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Energy conservation for the drift-wave system is shown to be separated into the wave-energy power balance equation and an ambient thermal-energy transport equation containing the anomalous transport fluxes produced by the fluctuations. The wave energy equation relates the wave energy density and wave energy flux to the anomalous transport flux and the dissipation of the fluctuations. The thermal balance equation determines the evolution of the temperature profiles from the divergence of the anomalous heat flux, the collisional heating and cooling mechanisms and the toroidal pumping effect.

1. INTRODUCTION

In this work, we describe the balance between the various fluctuation energy densities and the fluxes for systems containing drift wave turbulence. In conjunction with the fluctuation-energy balance, we consider the evolution of the background thermal energy or mean pressure profiles. The turbulently generated energy fluxes act to release the free energy stored in the background thermal energy.

The two energy densities of the fluctuation energies and the thermal energy may differ by orders of magnitude, but are freely exchanged with each other through the transfer terms calculated here. For example, the compressional work done on the system, that is a source increasing the thermal energy, is combined with the work done by the pressure gradient, that is a source term for the change of the fluctuation kinetic energy, to give a conservative energy flux. Also, in one interesting situation,[1] we will show in Sec. II that the so-called toroidal-pumping term in the thermal-energy evolution and the source term, that results from the finite-Larmor-radius thermal flux, in the total fluctuation energy are of the opposite signs. The names used for the various energy fluxes and the transfer mechanisms analyzed in this work are summarized in Table 1 for convenience.

The importance of the evolution of the total fluctuation energy is that all the transfer terms reduce to the conservative form (given by the divergence of a flux) and one can identify that there are two types of sources: one type is due to the interaction between the gradients of the mean fields and the fluctuations and the second type is the classical dissipation. Recently, Krommes and coworkers[2] have taken advantage of this generic form of the power balance in their calculations of upper bounds for the turbulent transport in bounded systems. They have used the global constraint that in the steady state, the "production" of fluctuation energy due to the interaction between the mean gradients and the fluctuations is balanced by the dissipation in the system. The resulting bounds are surprisingly good and the final state reasonably describes the physical properties of the system. In the present problem, one can, in principle, construct the similar variational principle from the equations developed in the present work.

Sometimes, the mean pressure equation is thought of as describing the evolution of the total energy because the fluctuation field energy and the kinetic energy are assumed to be far smaller than the internal thermal energy. Then, one is led to a paradox of whether he should call $\frac{3}{2}p\boldsymbol{V}$ or $\frac{5}{2}p\boldsymbol{V}$ as the convective energy flux per unit volume: The paradox arises because there exists the total energy conservation law

$$\frac{\partial}{\partial t}\left(E_{\mathrm{f}} + E_{\mathrm{k}} + \tfrac{3}{2}p\right) + \boldsymbol{\nabla}\cdot\mathcal{F} = 0, \tag{1.1}$$

where we define the total energy flux per unit volume

$$\mathcal{F} \doteq \boldsymbol{S}_{\mathrm{f}} + E_{\mathrm{k}}\boldsymbol{V} + \tfrac{5}{2}p\boldsymbol{V} + \boldsymbol{q},$$

E_{f} and E_{k} are the densities of the field energy and the kinetic energy respectively, $\boldsymbol{S}_{\mathrm{f}}$ is the field energy flux per unit volume, and $\boldsymbol{q}$ is the additional heat flux per unit volume counting for the conduction and other effects.[3] For simplicity, we neglect the viscosity and other source terms. As is known, $\frac{5}{2}p\boldsymbol{V}$ is the specific enthalpy flux, sum of the convective thermal energy flux and the energy flux due to the mechanical work performed on the system, including both the mean and the fluctuating parts. If one assumes that $E_{\mathrm{f}}, E_{\mathrm{k}} \ll p$, he obtains from Eq. (1.1)

$$\frac{\partial}{\partial t}\tfrac{3}{2}p + \boldsymbol{\nabla}\cdot\left(\tfrac{5}{2}p\boldsymbol{V} + \boldsymbol{q}\right) \approx 0, \tag{1.2}$$

which is directly contradictory to the thermal-energy equation

$$\frac{\partial}{\partial t}\tfrac{3}{2}p + \boldsymbol{\nabla}\cdot\left(\tfrac{3}{2}p\boldsymbol{V} + \boldsymbol{q}\right) = -p\boldsymbol{\nabla}\cdot\boldsymbol{V}, \tag{1.3}$$

where $\frac{3}{2}p\boldsymbol{V}$ is the convective thermal-energy flux per unit volume and $(-p\boldsymbol{\nabla}\cdot\boldsymbol{V})$ is the source term due to the compressional work. For the two equations (1.2) and (1.3) to be equal to each other, one must have $\boldsymbol{V}\cdot\boldsymbol{\nabla}p = 0$, which, in general, is not satisfied.

