrightarrow} \left[\frac{1}{|x|} + \frac{\pi}{2}\Delta_B^{TE}\right] sgn(x) \tag{6}$$

near a resonant surface r_m in coordinate space (with $x \equiv (r - r_m)nq'(r_m)$). This also corresponds to tearing modes in a cylinder.

On the other hand, Δ_B^{TW} is always positive and $\to \pi\lambda^2/2s$ as $\lambda \to 0$. Furthermore the expression (4a) is equivalent to the form

$$\phi^{TW}(x) \underset{|x|\to 0}{\longrightarrow} 2\pi\delta(x) - 2\Delta_B^{TW} \log|x| + \text{constant} \tag{7}$$

in coordinate space, which bears no resemblance to the form of the cylinder twisting mode

$$\phi_{\text{cyl}}^{TW}(x) \to \left[\frac{1}{|x|} + \Delta'\right] \tag{8}$$

To understand the cylinder limits we re-introduce the interchange term δ. The solution of eqn.(3) can then be expressed in associated Legendre functions and leads to the asymptotic forms for large $|\eta|$

$$\phi^{TW} \sim |\eta|^{\nu_+} + \Delta_B^{TW}|\eta|^{\nu_-} \tag{9a}$$

$$\phi^{TE} \sim \left[|\eta|^{\nu_+} + \Delta_B^{TE}|\eta|^{\nu_-}\right] sgn(\eta) \tag{9b}$$

with

$$\nu_\pm = -\frac{1}{2} \pm (\frac{1}{4} - \delta)^{1/2},$$

and

$$\Delta_B^{TW} = \frac{\pi\Gamma(1+\nu-\lambda)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{3}{2}+\nu)\Gamma(-\lambda-\nu)(2s)^{1+2\nu}}\left\{\tan\pi\nu - \tan[\frac{\pi}{2}(\lambda+\nu)]\right\} \tag{10a}$$

$$\Delta_B^{TE} = \frac{\pi\Gamma(1+\nu-\lambda)}{\Gamma(\frac{1}{2}+\nu))\Gamma(\frac{3}{2}+\nu)\Gamma(-\lambda-\nu)(2s)^{1+2\nu}}\left\{\tan\pi\nu + \cot\left[\frac{\pi}{2}(\lambda+\nu)\right]\right\} \tag{10b}$$

(where $\nu = \nu_+$).

The expressions (9) correspond to

$$\phi^{TW}(x) \sim |x|^{\nu_-} + \Delta^{TW}|x|^{\nu_+} \tag{11a}$$

$$\phi^{TE}(x) \sim \left[|x|^{\nu_-} + \Delta^{TE}|x|^{\nu_+}\right] sgn(x) \tag{11b}$$

in coordinate space, with [6]

$$\Delta^{TW} = \cot\left(\frac{\pi\nu}{2}\right)\frac{\Gamma(1-\nu)}{\nu\Gamma(1+\nu)}\Delta_B^{TW} \tag{12a}$$

$$\Delta^{TE} = \tan\left(\frac{\pi\nu}{2}\right)\frac{\Gamma(1-\nu)}{\nu\Gamma(1+\nu)}\Delta_B^{TE} \tag{12b}$$

When λ and δ are both small (note that $\nu \simeq -\delta$)

$$\Delta^{TW} \simeq -\frac{1}{s}\left[1 - \frac{\lambda^2}{\delta^2}\right] \tag{13a}$$

and

$$\Delta^{TE} \simeq \frac{-1}{s}\left[1 + 2\delta(1 - \gamma + \ell n\left(\frac{s}{2}\right) - \frac{\pi^2\lambda^2}{12}\right] \tag{13b}$$

with $\gamma = 0.5771\ldots$, the Euler constant. Thus we see that Δ^{TE} always approaches the cylinder limit (here$= -1/s$) when δ and λ are small, but Δ^{TW} does so only if $\lambda^2 < \delta^2$; otherwise it remains positive – corresponding to instability for simple layer models. (N.B. The fact that Δ^{TW} is negative when $\lambda^2 < \delta^2$ implies, on the extended Strauss model, a potential for tight-aspect-ratio stabilisation of resistive ballooning modes in a tokamak if $s < |\epsilon(1 - 1/q^2)|^{2/3}$.)

