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Abstract

Action principles for the Vlasov equation are presented. Four previously known action principles, which differ by the choice of dynamical variables, are described and the interelationship between them is discussed. A new action principle called the *leaf action*, which manifestly preserves the *Casimir invariants* and possess a single function as the dynamical variable, is presented. The relationship to the *noncanonical Hamitonian formalism* is also explored.

The main purpose of this communication is to present a new action principle for the Vlasov equation. We call this new action the *leaf action* because the theory is described in terms of a variable whose dynamics manifestly preserves certain invariants of the system, the so-called *Casimir invariants* that are associated with conservation of phase-space volume. The name leaf action arises because the constraint surfaces determined by these invariants are called *symplectic leaves*. In addition to preserving all the Casimir invariants, the leaf action has the novel and desirable feature of being variational with a single function as its dynamical variable.

Another purpose of this communication is to display four other action principles for the Vlasov equation. These action principles for the most part are not new, but we include them here for completeness. In the past they were given independently, but here the interelation between these action principles is discussed, along with their relationship to the noncanonical Hamiltonian formalism. We aim to provide a single easily accessible source for this material.

All of the action principles given here are for the Vlasov-Poisson equation. We choose to treat this system rather than the Maxwell-Vlasov equations, since it embodies the particle dynamics, where the major difficulty lies when attempting to make a collisionless kinetic theory variational; the electromagnetic field part has the standard form. This description where the electric field is eliminated via Poisson's equation results in a slight change in the particle part of action principle (e.g. the factor of $\frac{1}{2}$ in the last terms of Eqs. (10), (12), (17), (18), and (26) below), a change which to our knowledge is new. In any event the generalization from the results presented here to the Vlasov-Maxwell theory is straightforward.

It is well known by now that the Vlasov equation is a Hamiltonian system with so-called noncanonical or Lie-Poisson bracket structure. Traditionally the Vlasov equation is written as a partial differential equation for the smooth Vlasov distribution function $f(\mathbf{z}, t)$ (we use $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ to denote the phase-space coordinates, and suppress the species index):

$$\frac{\partial f}{\partial t} + [f, H] = 0 \ , \tag{1}$$

where $H(\mathbf{z}, t) = \mathbf{p}^2/2m + e\phi(\mathbf{q}, t)$ is the particle Hamiltonian, and

$$[f, g] = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{q}} \tag{2}$$

is the canonical particle Poisson bracket. Now if we consider observables $\mathcal{F}[f]$ which are functionals on the space of distribution function f, Morrison [1,2] discovered that the Vlasov equation (1) is equivalent to the Hamiltonian equation

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \ , \tag{3}$$

where $\mathcal{H}$ is the Hamiltonian whose functional derivative is H: $\delta\mathcal{H}/\delta f = H$, and $\{\,\cdot\,,\cdot\,\}$ is a Lie-Poisson bracket, defined by

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \int d^6\mathbf{z}\, f \left[\frac{\delta \mathcal{F}_1}{\delta f}, \frac{\delta \mathcal{F}_2}{\delta f}\right] \ . \tag{4}$$

For a derivation of this Lie-Poisson bracket from the canonical Hamiltonian formalism for particle motion see [2,3].

An important property of the Lie-Poisson bracket (4) is its infinite degeneracy. Consider observables of the form

$$\mathcal{C}[f] \doteq \int d^6\mathbf{z}\, C(f) \ , \tag{5}$$

where $C(f)$ is an arbitrary smooth function, one can easily show that $\mathcal{C}$ commutes with all functionals of f. Thus regardless of the actual Hamiltonian $\mathcal{H}$, $\mathcal{C}$ is conserved:

$$\frac{d\mathcal{C}}{dt} = \{\mathcal{C}, \mathcal{H}\} = \int d^6\mathbf{z}\, f[C'(f), H] = 0 \ . \tag{6}$$

The Casimir invariants $\mathcal{C}$ represent the degenerate directions of the Lie-Poisson bracket. Their level sets foliate the space of distribution into symplectic leaves, on which the dynamics is constrained. The physical meaning of these Casimir invariants was discussed by Gardner [4], Morrison [5], and Morrison and Pfirsch [6]. Simply put, it is as follows. Suppose we partition the particle phase space $\mathbf{z}$ into small cells of equal volume, and to each cell attach a certain value of f. Then specifying all Casimirs (thus a symplectic leaf) is equivalent to specifying the number of cells that have a given value of f, the latter is obviously conserved by Liouville's theorem.