The explanation for the paradox is that it is not right to neglect E_{f} and E_{k} in Eq. (1.1) just because they are small; although they may be small, the time-rate of change of E_{f} and E_{k} may not be smaller than that of p. For example, if the electrons respond to the electrostatic field adiabatically, after taking the average over the magnetic surface or the $y-z$ plane, we find that qualitatively

$$\frac{|E_{\mathrm{f}}|}{\tau_{\mathrm{f}}} \sim \frac{|S_{\mathrm{f},x}|}{L} \rightarrow \tau_{\mathrm{f}} \sim \left|\frac{\widetilde{n}_i}{n_i}\right| \frac{L}{\left|\widetilde{V}_x\right|},$$

$$\frac{|p_i|}{\tau_{\mathrm{p}}} \sim \frac{|\widetilde{p}_i\widetilde{V}_x|}{L} \rightarrow \tau_{\mathrm{p}} \sim \left|\frac{\widetilde{p}_i}{p_i}\right|^{-1} \frac{L}{\left|\widetilde{V}_x\right|},$$

therefore, if we take $\widetilde{T}_i = 0$, we find the ratio

$$\frac{\tau_{\mathrm{f}}}{\tau_{\mathrm{p}}} \sim \left|\frac{\widetilde{n}_i}{n_i}\right| \left|\frac{\widetilde{p}_i}{p_i}\right| \sim \left|\frac{\widetilde{n}_i}{n_i}\right|^2,$$

where $|\ldots|$ is understood as the root-mean-square value of the argument, the tildes denote the fluctuations, and the τ's and L are the typical scales of the time and the space. In other words,

$$\begin{aligned} \left|\widetilde{p}_{\mathrm{i}}\widetilde{V}_x\right| &\sim T_i \left|\widetilde{n}_{\mathrm{i}}\widetilde{V}_x\right| \\ &\sim \left(\frac{T_i}{T_{\mathrm{e}}}\right) |S_{\mathrm{f},x}|. \end{aligned}$$

Thus, indeed, one sees that $|\partial E_{\mathrm{f}}/\partial t| \sim |\partial p/\partial t|$. One reaches a similar conclusion for the kinetic-energy part of the fluctuation energy.

The reader should note that although we prefer to call $\frac{3}{2}\langle\widetilde{p}\widetilde{V}\rangle$ as the convective thermal flux due to the microscale convections, some authors[4] would call it the anomalous conductive heat flux. The reason for designating the terms as conductive is that as in the quasilinear regime the flux can be shown to have the form $\frac{3}{2}\langle\widetilde{p}\widetilde{V}\rangle = -\chi_{\mathrm{a}}\nabla\langle p\rangle$, where χ_{a} is a function of $\nabla\langle p\rangle$, if one coarse-grains the inhomogeneous direction into many cells with the size larger than a correlation length of the turbulence but smaller than a macroscopic length. Yet, this argument would break down if the correlation length is of the order of, or larger than the macroscopic length. However, this is not generally believed to be the case for the drift wave problem.[5]

As for the fluctuation energy balance for the electromagnetic fluctuations and the electrons we show in Sec. II that

$$\frac{\partial}{\partial t}\overline{\mathcal{E}_{\mathrm{f}}} = \overline{-\tfrac{2}{3}\left(\tfrac{3}{5}\boldsymbol{Q}_E + \boldsymbol{Q}_{\mathrm{m}}\right)\cdot\frac{\boldsymbol{\nabla}\langle p\rangle}{\langle p\rangle}} + \overline{\frac{\boldsymbol{V}_{\mathrm{g}}}{\langle n\rangle}\cdot\langle\widetilde{p}\boldsymbol{\nabla}\widetilde{n}\rangle} - (\text{dissipation terms}). \tag{1.4}$$

Definitions of all the terms are left to be explained in Sec. II and it is sufficient here to say that the first overlined term on the right-hand side of Eq. (1.4) describes the production of the fluctuation energy $\overline{\mathcal{E}_{\mathrm{f}}}$ due to the heat flow across the nonuniform mean pressure and the second overlined term represents the production because the magnetic field is not uniform. In the steady state, these two terms balance the absorption of the fluctuaion energy due to the dissipation. As we will show in Sec. II, the second overlined term on the right-hand side of Eq. (1.4) is of the opposite sign to the source term due to the $\boldsymbol{E}\times\boldsymbol{B}$ compression in the thermal energy equation if $T_i = T_e$ and the ions are assumed to follow the electrostatic field adiabatically; again, this shows the energy exchange between the thermal energy and the fluctuation energies.