3 Calculation of Δ_B^{TW} in Configuration Space

In a toroidal system with $\delta \neq 0$ the matching condition for high-n modes may be applied either in coordinate space, using Δ from the form $(|x|^{-1+\delta} + \Delta|x|^{-\delta})$ as $|x| \to 0$, or in ballooning-transform space using Δ_B from the form $\left(|\eta|^{-\delta} + \Delta_B|\eta|^{-1+\delta}\right)$ as $|\eta| \to \infty$.

For the tearing parity mode a smooth limit exists in coordinate space and in ballooning space as $\delta \to 0$. However for the twisting parity mode the limit $\delta \to 0$ is singular in coordinate space. This singular behaviour arises because, in addition to the discontinuous components $|x|^{-1+\delta}$ and $|x|^{-\delta}$, there is a continuous component (due to toroidal coupling) which becomes indistinguishable from the 'small' component $|x|^{-\delta}$ as $\delta \to 0$. However the presence of a continuous component at $x = 0$ does not affect the asymptotic, $|\eta| \to \infty$, behaviour in ballooning-space; this reflects only singularities at $x = 0$. Consequently Δ_B^{TW} is unaffected by the continuous component.

In the remainder of this section we show that, despite the singular behaviour of Δ when $\delta = 0$, the matching may still be carried out in coordinate space. This will open the way for a similiar calculation of low-n twisting modes, (for which the ballooning-transformation is not applicable) in the next section.

In coordinate space the singular part of the twisting parity mode is given by,

$$\int_{-\infty}^{-\eta_0} + \int_{\eta_0}^{\infty} d\eta \; e^{ix\eta}\left(|\eta|^{-\delta} + \Delta_B^{TW}|\eta|^{-1+\delta}\right) \tag{14}$$

where $\eta_0 \gg 1$. For $\delta \neq 0$ this takes the form $\left(|x|^{-1+\delta} + \Delta|x|^{-\delta}\right)$ but for $\delta = 0$ it becomes

$$2\pi\,\delta(x) - 2\Delta_B^{TW}\left[\log|x| + \log\eta_0\right] \tag{15}$$

as in eqn.(7). This indicates that calculation of toroidal twisting modes in coordinate space should be based not on coupling of the usual cylinder modes, but on coupling of δ-function modes. Then we can recognise Δ_B^{TW} in the behaviour near a critical surface as

$$\Delta_B^{TW} = -\frac{\pi \times \text{Coeff.of} \log|x|}{\text{Coeff.of}\,\delta(x)} \tag{16}$$

To illustrate this, we consider the well-known "$s-\alpha$" model[7] of high-n modes in a large aspect ratio tokamak. In ballooning space this is given by

$$\frac{d}{d\eta}\left[1+(s\eta-\alpha\sin\eta)^2\right]\frac{d\phi}{d\eta}+\alpha\left[\cos\eta+\sin\eta(s\eta-\alpha\sin\eta)\right]\phi=0 \tag{17}$$

Drake and Antonsen[8] obtained an asymptotic solution of eqn.(17) by expansion in α, as

$$\phi(\eta) \sim 1+\Delta_B^{TW}/|\eta| \tag{18}$$

with

$$\Delta_B^{TW} = \frac{\pi}{4}\frac{\alpha^2}{s^2}(s+2)\left[1-\frac{(s+2)}{s}\exp\left(\frac{-2}{s}\right)\right] \tag{19}$$

(Note that $\delta = 0$ in the $s-\alpha$ model and eqn.(19) agrees with the small α limit of the Strauss model as $s \to 0$.).