It is desirable to restrict the Vlasov equation to a symplectic leaf on which all points are dynamically accessible (subject only to the energy constraint). This may be of particular importance in the study of plasma turbulence, where statistical mechanical description is widely used. Crawford and Hislop [7] considered such a restriction for the one-dimensional electrostatic case. They first introduced a leaf coordinate, $W(q,p,t)$, to represent all states close to an equilibrium $f_0(p)$:

$$f(q,p) = e^{\{W,\cdot\}} f_0 \tag{7}$$

(the subset $W = W(f_0)$ is excluded for obvious reasons). Then they derived from the Vlasov equation (1), by an iterative scheme, the equation for W: $\frac{\partial W}{\partial t} = X_H(W)$, where X_H is a formal infinite series.

In the present work we consider the restriction of the Vlasov equation to a symplectic leaf in the general case, by utilizing the power of an action principle. We first give the four old action principles for the Vlasov equation known to us. Of these four action principles, the first two are in terms of Lagrangian variables which are the particle coordinates, the third is in terms of Eulerian variables which are functions on particle phase space, and the fourth uses a mixed representation. The leaf action principle, which we finally present, also uses a mixed representation.

The first is the *Low Lagrangian* [8–11]. The dynamical variables are the particle position $\mathbf{q}(\mathbf{z}_0, t)$ only, where $\mathbf{z}_0 = (\mathbf{Q}, \mathbf{P})$ labels the particles. Poisson's equation is treated as a constraint because it does not contain any time derivative. The electrostatic potential $\phi(\mathbf{q}, t)$ is then solved by the Green's function method (the sum over species is implied):

$$\phi(\mathbf{q},t) = \int d^6\mathbf{z}_0'\, K(\mathbf{q}(\mathbf{z}_0,t)|\mathbf{q}(\mathbf{z}_0',t)) f_0(\mathbf{z}_0') \tag{8a}$$

$$= \int d^6\mathbf{z}'\, K(\mathbf{q}|\mathbf{q}') f(\mathbf{z}',t) \tag{8b}$$

where $K(\mathbf{q}|\mathbf{q}') = K(\mathbf{q}'|\mathbf{q})$ is the Green's function for Poisson's equation:

$$\nabla^2 K(\mathbf{q}|\mathbf{q}') = -4\pi e\, \delta^3(\mathbf{q}-\mathbf{q}')\ , \tag{9}$$

$f_0(\mathbf{z}_0)$ is a given Vlasov distribution in the labeling space, and $f(\mathbf{z},t) = f_0(\mathbf{z}_0)$. The Low Lagrangian then reads

$$\mathcal{A}[\mathbf{q}] = \int dt \int d^6\mathbf{z}_0\, f_0(\mathbf{z}_0) \left[\frac{m}{2}\dot{\mathbf{q}}^2(\mathbf{z}_0,t) - \frac{e}{2}\phi(\mathbf{q}(\mathbf{z}_0,t),t)\right]\ , \tag{10}$$

where $f_0(\mathbf{z}_0)$ is a smooth Vlasov distribution in the labeling space, and ϕ is to be viewed as a shorthand for the expression of Eq. (8a). Note that (10) is simply the continuum version of Hamilton's principle. Variation yields the equations of motion

$$m\ddot{\mathbf{q}} = -e\nabla\phi\ , \tag{11}$$

which can be shown to be equivalent to the Vlasov equation by standard manipulations.