In this work, we use the two-component fluid equations derived by Braginskii because it is believed that other than some kinetic effects like Landau damping, they model the drift-wave problems fairly well. As fundamental assumptions, it is assumed that in the case when the ion dynamics are important, the electrons are adiabatic with a finite phase difference between the electrostatic potential φ and the density n and the electron temperature is constant and in the case when the electron dynamics are considered the roles of the electrons and the ions are reversed in the dynamical equations. Also, in the case of the electromagnetic fluctuations it is assumed that the magnetic fluctuations are perpendicular to the mean field, thereby, eliminating the fast compressional Alfvén waves. As usual, the condition of the quasineutrality is enforced.

2. ENERGY BALANCE

In this section we derive the evolution equation of the thermal-energy density and the balance equation for the fluctuation energy density. In the present work, we will work in the slab geometry. The generalization to arbitrary geometry is straightforward. Before we move on, we need to clarify the averages we take in this work. First, we take the average over the magnetic surface. We denote the surface average of f with an angular bracket $\langle f\rangle$. The fluctuations from the surface averages are represented with the tildes $\widetilde{f} \doteq f - \langle f\rangle$. Another average we take is the average over the direction perpendicular to the magnetic surface, which is denoted by the bar. Thus, $\overline{\langle f\rangle}$ means the volume average of f.

To start with we approximate the perpendicular velocity $\boldsymbol{V}_\perp$ for the short spatial scale and low frequency fluctuations as the sum of the $\boldsymbol{E}\times\boldsymbol{B}$, the diamagnetic, and the polarization drifts

$$\begin{aligned}
\boldsymbol{V}_\perp &= \boldsymbol{V}_\perp^{(1)} + \boldsymbol{V}_\mathrm{p},\\
\boldsymbol{V}_\perp^{(1)} &\doteq \boldsymbol{V}_E + \boldsymbol{V}_\mathrm{d},\\
\boldsymbol{V}_E &\doteq \frac{c\boldsymbol{E}\times\boldsymbol{B}}{B^2},\\
\boldsymbol{V}_\mathrm{d} &\doteq \frac{c\boldsymbol{B}\times\nabla p}{qnB^2},\\
\boldsymbol{V}_\mathrm{p} &\doteq \Omega^{-1}\hat{\boldsymbol{b}}\times\left[\left(\frac{\partial}{\partial t} + \boldsymbol{V}^{(1)}\cdot\nabla\right)\boldsymbol{V}_\perp^{(1)} + \frac{1}{mn}\left(\nabla\cdot\boldsymbol{\Pi}^{(1)}\right) - \frac{1}{mn}\boldsymbol{R}\right],\\
\boldsymbol{V}^{(1)} &\doteq \boldsymbol{V}_\perp^{(1)} + \boldsymbol{V}_\parallel,
\end{aligned} \tag{2.1}$$

where $\boldsymbol{\Pi}$ is the momentum stress tensor,[3] and $\boldsymbol{R}$ is the transfer of the momentum between the ions and the electrons by collisions. Here, $\hat{\boldsymbol{b}}$ is the unit vector along the magnetic field $\boldsymbol{B}$, Ω is the gyrofrequency, m is the mass, and n is the particle density. For the fluctuations with low frequency $\partial/\partial t < \Omega$ and the wavelength longer than the gyroradius, $\left|\boldsymbol{V}_\perp^{(1)}\right| \gg |\boldsymbol{V}_\mathrm{p}|$.

2.1 Particle Flux

Before studying the energy transport, let us consider briefly the particle transport. The evolution of the mean particle density is described by the continuity equation:

$$\frac{\partial\langle n\rangle}{\partial t} + \nabla\cdot\boldsymbol{\Gamma} = 0, \tag{2.2}$$

where we define the particle flux $\boldsymbol{\Gamma}$ across the magnetic surface

$$\boldsymbol{\Gamma} \doteq \langle n\boldsymbol{V} \rangle,$$

since $|V_{\rm p}| \ll |\boldsymbol{V}_E|$ and $\langle n\boldsymbol{V}_{\rm d} \rangle$ is dominantly parallel to the magnetic surface, $\boldsymbol{\Gamma}$ reduces to

$$\begin{aligned}
&\approx \left\langle \widetilde{n}\widetilde{\boldsymbol{V}_E} \right\rangle + \langle n\boldsymbol{V}_{\|} \rangle, \\
&\approx \left\langle \widetilde{n}\widetilde{\boldsymbol{V}_E} \right\rangle + \langle n \rangle \left\langle \widetilde{V}_{\|} \frac{\widetilde{\boldsymbol{B}}}{B} \right\rangle \\
&\doteq \boldsymbol{\Gamma}_E + \boldsymbol{\Gamma}_{\rm m},
\end{aligned}$$

where we assume that $\langle \boldsymbol{E} \rangle = \langle V_{\|} \rangle = 0$ and $|\widetilde{n}| \ll |n|$. Here, the fluxes $\boldsymbol{\Gamma}_E$ and $\boldsymbol{\Gamma}_{\rm m}$ are due to the $\boldsymbol{E}\times\boldsymbol{B}$ drifts and the flow along the fluctuating magnetic-field lines, respectively.