The $s-\alpha$ model in ballooning space can be considered as the fourier-transform of an equation for $\phi_m(x) = \phi(x-m)$ in coordinate space, namely

$$\begin{aligned}
&x\frac{d^2}{dx^2}(x\phi(x)) - \frac{x^2}{s^2}\phi(x)\\
&-\frac{\alpha}{s^2}\Big\{s(x^2+\frac{1}{2})\frac{d}{dx}[\phi(x+1)-\phi(x-1)] + sx\frac{d}{dx}[\phi(x+1)+\phi(x-1)]\\
&+sx[\phi(x+1)-\phi(x-1)] - \frac{1}{2}[\phi(x+1)+\phi(x-1)]\Big\}\\
&-\frac{\alpha^2}{2s^2}\Big\{(x^2+1)[\phi(x)-\frac{1}{2}(\phi(x+2)+\phi(x-2))] - x[\phi(x+2)-\phi(x-2)]\Big\} = 0
\end{aligned} \tag{20}$$

[This equation can also be derived from the general toroidal equations discussed in the next section.]

In the light of the preceding discussion we now seek a solution of eqn.(20) in the form $\phi = \phi^{(0)} + \alpha\phi^{(1)} + \alpha^2\phi^{(2)} + \ldots$ where $\phi^{(0)} = \delta(x)$ and $\phi^{(1)}, \phi^{(2)}$ arise from toroidal coupling. In first order we find that $\psi^{(1)}(x) = x\phi^{(1)}(x)$ has discontinuities at the 'sideband' positions $x \pm 1 = 0$,

$$\psi^{(1)} = \frac{\alpha}{4}\exp\left(\frac{1-|x|}{s}\right)\left[1-\frac{(s+2)}{s}\exp\left(\frac{-2}{s}\right)\right] sgn(x) \quad \text{for} |x| > 1 \tag{21}$$

and

$$\psi^{(1)} = \frac{\alpha}{2}\left(\frac{s+2}{s}\right)\exp\left(\frac{-1}{s}\right)\sinh\left(\frac{x}{s}\right) \qquad \text{for } |x| < 1. \tag{22}$$

In second order $\psi^{(2)}(x)$ is given, near $x = 0$, by

$$x\frac{d^2}{dx^2}\psi^{(2)}(x) = R^{(1)} \tag{23}$$

where

$$R^{(1)} = \frac{-\alpha}{2s^2}\left\{\left[(1+s)\psi^{(1)} - s\frac{d\psi^{(1)}}{dx}\right]_1 - \left[(1+s)\psi^{(1)} + s\frac{d\psi^{(1)}}{dx}\right]_{-1}\right\} \tag{24}$$

and the logarithmic contribution to ψ is $R^{(1)}x\log|x|$. Then, from eqn.(16), evaluating $R^{(1)}$ we have

$$\Delta_B^{TW} = \frac{\pi\alpha^2}{4s^2}(s+2)\left[1 - \frac{(s+2)}{s}\exp\left(\frac{-2}{s}\right)\right] \tag{25}$$

in accord with the calculation in ballooning space.

Note that the essential features of the high-n calculation in coordinate space are that toroidal coupling induces discontinuous 'sidebands' of a δ-function perturbation on a resonant surface and that these sidebands in turn induce a logarithmic singularity at the original resonant surface. These features will also appear in the calculation of **low-n** modes to be described in the next section.

4 Low-n Resistive Ballooning Modes

We now turn to the main topic of this work, the description of low-n ballooning (twisting parity) modes in a large aspect ratio tokamak ($\delta = 0$). As in part I, the marginal ideal mhd equations are

$$r\frac{d\psi_m}{dr} = \frac{L_m^m\chi_m}{(m-nq)} + \epsilon\sum_{\pm}\left[\frac{L_m^{m\pm1}\chi_{m\pm1} + M_m^{m\pm1}\psi_{m\pm1}}{m\pm1-nq}\right] \tag{26}$$

$$r(m-nq)\frac{d}{dr}\frac{\chi_m}{(m-nq)} = \frac{P_m^m\psi_m}{(m-nq)} + \epsilon\sum_{\pm}\left[\frac{N_m^{m\pm1}\chi_{m\pm1} + P_m^{m\pm1}\psi_{m\pm1}}{m\pm1-nq}\right] \tag{27}$$

where $\psi_m = (m-nq)\phi_m$ and the coefficients L,M,N,P have been given in part 1. The ordering parameter ϵ has been introduced to identify terms representing toroidal coupling.