The second action is a close cousin of the Low Lagrangian. It is known as the *phase-space action*, and is obtained by a Legendre transform $(\mathbf{q}, \dot{\mathbf{q}}) \mapsto (\mathbf{q}, \mathbf{p})$, where $\mathbf{p} = m\dot{\mathbf{q}}$. The action then becomes

$$\mathcal{A}[\mathbf{q},\mathbf{p}] = \int dt \int d^6\mathbf{z}_0\, f_0(\mathbf{z}_0) \left[\mathbf{p}\cdot\dot{\mathbf{q}} - \frac{\mathbf{p}^2}{2m} - \frac{e}{2}\phi(\mathbf{q},t)\right]\ . \tag{12}$$

Variations are made with respect to $\mathbf{q}$ and $\mathbf{p}$ independently, resulting in Hamilton's equations

$$\dot{\mathbf{q}} = \frac{\mathbf{p}}{m} , \qquad \dot{\mathbf{p}} = -e\nabla\phi , \tag{13}$$

which are of course equivalent to (11). This action possesses a clear geometrical meaning (the integrand is a one-form in phase space) which makes it amenable to the powerful Lie transform technique [12]. It was used by Littlejohn [13] to greatly simplify the guiding-center theory, and has been successfully applied to the oscillating-center theory by Grebogi *et al* [14].

There is also a variant of the two actions just discussed that is worth mentioning. One can invert the coordinates $\mathbf{z}(\mathbf{z}_0, t)$ and re-express the action in terms of the labeling fields $\mathbf{z}_0(\mathbf{z}, t)$. This form has proven useful in formulating variational fluid theories (see e.g. [15].

The third action is called the *Clebsch action*, by analogy with its counterpart in fluid theories [16,2]. Define the potentials $\alpha(\mathbf{z}, t)$ and $\beta(\mathbf{z}, t)$ according to

$$f = \{\alpha, \beta\} = \frac{\partial\alpha}{\partial\mathbf{q}} \cdot \frac{\partial\beta}{\partial\mathbf{p}} - \frac{\partial\alpha}{\partial\mathbf{p}} \cdot \frac{\partial\beta}{\partial\mathbf{q}} . \tag{14}$$

One can then show that if α and β solve

$$\frac{\partial\alpha}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial\alpha}{\partial\mathbf{q}} - e\nabla\phi \cdot \frac{\partial\alpha}{\partial\mathbf{p}} = 0 , \tag{15}$$

$$\frac{\partial\beta}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial\beta}{\partial\mathbf{q}} - e\nabla\phi \cdot \frac{\partial\beta}{\partial\mathbf{p}} = 0 , \tag{16}$$

then f as constructed in (14) solves the Vlasov equation (1). Be reminded that in (15) and (16) α and β are coupled through ϕ, as given by (8b). Equations (15) and (16) are derivable from the action principle defined by

$$\mathcal{A}[\alpha, \beta] = \int dt \int d^6\mathbf{z} \left(\alpha\frac{\partial\beta}{\partial t} - \frac{\mathbf{p}^2}{2m}\{\alpha, \beta\} - \frac{e}{2}\int d^6\mathbf{z}' \{\alpha, \beta\} K(\mathbf{q}|\mathbf{q}')\{\alpha', \beta'\} \right) , \tag{17}$$

where $\alpha' \equiv \alpha(\mathbf{z}', t)$, and the same for β'. This action has a notable feature: the number of particles in a region of phase space, given by $\int d^6\mathbf{z}\ f(\mathbf{z}, t)$, is determined by the value of α

and β on the boundary of that region. This suggests its potential applications to problems that involve particle injection, which is not possible with other actions.