The electrostatic component of the particle flux $\boldsymbol{\Gamma}_E$ depends on the phase angle between the $\widetilde{n}$ and $\boldsymbol{E}_\perp = -\boldsymbol{\nabla}_\perp\varphi$ fluctuation and vanishes when $\widetilde{n}$ and $\widetilde{\varphi}$ are in phase or have a 180° phase shift. These two important special cases occur for the ion pressure gradient (η_i) mode where the electrons are adiabatic and the electron pressure gradient (η_e) mode where the ions are adiabatic.[1] However, in general, there exists a phase difference between $\widetilde{\varphi}$ and $\widetilde{n}$, which gives rise to finite $\boldsymbol{\Gamma}_E$ across the surface.[6] The flux is positive or negative depending on whether the phase of $\widetilde{n}$ leads or follows that of $\widetilde{\varphi}$. While the electrostatic fluctuations are relatively well understood, the transport due to the electromagnetic fluctuations are still not thoroughly studied. On the one hand, the role of the stochastic magnetic fields (passive situation) with regard to the transport has been studied extensively[7] and is understood fairly well. On the other hand, the self-consistent treatment of the fluctuating magnetic field $\widetilde{\boldsymbol{B}}$ poses a very difficult problem. Recently, there have been controversies over the role of the magnetic fluctuations, whether they yield a sizable or experimentally measured transport. While Terry *et al.*[8] claim that the magnetic fluctuations do not contribute to the transport at any level, Krommes and Kim[9] and Thoul *et al.*[10] stress that the magnetic fluctuations do, in general, lead to the transport. In order to answer the questions about the magnetic fluctuations one must know how the fluctuations evolve or, at least, the informations about the correlations between the fluctuations. This can be done by solving a set of statistical equations that may be obtained by exploiting a closure scheme, like the weak-turbulence theory or, the direct-interaction approximation.[11] However, this topic is beyond the scope of the present study. In a relatively simple case, the electron thermal transport is explicitly calculated in the quasilinear limit from the self-consistent field equation by Hong and Horton.[12]

The magnetic component of the particle flux $\boldsymbol{\Gamma}_{\mathrm{m}}^{(\mathrm{e})}$ for the electrons further reduces when the ion parallel flow is negligible $\left|qnV_\parallel^i/\widetilde{j}_\parallel\right| \ll 1$:

$$\boldsymbol{\Gamma}_{\mathrm{m}}^{(\mathrm{e})} \approx -\frac{1}{e}\left\langle \widetilde{j}_\parallel \frac{\widetilde{\boldsymbol{B}}}{B}\right\rangle,$$

by substituting $\widetilde{j}_\parallel \approx (c/4\pi)\hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\widetilde{\boldsymbol{B}}$ in the slab geometry, one shows that the particle flux is driven by the gradient of the magnetic stress,

$$\Gamma_{\mathrm{m},x}^{(\mathrm{e})} \approx -\left(\frac{c}{4\pi eB}\right)\frac{\partial}{\partial x}\left\langle \widetilde{B}_x\widetilde{B}_y\right\rangle.$$

Unless the field $\widetilde{\boldsymbol{B}}$ is connected to external electromagnetic fields, the boundary terms are dropped for the local modes and the volume average of the flux vanishes,

$$\overline{\Gamma_{\mathrm{m},x}^{(\mathrm{e})}} = 0,$$

showing that the local inward and outward fluxes of $\boldsymbol{\Gamma}_E(x)$ cancel each other. From the point of view of the ambipolarity, the vanishing of the volume average of $\Gamma_{\mathrm{m},x}^{(\mathrm{e})}(x)$ is expected because $\boldsymbol{\Gamma}_E$ is the same for both the ions and the electrons and the ion flux due to $\widetilde{\boldsymbol{B}}$ vanishes $\overline{\Gamma_{\mathrm{m},x}^{(\mathrm{i})}} = 0$. [For more discussions about the ambipolarity, see Ref. 13.] For the finite ion parallel current, the same argument shows that $\overline{\Gamma_{\mathrm{m},x}^{(\mathrm{e})}} = \overline{\Gamma_{\mathrm{m},x}^{(\mathrm{i})}}$.