Before discussing the solutions of eqns.(26) and (27) it is necessary to reconsider the matching problem. We recall that when $\delta \neq 0$ the outer, ideal mhd, solution near a critical surface r_m is

$$\phi \sim A_{L,R}\left\{|r-r_m|^{-1+\delta} + \Delta_{L,R}|r-r_m|^{-\delta}\right\} \tag{28}$$

where the subscripts L,R denote left and right of the critical surface. This can be written as a symmetric part

$$(A_R + A_L)|r - r_m|^{-1+\delta} + (A_R\Delta_R + A_L\Delta_L)\,|r - r_m|^{-\delta} \tag{29}$$

and an antisymmetric part

$$\left\{(A_R - A_L)|r - r_m|^{-1+\delta} + (A_R\Delta_R - A_L\Delta_L)|r - r_m|^{-\delta}\right\} sgn(r - r_m). \tag{30}$$

These have to be matched to the symmetric and antisymmetric inner layer solutions

$$A(|y|^{-1+\delta} + \Delta_-(\omega)|y|^{-\delta}) + B\left(|y|^{-1+\delta} + \Delta_+(\omega)|y|^{-\delta}\right) sgn(y) \tag{31}$$

where $y = (r - r_m)/\sigma$, with σ the layer width. The matching leads, as in part 1, to a dispersion equation

$$\Delta_+(\omega)\Delta_-(\omega) - \tfrac{1}{2}\sigma^{1-2\delta}(\Delta_R + \Delta_L)(\Delta_+(\omega) + \Delta_-(\omega)) + \sigma^{2(1-2\delta)}\Delta_L\Delta_R = 0. \tag{32}$$

In general, the eigenmodes need not have definite overall parity. However, since it is implicit in the matching that $\sigma \to 0$, separation into eigenmodes with definite parity **in the layer** occurs so long as $\Delta_+(\omega)$ and $\Delta_-(\omega)$ do not vanish simultaneously. Then we find an eigenmode of tearing parity (odd ϕ) in the layer, for which

$$\Delta_+(\omega) = \tfrac{1}{2}\sigma^{1-2\delta}(\Delta_R + \Delta_L) \text{ and } A_R \simeq -A_L. \tag{33}$$

and an eigenmode with twisting (even ϕ) parity, for which

$$\Delta_-(\omega) = \tfrac{1}{2}\sigma^{1-2\delta}(\Delta_R + \Delta_L) \text{ and } A_R \simeq +A_L. \tag{34}$$

When $\delta \to 0$, we will see that, as in the high-n limit, the twisting and tearing modes are no longer distinguished merely by their parity; **each then has a different functional form.** The antisymmetric part of the solution becomes

$$\phi_A \sim \left\{\frac{1}{|r - r_m|} + D\right\} sgn(r - r_m) \tag{35}$$

while the symmetric part becomes

$$\phi_s \sim \{2\pi\delta(r - r_m) - C\ell n|r - r_m|\}\,. \tag{36}$$

These have to be matched to the inner solutions which themselves become

$$\left(2\pi\delta(y) - 2\Delta_B^{TW}(\omega)\ell n|y|\right) \quad \text{and} \quad \left(\frac{1}{|y|} + \frac{\pi\Delta_B^{TE}(\omega)}{2}\right) sgn(y) \tag{37}$$

Again assuming that $\Delta_B^{TE}(\omega)$ and $\Delta_B^{TW}(\omega)$ do not vanish simultaneously, this leads to an eigenmode of tearing parity in the layer, with

$$\Delta_B^{TE}(\omega) = 2\sigma D/\pi \tag{38}$$

and an eigenmode of twisting parity in the layer, with

$$\Delta_B^{TW}(\omega) = \sigma C/2. \tag{39}$$

As the twisting parity modes are uniquely identified by the appearance of δ-function and logarithmic singularities at resonant surfaces we can calculate low-n twisting modes in a large aspect ratio tokamak by finding the logarithmic response of ψ_m, induced by toroidal coupling, to a solution

$$\phi^{(0)} = \sum \beta_m \delta(m - nq(r)) \tag{40}$$

of the lowest order (uncoupled) equations - just as we did for high-n modes in section (3).