The fourth action in this brief review is called the *Hamilton-Jacobi action.* It was first constructed by Pfirsch [17], and has been generalized and applied by Pfirsch and Morrison [18,6,19] to derive unambiguous energy principles for kinetic guiding-center theories. It uses as its dynamical variables a mixed-variable generating function $S(\mathbf{q}, \mathbf{P}, t)$ for the particles, and a density function $\varphi(\mathbf{q}, \mathbf{P}, t)$ representing the number of particles on an orbit. The action reads

$$\mathcal{A}[\varphi, S] = -\int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\ \varphi \left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial \mathbf{q}}\right)^2 + \frac{e}{2}\phi(\mathbf{q}, t) - H_0\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right)\right], \tag{18}$$

where ϕ is defined by (8b) with $f(\mathbf{z}, t)$ given by

$$\varphi(\mathbf{q}, \mathbf{P}, t) = f\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}\right)\left|\frac{\partial^2 S}{\partial \mathbf{q}\partial \mathbf{P}}\right|, \tag{19}$$

and $H_0(\mathbf{Q}, \mathbf{P})$ is an arbitrary function. Variation with respect to φ immediately yields the Hamilton-Jacobi equation; variation with respect to S yields an equation that can be manipulated into the Vlasov equation for $f(\mathbf{z}, t)$. Similar manipulations show that $f_0(\mathbf{z}_0) \equiv f(\mathbf{z}, t)$ must be chosen so that

$$\{f_0, H_0\}_{(\mathbf{Q},\mathbf{P})} \equiv \frac{\partial f_0}{\partial \mathbf{Q}} \cdot \frac{\partial H_0}{\partial \mathbf{P}} - \frac{\partial f_0}{\partial \mathbf{P}} \cdot \frac{\partial H_0}{\partial \mathbf{Q}} = 0 \tag{20}$$

is satisfied. The detailed calculations can be found in [19].

The *leaf action*, which is derived from the phase-space action (12), bears a close resemblance to the Hamilton-Jacobi action (18). We start from the phase-space action in its general form

$$\mathcal{A}[\mathbf{q}, \mathbf{p}] = \int dt \int d^6\mathbf{z}_0 f_0(\mathbf{z}_0)(\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)) . \tag{21}$$

Here we concentrate on the particle part of the action only, which is why the ϕ term of (21) differs from the ϕ term of Eq. (12) by a factor of $\frac{1}{2}$. The function $f_0(\mathbf{z}_0)$ is a smooth Vlasov distribution in the labeling space which specifies a symplectic leaf; it is considered to be given. One can think of $\mathbf{z}_0$ as the initial particle position in phase space:

$$\mathbf{q}(\mathbf{z}_0, t=0) = \mathbf{Q} \ , \qquad \mathbf{p}(\mathbf{z}_0, t=0) = \mathbf{P} \ , \tag{22}$$

then $f_0(\mathbf{z}_0)$ would be the initial Vlasov distribution function. But one is by no means constrained to such an interpretation, which is sometimes inconvenient.

Now let $(\mathbf{q}, \mathbf{p})$ be generated by a single mixed-variable generating function $S(\mathbf{q}, \mathbf{P}, t)$:

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \ , \qquad \mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}} \ , \tag{23}$$

then the Vlasov distribution f, defined by $f(\mathbf{q}, \mathbf{p}, t) \equiv f_0(\mathbf{Q}, \mathbf{P})$, always stays on the same symplectic leaf determined by f_0. Thus by using S we have in effect restricted the variations to a given leaf. Here for explicitness we choose the F_2-type generating function, but one can also use any other type and whole calculation, which follows, carries through. In fact it is well known that a given type of generating function may develop singularities (caustics), so in practice one may have to switch between the various types of generating functions. Locally a generating function always exists, as shown e.g. by Arnol'd [20]. Equations (23) suggests us to view the phase space as foliated by Lagrangian manifolds (a Lagrangian submanifold is an n-dimensional subspace in the $2n$-dimensional phase space defined by the first of equation (23); in our problem $n = 3$), labeled by $\mathbf{P}$, whereas $\mathbf{Q}$ serves as coordinates on each manifold (in contrast to Eq. (22)).