2.2 Field and Kinetic Energy

Now, we study the energy balances. Let us begin by calculating the rate of change of the electrostatic energy density $\frac{1}{2}q\langle\widetilde{n}\widetilde{\varphi}\rangle$. Upon multiplying the continuity equation with $q\widetilde{\varphi}$ and taking the average over the surface one obtains the equation for $\frac{1}{2}q\langle\widetilde{n}\widetilde{\varphi}\rangle$ as

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{1}{2}q\langle\widetilde{n}\widetilde{\varphi}\rangle\right)+\boldsymbol{\nabla}\cdot\left(q\left\langle n\widetilde{\varphi}\widetilde{\boldsymbol{V}}\right\rangle\right)&=q\left\langle n\widetilde{\boldsymbol{V}}\cdot\boldsymbol{\nabla}\widetilde{\varphi}\right\rangle\\&=-q\left\langle n\widetilde{\boldsymbol{V}}\cdot\left(\widetilde{\boldsymbol{E}}+\frac{1}{c}\frac{\partial\widetilde{\boldsymbol{A}}}{\partial t}\right)\right\rangle\end{aligned}$$

With the neglect of the compressional modes $\boldsymbol{A}_\perp=0$, we obtain the general result

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}q\langle\widetilde{n}\widetilde{\varphi}\rangle\right)+\boldsymbol{\nabla}\cdot\left(q\left\langle n\widetilde{\varphi}\widetilde{\boldsymbol{V}}\right\rangle\right)=-q\left\langle n\boldsymbol{V}_\perp\cdot\widetilde{\boldsymbol{E}}\right\rangle-q\left\langle n\widetilde{V}_\parallel\widetilde{E_\parallel}\right\rangle-\frac{q}{c}\left\langle n\widetilde{V}_\parallel\frac{\partial\widetilde{A_\parallel}}{\partial t}\right\rangle, \tag{2.3}$$

by using $\widetilde{\boldsymbol{A}}=\hat{\boldsymbol{b}}\widetilde{A_\parallel}$ and $\hat{\boldsymbol{b}}\cdot\partial\widetilde{\boldsymbol{A}}/\partial t\approx\partial\widetilde{A_\parallel}/\partial t$.

One can further reduce the last term on the right-hand side of Eq. (2.3) in two important special cases: First, in the η_i turbulence problem, the field fluctuation is of the electrostatic nature and, thus, $\langle n\widetilde{V}_\parallel\partial\widetilde{A_\parallel}/\partial t\rangle=0$. Second, in the η_e turbulence problem, the fluctuations are electromagnetic[1,14] with $j_\parallel\approx-enV_\parallel$. Then, the last term in Eq. (2.3) can be reexpressed as

$$\frac{q}{c}\left\langle n\widetilde{V}_\parallel\frac{\partial\widetilde{A_\parallel}}{\partial t}\right\rangle=\frac{1}{c}\left\langle\widetilde{j}_\parallel\frac{\partial\widetilde{A_\parallel}}{\partial t}\right\rangle,$$

and by using $\widetilde{j_\parallel}=(c/4\pi)\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\hat{\boldsymbol{b}}\widetilde{A_\parallel})$

$$\approx-\frac{1}{4\pi}\left\langle\nabla_\perp^2\widetilde{A_\parallel}\frac{\partial\widetilde{A_\parallel}}{\partial t}\right\rangle,$$

and, finally, after integration by parts, we obtain

$$=\frac{\partial}{\partial t}\left(\frac{\langle\widetilde{B}^2\rangle}{8\pi}\right)+\boldsymbol{\nabla}\cdot\left[\frac{c}{4\pi}\left\langle(\boldsymbol{E}+\boldsymbol{\nabla}\varphi)\times\widetilde{\boldsymbol{B}}\right\rangle\right], \tag{2.4}$$

which correctly represents the change of the magnetic-field energy density from the Poynting flux vector where only the inductive or transversely polarized part of the electric field produces an electromagnetic flux.

In order to study the change of the parallel kinetic energy we consider the equation of motion along the magnetic field lines. By taking a scalar product between the equation of motion and $\hat{\boldsymbol{b}}$ and by using the continuity equation we have

$$\begin{aligned}\frac{\partial}{\partial t}(mnV_{\|}) + \boldsymbol{\nabla}\boldsymbol{\cdot}(mnV_{\|}\boldsymbol{V}) - \underline{mn\boldsymbol{V}\boldsymbol{\cdot}\frac{\partial\hat{\boldsymbol{b}}}{\partial t}} - \underline{mn\boldsymbol{V}\boldsymbol{\cdot}(\boldsymbol{V}\boldsymbol{\cdot}\boldsymbol{\nabla})\hat{\boldsymbol{b}}} \\ = qnE_{\|} - \nabla_{\|}p - \hat{\boldsymbol{b}}\boldsymbol{\cdot}(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\Pi}) + R_{\|},\end{aligned} \tag{2.5}$$