The logarithmic response is second order in toroidal coupling. The first order response $\psi_{m\pm1}^{(1)}$ contains contributions from β_m and $\beta_{m\pm2}$ and to describe these it is convenient to introduce functions $\psi_{m\pm1}^L, \psi_{m\pm1}^R$ which satisfy the **uncoupled** equations, with $\psi^L = 0$ at $r = 0$ and $\psi^R = 0$ at $r = a$. These functions are continuous and have unit amplitude at their resonant surfaces. Then between resonant surfaces $\psi_{m\pm1}^{(1)}$ can be expressed in terms of ψ^L and ψ^R and it has the following discontinuities at r_m and $r_{m\pm2}$:

$$\begin{aligned}\left[r\frac{d\psi_{m\pm1}^{(1)}}{dr}\right]_{r_m} &= \beta_m \tau_{m\pm1}^m, \quad \left[\psi_{m\pm1}^{(1)}\right]_{r_m} = \beta_m \sigma_{m\pm1}^m \\ \left[r\frac{d\psi_{m\pm1}^{(1)}}{dr}\right]_{r_{m\pm2}} &= \beta_{m\pm2}\tau_{m\pm1}^{m\pm2}, \quad \left[\psi_{m\pm1}^{(1)}\right]_{r_{m\pm2}} = \beta_{m\pm2}\sigma_{m\pm1}^{m\pm2}\end{aligned} \tag{41}$$

where

$$\begin{aligned}\tau_{m\pm1}^m &\equiv \pm\frac{L_{m\pm1}^{m\pm1}P_{m\pm1}^m}{nq'}\bigg|_{r_m} = \pm\frac{(m\pm1)^2}{m}\frac{\alpha}{2s}(1+s)\bigg|_{r_m} \\ \tau_{m\pm1}^{m\pm2} &= \pm\frac{L_{m\pm1}^{m\pm1}P_{m\pm1}^{m\pm2}}{nq'}\bigg|_{r_{m\pm2}} = \mp\frac{(m\pm1)^2}{(m\pm2)}\frac{\alpha}{2s}(1+s)\bigg|_{r_{m\pm2}} \\ \sigma_{m\pm1}^m &= M_{m\pm1}^m(r_m) = -\frac{(m\pm1)}{m}\frac{\alpha}{2s}(r_m) \\ \sigma_{m\pm1}^{m\pm2} &= M_{m\pm1}^{m\pm2}(r_{m\pm2}) = -\frac{(m\pm1)}{(m\pm2)}\frac{\alpha}{2s}(r_{m\pm2})\end{aligned} \tag{42}$$

and, as defined earlier, $s = rq'/q$ and $\alpha = -2Rp'q^2/B^2$.

If the resonant surfaces r_m and r_{m-2} both lie within the plasma, $\psi_{m-1}^{(1)}$ is given in terms of ψ_{m-1}^L for $0 < r < r_{m-2}$, by a combination of ψ_{m-1}^L and ψ_{m-1}^R for $r_{m-2} < r < r_m$, and in terms of ψ^R_{m-1} for $r_m < r \leq a$. The discontinuities (41) determine the coefficients of ψ^L_{m-1} and ψ^R_{m-1} in the three regions. If the resonance r_{m-2} lies outside the plasma, $\beta_{m-2} = 0$ and $\psi^{(1)}_{m-1}$ is given in terms of ψ^L_{m-1} for all $0 < r < r_m$. Similar remarks apply to $\psi^{(1)}_{m+1}$.