We want S to generate the dynamics, not just a relabeling of particles; this is partially fulfilled by requiring

$$\frac{\partial S}{\partial t} \neq 0 \tag{24}$$

in a non-trivial way. The more precise criterion will be given later on. From (23) we have

$$\mathbf{p} \cdot \dot{\mathbf{q}} = \frac{\partial S}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} = \frac{dS}{dt} - \frac{\partial S}{\partial t} \ , \tag{25}$$

where dS/dt means total time derivative of S holding $\mathbf{z}_0$ fixed. Inserting this equation into (21) and changing the integration variables from $(\mathbf{Q}, \mathbf{P})$ to $(\mathbf{q}, \mathbf{P})$, we see that the dS/dt term drops out, and the action becomes

$$\mathcal{A}[S] = -\int dt \int d^3\mathbf{q}\, d^3\mathbf{P} \left|\frac{\partial^2 S}{\partial \mathbf{q} \partial \mathbf{P}}\right| f_0\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right)\left[\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right)\right] . \tag{26}$$

The van Vleck determinant $\omega = |\partial^2 S/\partial \mathbf{q}\partial \mathbf{P}|$, due to the mixed-variable representation, is required to be finite. The Jacobian matrix of this transformation

$$\omega_{ij} = \frac{\partial^2 S}{\partial q_i \partial P_j} \tag{27}$$

in fact constitutes the symplectic two-form in the mixed-variable space, because by (23) we have

$$\omega_{ij} dq_i \wedge dP_j = dq_i \wedge dp_i = dQ_j \wedge dP_j . \tag{28}$$

Therefore its inverse J^{ij}, where $J^{ij}\omega_{jk} = \delta^i_k$, defines a Poisson bracket in this space:

$$\{f, g\}_{(\mathbf{q},\mathbf{P})} = J^{ij}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_j}\frac{\partial g}{\partial q_i}\right) . \tag{29}$$

As a general property of cosymplectic forms the following holds true:

$$\frac{\partial}{\partial q_j}(\omega J^{ij}) = 0 = \frac{\partial}{\partial P_i}(\omega J^{ij}) . \tag{30}$$

It is straightforward to carry out the variation. After some algebra we find

$$0 = \frac{\delta \mathcal{A}}{\delta S} = \omega\left\{f_0\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right), \frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right)\right\}_{(\mathbf{q},\mathbf{P})} . \tag{31}$$

Since $\omega \neq 0$, and the bracket $\{\,\cdot\,,\,\cdot\,\}_{(\mathbf{q},\mathbf{P})}$ is non-degenerate, a general solution ("first integral") of (31) is

$$\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) = H_0\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right), \tag{32}$$

where $H_0(\mathbf{z}_0)$ is an arbitray function that commutes with $f_0(\mathbf{z}_0)$. This generalized form of the Hamilton-Jacobi equation was introduced by Pfirsch and Morrison [19]. It has the following interpretation: if we regard S as generating the canonical transformation from $(\mathbf{q},\mathbf{p})$ to $(\mathbf{Q},\mathbf{P})$, then H_0 is the Hamiltonian in the labeling space; that f_0 commutes with H_0 tells us that it is a solution of the Vlasov equation in that space.

From (23) and using the chain rule we can easily obtain

$$\left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = -\mathbf{J}\cdot\frac{\partial^2 S}{\partial \mathbf{q}\partial t}, \tag{33}$$

$$\left(\frac{\partial \mathbf{Q}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \frac{\partial^2 S}{\partial \mathbf{P}\partial t} + \frac{\partial^2 S}{\partial \mathbf{P}\partial \mathbf{P}}\cdot\left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} \tag{34}$$

and

$$\frac{\partial f_0}{\partial \mathbf{Q}} = \mathbf{J}\cdot\frac{\partial}{\partial \mathbf{q}} f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right) . \tag{35}$$

$$\frac{\partial f_0}{\partial \mathbf{P}} = \frac{\partial}{\partial \mathbf{P}} f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right) - \frac{\partial^2 S}{\partial \mathbf{P}\partial \mathbf{P}}\cdot\frac{\partial f_0}{\partial \mathbf{Q}} . \tag{36}$$

These relationships together with definition of f lead to

$$\left(\frac{\partial f}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \frac{\partial f_0}{\partial \mathbf{Q}}\cdot\left(\frac{\partial \mathbf{Q}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} + \frac{\partial f_0}{\partial \mathbf{P}}\cdot\left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \left\{ f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right), \frac{\partial S}{\partial t}\right\}_{(\mathbf{q},\mathbf{P})} . \tag{37}$$

Thus it becomes clear that (31) when re-expressed in $(\mathbf{q},\mathbf{p})$-space is precisely the Vlasov equation (1). Eq. (37) also provides a more precise criterion than (24): we must require that $\frac{\partial S}{\partial t}$ does not commute with f_0.