where the underlined terms are negligible because after being compared with the first two terms in the left-hand side of Eq. (2.5) they are found to be either of higher order in the fluctuations or they are smaller by a factor of the ratio of the fluctuation scale length or time to those of the mean magnetic field. Since we assume that there is no mean flow, we derive the equation for the parallel kinetic-energy density by multiplying Eq. (2.5) with $\widetilde{V}_{\|}$ and taking the surface average to obtain:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{1}{2}m\left\langle n\widetilde{V}_{\|}^{2}\right\rangle\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\tfrac{1}{2}m\left\langle n\widetilde{V}_{\|}^{2}\boldsymbol{V}\right\rangle + \langle p\boldsymbol{V}_{\|}\rangle + \langle\boldsymbol{\Pi}\boldsymbol{\cdot}\boldsymbol{V}_{\|}\rangle\right) \\ = q\left\langle n\widetilde{V}_{\|}\widetilde{E}_{\|}\right\rangle + \langle p\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{V}_{\|}\rangle + \langle\boldsymbol{\Pi} : \boldsymbol{\nabla}\boldsymbol{V}_{\|}\rangle + \left\langle R_{\|}\widetilde{V}_{\|}\right\rangle.\end{aligned} \tag{2.6}$$

2.3 Thermal Balance Equation

For the heat transport we have

$$\frac{\partial}{\partial t}\tfrac{3}{2}p + \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\tfrac{3}{2}p\boldsymbol{V} + \boldsymbol{q}\right) = -p\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{V} - \boldsymbol{\Pi} : \boldsymbol{\nabla}\boldsymbol{V} + \mathcal{Q}, \tag{2.7}$$

where

$$\begin{aligned}p &\doteq nT, \\ \boldsymbol{q} &\doteq \boldsymbol{q}_{\mathrm{d}} + \boldsymbol{q}', \\ \boldsymbol{q}_{\mathrm{d}} &\doteq \tfrac{5}{2}\left(\frac{cp}{qB}\right)\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T\end{aligned}$$

due to the finite-Larmor-radius effect,

$$
\boldsymbol{q}' \doteq -\chi_\parallel \boldsymbol{\nabla}_\parallel T - \chi_\perp \boldsymbol{\nabla}_\perp T + \boldsymbol{q}_u,
$$

$\boldsymbol{q}_u$ represents the electron heat flux due to the nonzero relative velocity between the electrons and the ions ($\boldsymbol{q}_u = 0$ for the ions), and $\mathcal{Q}$ is the heat source due to the collisions between the ions and the electrons, auxiliary heating, and atomic physics processes.

For later use, we examine the compressional source term $(-p\boldsymbol{\nabla}\cdot\boldsymbol{V}_E)$. After a little algebra, we find that

$$
-p\boldsymbol{\nabla}\cdot\boldsymbol{V}_E = \left(\frac{p}{B}\right)\frac{\partial B}{\partial t} + nq\boldsymbol{E}\cdot\boldsymbol{V}_{\mathrm{g}}, \tag{2.8}
$$

where we define the guiding-center drift velocity $\boldsymbol{V}_{\mathrm{g}}$

$$
\begin{aligned}
\boldsymbol{V}_{\mathrm{g}} &\doteq \left(\frac{cT}{q}\right)\boldsymbol{\nabla}\times\left(\frac{\hat{\boldsymbol{b}}}{B}\right) \\
&= \boldsymbol{V}_{\parallel\mathrm{Ban\tilde{o}s}} + \boldsymbol{V}_{\mathrm{curv}} + \boldsymbol{V}_{\nabla B},
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{V}_{\parallel\mathrm{Ban\tilde{o}s}} &\doteq \left(\frac{cT}{qB}\right)\hat{\boldsymbol{b}}\left(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\hat{\boldsymbol{b}}\right), \\
\boldsymbol{V}_{\mathrm{curv}} &\doteq \left(\frac{cT}{qB}\right)\hat{\boldsymbol{b}}\times\left(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\right)\hat{\boldsymbol{b}}, \\
\boldsymbol{V}_{\nabla B} &\doteq \left(\frac{cT}{qB^2}\right)\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}B.
\end{aligned}
$$

However, one can simplify Eq. (2.8) for the case

$$
\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{1}{c}\hat{\boldsymbol{b}}\frac{\partial A_\parallel}{\partial t}
$$

and

$$
\boldsymbol{\nabla}\cdot\boldsymbol{V}_E = \frac{q}{T}\boldsymbol{V}_{\mathrm{g}}\cdot\boldsymbol{\nabla}\varphi \tag{2.9}
$$

by using the identity $\boldsymbol{\nabla}\cdot\left[(1/B)\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}f\right] = (q/cT)\boldsymbol{\nabla}f\cdot\boldsymbol{V}_{\mathrm{g}}$. Also, for an intermediate step needed for later use, one can combine the thermal fluxes $p\boldsymbol{V}_{\mathrm{d}}$ and $\boldsymbol{q}_{\mathrm{d}}$ to find that

$$
\begin{aligned}
\tfrac{5}{3}p\boldsymbol{\nabla}\cdot\boldsymbol{V}_{\mathrm{d}} + \tfrac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{q}_{\mathrm{d}} &= \tfrac{5}{3}\boldsymbol{\nabla}\cdot\left[\left(\frac{c}{qB}\right)\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}(pT)\right] \\
&= \tfrac{5}{3}\left(\boldsymbol{\nabla}p + n\boldsymbol{\nabla}T\right)\cdot\boldsymbol{V}_{\mathrm{g}}.
\end{aligned} \tag{2.10}
$$