In the region $r_{m-2} < r < r_m$ we find, for example,

$$\psi^{(1)}_{m-1}(r) = \frac{\beta_m}{r_{m-1}\Delta^0_{m-1}}\left[\frac{(m-1)}{m}\frac{\alpha}{2s}\left(r\frac{d}{dr}\psi^R_{m-1} - (m-1)(1+s)\psi^R_{m-1}\right)\right]_{r_m}\psi^L_{m-1}(r)$$

$$+\frac{\beta_{m-2}}{r_{m-1}\Delta^0_{m-1}}\left[\frac{(m-1)}{(m-2)}\frac{\alpha}{2s}\left(r\frac{d}{dr}\psi^L_{m-1} + (m-1)(1+s)\psi^L_{m-1}\right)\right]_{r_{m-2}}\psi^R_{m-1}(r) \tag{43}$$

$$\psi^{(1)}_{m+1} = \psi^L_{m+1}\left\{\frac{\beta_m}{r_{m+1}\Delta^0_{m+1}}\left[\frac{(m+1)}{m}\frac{\alpha}{2s}\left(r\frac{d}{dr}\psi^R_{m+1} + (m+1)(1+s)\psi^R_{m+1}\right)\right]_{r_m}\right.$$

$$\left. + \frac{\beta_{m+2}}{r_{m+1}\Delta^0_{m+1}}\left[\frac{(m+1)}{(m+2)}\frac{\alpha}{2s}\left(r\frac{d}{dr}\psi^R_{m+1} - (m+1)(1+s)\psi^R_{m+1}\right)\right]_{r_{m+2}}\right\} \tag{44}$$

where $\Delta^0_{m\pm1}$ are the conventional **tearing** mode discontinuities in the absence of coupling

$$\Delta^0_m = \left[\frac{1}{\psi_m}\left(\frac{d\psi^R_m}{dr} - \frac{d\psi^L_m}{dr}\right)\right]_{r=r_m}. \tag{45}$$

(If a sideband $m \pm 1$ is non-resonant it is convenient to normalise the two solutions to unity at r_m and then $r_{m\pm1}\Delta^0_{m\pm1}$ is replaced by $r_m W$ where W is the Wronskian of the solutions $\psi^R_{m\pm1}$ and $\psi^L_{m\pm1}$.)

In second order we have for $\psi^{(2)}_m(r)$

$$\left[r\frac{d}{dr}r\frac{d}{dr} - m^2 - \frac{rmq\sigma'}{(m-nq)}\right]\psi^{(2)}_m(r) = \frac{R^{(1)}_m(r)}{(m-nq)} \tag{46}$$

where $R^{(1)}_m(r)$ is given in terms of $\psi^{(1)}_{m\pm1}$ and $\sigma = (1/r)(r^2/q)'$ (see part I). The logarithmic singularity in $\psi^{(2)}_m$ at $(m-nq) = 0$ is therefore given by

$$\psi^{(2)}_m(r) = \frac{R^{(1)}_m(r_m)}{msr_m}.(r - r_m)\log|r - r_m| \tag{47}$$

so that, in eqn.(36),

$$C = \frac{2\pi R^{(1)}_m(r_m)}{\beta_m msr_m}. \tag{48}$$

Writing $\hat{\Delta}_m$ for Δ^{TW}_B, and expressing $R^{(1)}_m$ explicitly in terms of $\psi^{(1)}_{m\pm1}$, this leads to the relation

$$\beta_m\hat{\Delta}_m = \sum_{\pm}\frac{\pi\alpha m}{2s(m\pm1)}\left[r\frac{d}{dr}\psi^{(1)}_{m\pm1} \pm (m\pm1)(s+1)\psi^{(1)}_{m\pm1}\right]_{r=r_m}. \tag{49}$$

Recalling that $\psi^{(1)}_{m\pm1}$ involve only β_m and $\beta_{m\pm2}$, eqn.(49) leads to a three-term recurrence relation

$$\beta_{m-2}E_{m-2} + \beta_m\left(E_m - \hat{\Delta}_m\right) + \beta_{m+2}E_{m+2} = 0 \tag{50}$$