The leaf action (26) uses a single function S as variable. Comparing it with the Hamilto-Jacobi action (18) leads us to seek a Hamiltonian description. Since (26) is linear in $\frac{\partial S}{\partial t}$, Legendre transformation cannot be used directly; instead we must use the Dirac constraint method. Define momentum conjugate to S by

$$\Pi = -\left|\frac{\partial^2 S}{\partial \mathbf{q}\partial \mathbf{P}}\right| f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right) , \tag{38}$$

and use it as a constraint. By introducing a Lagrange multiplier $\lambda(\mathbf{q},\mathbf{P},t)$ for this constraint we then obtain a three-variable action

$$\mathcal{A}[\Pi,S,\lambda] = \int dt \int d^3\mathbf{q}\, d^3\mathbf{P} \left[\Pi\left(\frac{\partial S}{\partial t}+H\right) - \lambda\left(\Pi + \omega f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right)\right)\right] . \tag{39}$$

Variations yield three equations of motion. The requirement of inner-consistency leads to an equation for λ:

$$\{\lambda, f_0\}_{(\mathbf{q},\mathbf{P})} = 0 \ . \tag{40}$$

Thus the general solution for λ is H_0, same as that in (32). Putting this back into the action (39) reduces it to a two-variable action:

$$\mathcal{A}[\Pi, S] = \int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\ \Pi \left[\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) - H_0\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right)\right] \ . \tag{41}$$

Note that in (41) all reference to f_0 has disappeared. We can relax the constraint (38) and treat it as an initial condition, for if it holds at one time, then it will hold for all times. With this argument we delegate the specification of a leaf to the initial condition, and the action (41) applys to any such choice. The two actions (18) and (41) are identical if we equate Π with $-\varphi$. The geometrical setting for link between the spaces of S and (Π, S) is also discussed in [21].

We conclude this paper by establishing a relationship between the Hamilton-Jacobi action (41), and the Clebsch action (17). Define $\alpha(\mathbf{q}, \mathbf{p}, t)$ by

$$\alpha\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) = H_0\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right) \ , \tag{42}$$

then similar to (37) we find

$$\left(\frac{\partial \alpha}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \left\{\alpha\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right), \frac{\partial S}{\partial t}\right\}_{(\mathbf{q},\mathbf{P})} \ . \tag{43}$$

Therefore we have, for any function $\beta(\mathbf{q}, \mathbf{p}, t)$,

$$\int d^6\mathbf{z}\, \beta \frac{\partial \alpha}{\partial t} = -\int d^3\mathbf{q}\, d^3\mathbf{P} \left|\frac{\partial^2 S}{\partial \mathbf{q} \partial \mathbf{P}}\right| \left\{\alpha\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right), \beta\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right)\right\}_{(\mathbf{q},\mathbf{P})} \frac{\partial S}{\partial t} \ . \tag{44}$$

Now suppose we choose β to satisfy

$$\left\{\alpha\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right), \beta\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right)\right\}_{(\mathbf{q},\mathbf{P})} = -\Pi \Big/ \left|\frac{\partial^2 S}{\partial \mathbf{q} \partial \mathbf{P}}\right| \ , \tag{45}$$

then (44) becomes the first term of the action (41). Inserting the function Π as obtained from (45) into (41), and expressing the integral in terms of $(\mathbf{q}, \mathbf{p})$, we obtain the Clebsch action. The H_0 term contributes a constant to the action so can be omitted. Observe also that $\Pi = -\varphi$ together with (45) and (19) give the relationship expressed in Eq. (14).

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