After substituting $\boldsymbol{V}_\perp \approx \boldsymbol{V}_E + \boldsymbol{V}_{\rm d} + \boldsymbol{V}_{\rm p}$ into Eq. (2.7), using Eqs. (2.9) and (2.10), and taking the surface average, one obtains the equation for the mean thermal-energy density

$$\begin{aligned}\frac{\partial}{\partial t}\tfrac{3}{2}\langle p\rangle + \boldsymbol{\nabla}\cdot\left[\tfrac{3}{2}\left(\left\langle \widetilde{p}\widetilde{\boldsymbol{V}_E}\right\rangle + \langle p\boldsymbol{V}_{\rm p}\rangle + \langle p\boldsymbol{V}_\parallel\rangle\right) + \langle \boldsymbol{q}'\rangle\right] \\ \approx -\frac{q}{T}\boldsymbol{V}_{\rm g}\cdot\langle\widetilde{p}\boldsymbol{\nabla}\widetilde{\varphi}\rangle - \langle p\boldsymbol{\nabla}\cdot\boldsymbol{V}_{\rm p}\rangle - \langle p\boldsymbol{\nabla}\cdot\boldsymbol{V}_\parallel\rangle - \langle \boldsymbol{\Pi} : \boldsymbol{\nabla}\boldsymbol{V}\rangle + \langle \mathcal{Q}\rangle.\end{aligned} \tag{2.11}$$

The terms associated with the polarization drift $\boldsymbol{V}_{\rm p}$ are smaller than those with $\boldsymbol{V}_E$ in Eq. (2.11). It will be shown later, however, that the polarization drift flux terms are necessary for the perpendicular kinetic energy to be included in the total energy conservation. [See Eq. (2.13).] The importance of Eq. (2.11) is that it rigorously seperates out the small but important energy densities associated with the fluctuations from the total thermal balance equation.

By defining the total energy density $\mathcal{E}_{\rm t}$ and the total energy flux $\boldsymbol{F}_{\rm t}$ per unit volume as

$$\mathcal{E}_{\rm t} \doteq -\tfrac{1}{2}e\langle\widetilde{n}\widetilde{\varphi}\rangle + \frac{\langle\widetilde{B}^2\rangle}{8\pi} + \underline{\tfrac{1}{2}m\langle n\rangle\left[\left\langle {V_\perp^{(1)}}^2\right\rangle + \langle V_\parallel^2\rangle\right]} + \tfrac{3}{2}\langle p\rangle,$$

and

$$\begin{aligned}\boldsymbol{F}_{\rm t} \doteq \boldsymbol{Q}_E + \gamma\boldsymbol{Q}_{\rm m} + \underline{\left\langle \boldsymbol{\Pi}\cdot\boldsymbol{V}^{(1)}\right\rangle} + \langle\boldsymbol{q}'\rangle \\ \underline{- e\langle n\rangle\left\langle \widetilde{\varphi}V_\parallel\left(\frac{\widetilde{\boldsymbol{B}}}{B}\right)\right\rangle} + \underline{\frac{c}{4\pi}\left\langle(\boldsymbol{E} + \boldsymbol{\nabla}\varphi)\times\widetilde{\boldsymbol{B}}\right\rangle} + \underline{\tfrac{1}{2}m\langle n\rangle\left\langle {V^{(1)}}^2\boldsymbol{V}^{(1)}\right\rangle},\end{aligned}$$

one can work out the total energy conservation from Eqs. (2.3), (2.4), (2.6), and (2.11) for the electrons and the electromagnetic fluctuations as

$$\frac{\partial\mathcal{E}_{\rm t}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_{\rm t} = \left\langle \boldsymbol{R}\cdot\boldsymbol{V}^{(1)}\right\rangle + \langle\mathcal{Q}\rangle. \tag{2.12}$$