and hence to a tridiagonal "E-matrix" for toroidal twisting modes

$$\left|E - \hat{\Delta}_m\right| = 0 \tag{51}$$

where $\hat{\Delta}_m = \{\text{diagonal}\hat{\Delta}_m\}$ and

$$\begin{aligned}
E_{m,m+2} &= -\frac{\pi}{4(m+2)}\frac{1}{r_{m+1}\Delta^0_{m+1}s(r_m)}\; D^{L,+}_{m+1}(r_m)D^{R,-}_{m+1}(r_{m+2}) \\
E_{m,m-2} &= -\frac{\pi}{4(m-2)}\frac{1}{r_{m-1}\Delta^0_{m-1}s(r_m)}\; D^{R,-}_{m-1}(r_m)D^{L,+}_{m-1}(r_{m-2}) \\
E_{m,m} &= -\frac{\pi}{4m}\sum_{\pm}\frac{1}{r_{m\pm1}\Delta^0_{m\pm1}s(r_m)}\; D^{R,\pm}_{m\pm1}(r_m)D^{L,\pm}_{m\pm1}(r_m)
\end{aligned} \tag{52}$$

where

$$D^{L,\pm}_k = \frac{\alpha}{s}\left[r\frac{d}{dr}\psi^L_k \pm k(1+s)\psi^L_k\right] \text{ etc.} \tag{53}$$

Note that, despite the fact that the toroidal twisting modes are not directly related to the cylinder modes, the elements of the E-matrix are given in terms of uncoupled cylinder solutions ψ^L and ψ^R and the cylinder **tearing** mode quantities Δ^0_m.

It is interesting to see the relation of the E-matrix (52) for twisting modes of general n to the high-n limit discussed in section (3). At large m

$$\psi^L_m(r) \sim \left(\frac{r}{r_m}\right)^m \text{ and } \psi^R_m(r) \sim \left(\frac{r}{r_m}\right)^{-m} \tag{54}$$

and

$$\frac{r_m}{r_{m\pm1}} \sim \left(1 \mp \frac{1}{ms}\right). \tag{55}$$

Therefore

$$E_{m,m} \sim \frac{\pi}{4}\frac{\alpha^2}{s^2}(s+2) \equiv E_0. \tag{56}$$

and

$$E_{m,m\pm2} \sim \frac{-\pi}{8}\frac{\alpha^2}{s^3}(s+2)^2\exp\left(\frac{-2}{s}\right) \equiv E_2. \tag{57}$$

The harmonic number m no longer appears in the E-matrix and all resonant surfaces are now equivalent. The coefficients β_m therefore differ only by a phase factor so that $\beta_m \sim \exp(iu)$. Then from eqn.(50)

$$\Delta = E_0 + 2E_2\cos(2u). \tag{58}$$

This is in agreement with the value given in section (3) if one sets $u = 0$, ie, $\beta_m = 1$.

5 Summary and Conclusions

In parts I and II of this work we have discussed resonant toroidal modes described by ideal mhd equations outside critical layers at $nq = m$, where there are discontinuities Δ_m in the solution. These discontinuities are matched to corresponding $\Delta(\omega)$ calculated from the critical layer equations, which contain non-ideal terms such as resistivity.

In a torus the ideal mhd equations do not specify the individual Δ_m, as they do for a cylinder. Instead they specify a single relation between the Δ_m in the form of an "E-matrix" such that $|E - \Delta| = 0$, where $\Delta = \{\text{diagonal } \Delta_m\}$.

Part I concerned toroidal tearing modes, in which the perturbed magnetic field has even parity in the critical layers. We showed that in a large aspect ratio torus these modes are, as one would expect, a natural extension of the cylinder tearing modes.

In the present paper we have considered toroidal twisting modes, in which the perturbed magnetic field has odd parity in the critical layer. We have shown that, when the interchange term δ is small, these modes are intrinsically toroidal. Even when the toroidal coupling is weak they are not related to the cylinder twisting modes.

The different character of the toroidal tearing and twisting modes is already apparent in the high-n limit – when modes of either parity can be described using the ballooning transformation. Then a single quantity Δ_B, defined through the asymptotic form of the ideal mhd solution in η-transform space, replaces the Δ_m. As the interchange term $\delta \to 0, \Delta_B$ remains finite for both twisting and tearing modes but the corresponding coordinate space quantity Δ has a singular limit for twisting modes. Nevertheless we have shown that high-n twisting modes **can** be calculated in coordinate space. Essentially, for the twisting mode the usual $|x|^{\nu_\pm}$ forms near a critical surface are replaced by a δ-function and a logarithmic singularity. The ratio of the coefficients of these terms is related to the ballooning space quantity Δ_B, not to the coordinate space quantity Δ.

The most important result of the present work is that this calculation of high-n twisting modes in coordinate space, and the identification of Δ_B as the ratio of coefficients of singular terms, can be extended to deal with **low-n** twisting modes – for which the ballooning transformation itself is not applicable. Thus in a large aspect ratio tokamak, low-n twisting-ballooning modes are calculated, not by expanding about uncoupled cylinder twisting modes, but by expanding about singular δ-function modes on each critical surface. Then the toroidal coupling induces, in second order, a logarithmic contribution at the same resonant surface – so defining Δ_B^{TW} and hence the appropriate E-matrix. Furthermore the elements of the E-matrix can be computed from the uncoupled cylinder equations – despite the fact that the toroidal twisting modes lack any resemblance to cylinder modes. Finally, this E matrix can be used in conjunction with the ballooning space quantity $\Delta_B^{TW}(\omega)$, containing the influence of non-ideal effects such as resistivity, to form the dispersion equation for low n-modes.

Appendix Δ' notation

The symbol Δ, carrying suffix, superfix or argument, appears in many places throughout this paper. Sometimes it refers to a quantity in configuration space, within a resonant layer or external to it, and sometimes it refers to a quantity in ballooning space. While the meaning of each usage should be self-evident from the context, this appendix gathers them together, along with their definitions.

The stability index for slab (or cylinder) tearing modes, originally introduced by Furth, Killeen and Rosenbluth[10] as the discontinuity in logarithmic derivative, is denoted by Δ'. For a particular poloidal harmonic in a cylinder this quantity is denoted by Δ_m° (as in Eq.45), while Δ_m denotes the equivalent discontinuity at the resonant surface r_m for a **toroidal** mode. [The definition of $\Delta', \Delta_m^{\circ}$ and Δ_m as the discontinuity in logarithmic derivative is only valid when $\delta = 0$. When $\delta \neq 0$ they are defined, as by Coppi, Greene and Johnson[6], as the discontinuity in the "small" solution at the singularity. See part I.]

The quantities corresponding to Δ_m obtained from solutions of the equations **within** the critical layers are indicated by explicitly displaying the eigenvalue ω as an argument. Thus $\Delta_m(\omega)$ is the layer quantity to be matched to the external quantity Δ_m at each critical surface.

In ballooning space, solutions of the ideal equations have the general structure

$$\phi(\eta) \simeq |\eta|^{\nu_+} + \Delta_B |\eta|^{\nu_-} \quad \text{for } |\eta| \to \infty \tag{A.1}$$

and are of odd parity for tearing modes and even parity for twisting modes, thus defining (as in Eqs.9) the ballooning space quantities Δ_B^{TE} and Δ_B^{TW}. The form corresponding to (A.1) in configuration space is given in Eqs.(11) which similarly define the configuration space quantities Δ^{TE} and Δ^{TW}. Because both η and $x = (r - r_m)nq'$ are dimensionless variables, the quantities $\Delta_B, \Delta_B^{TE}, \Delta_B^{TW}, \Delta^{TE}$ and Δ^{TW} are all dimensionless.

In section 4, the quantities Δ_L and Δ_R describe the **ratio** of the small to the large solution on the left and on the right of the resonant surface. They have the dimension $L^{(\nu_- - \nu_+)}$ since the configuration space variable used in section 4 is $(r - r_m)$. Using Eqs.(29) and (30)

$$\Delta^{TE} = \frac{1}{2}(\Delta_L + \Delta_R)(nq')^{(\nu_- - \nu_+)} \tag{A.2}$$

and in the limit $\delta \to 0$the original Δ' of Furth, Killeen and Rosenbluth is

$$\Delta' = (\Delta_L + \Delta_R) \tag{A.3}$$

Finally, Eq.(31) defines $\Delta_+(\omega)$ and $\Delta_-(\omega)$ as the ratio of small solution to large solution for the resonant layer equations, having tearing parity ($\Delta_-(\omega)$) and twisting parity ($\Delta_+(\omega)$) respectively. These quantities are dimensionless.

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