The electrostatic potential energy density $-\frac{1}{2}e\langle\widetilde{n}\widetilde{\varphi}\rangle$ is positive definite because $\widetilde{n} = -(e\langle n\rangle/T_i)\widetilde{\varphi}$. In the kinetic energy, the contribution due to the diamagnetic drift is included because it represents part of the fluid kinetic energy even though each individual particle does not have execute such a drift motion. In Eq. (2.12) $\boldsymbol{Q}_E$ is the thermal-energy flux per unit volume due to $\boldsymbol{E}\times\boldsymbol{B}$ drift

$$\boldsymbol{Q}_E \doteq \tfrac{3}{2}\left\langle \widetilde{p}\widetilde{\boldsymbol{V}_E} \right\rangle,$$

$\boldsymbol{Q}_{\mathrm{m}}$ is the flux due to the magnetic fluctuations

$$\boldsymbol{Q}_{\mathrm{m}} \doteq \tfrac{3}{2}\langle p\rangle \left\langle \left(\frac{\widetilde{\boldsymbol{B}}}{B}\right) V_{\parallel} \right\rangle,$$

and $\gamma = 5/3$. The fluxes $\boldsymbol{Q}_E$ and $\boldsymbol{Q}_{\mathrm{m}}$ are cross correlation functions between the φ and $A_{\parallel}$ fields and the particle fields $\widetilde{p}$ and $V_{\parallel}$.

In obtaining Eq. (2.12) we have done as follows: We have neglected the fluxes due to $\boldsymbol{V}_{\mathrm{p}}$; we have neglected the divergence of the flux of the electrostatic potential energy due to the $\boldsymbol{E}\times\boldsymbol{B}$ advection because

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left\langle n\widetilde{\varphi}\widetilde{\boldsymbol{V}_E}\right\rangle &\approx \boldsymbol{\nabla}\cdot\left(\langle n\rangle\left\langle\widetilde{\varphi}\widetilde{\boldsymbol{V}_E}\right\rangle\right)\\ &\approx \boldsymbol{\nabla}\cdot\left(c\langle n\rangle\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\langle\widetilde{\varphi}^2\rangle/2B\right)\\ &\approx 0;\end{aligned}$$

in Eq. (2.2), by assuming that the velocity $\boldsymbol{V}_{\mathrm{g}}$ has only the mean component $\boldsymbol{V}_{\mathrm{g}} = \langle\boldsymbol{V}_{\mathrm{g}}\rangle$ and is perpendicular to the mean gradients we have employed the relation

$$\langle\widetilde{\varphi}\boldsymbol{\nabla}\cdot(n\boldsymbol{V}_{\mathrm{d}})\rangle \approx \frac{1}{T}\boldsymbol{V}_{\mathrm{g}}\cdot\langle\widetilde{\varphi}\boldsymbol{\nabla}\widetilde{p}\rangle$$

that cancels $(-\langle\widetilde{p}\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{V}_E}\rangle)$ in Eq. (2.11); we have used the important relation

$$\begin{aligned}&(-qn\boldsymbol{E}+\boldsymbol{\nabla}p)\cdot\boldsymbol{V}_{\mathrm{p}}\\ &\quad\approx -\left[\frac{\partial}{\partial t}\left(\tfrac{1}{2}mnV_{\perp}^{(1)^2}\right)+\boldsymbol{\nabla}\cdot\left(\tfrac{1}{2}mnV_{\perp}^{(1)^2}\boldsymbol{V}^{(1)}\right)+\boldsymbol{V}_{\perp}^{(1)}\cdot(\boldsymbol{\nabla}_{\perp}\cdot\boldsymbol{\mathit{\Pi}})-\boldsymbol{V}_{\perp}^{(1)}\cdot\boldsymbol{R}\right],\end{aligned} \tag{2.13}$$

that results from approximating the continuity equation to $\partial n/\partial t + \boldsymbol{\nabla}\cdot\left(n\boldsymbol{V}^{(1)}\right) \approx 0$.

It is important to notice that only 3/2, not 5/2, of $\langle\widetilde{p}\widetilde{\boldsymbol{V}_E}\rangle$ contributes to the energy flux $\boldsymbol{F}_{\mathrm{t}}$ because the driving term $qn\boldsymbol{V}_{\mathrm{d}}\cdot\boldsymbol{E}$ for the field energy in Eq. (2.3) and the source term $\boldsymbol{V}_E \cdot \boldsymbol{\nabla} p$ in Eq. (2.11) are balanced out and the electrostatic-field-energy flux $q\langle n \boldsymbol{V}_{\mathrm{d}} \widetilde{\varphi}\rangle$ due to the diamagnetic drift cancels out one of the $\langle \widetilde{p} \widetilde{\boldsymbol{V}_E}\rangle$'s.

This point, now, brings us to the discussions of the energy flux in Ref. 4: there, for the electrostatic case, two different formulas [Eq. (12) and Eq. (12′)] for the energy flux that are found in the literature are discussed. Basically, they are $Q_1^{\mathrm{R}} = 3/2\langle \widetilde{V_{Ex}} \widetilde{p}\rangle$ and $Q_2^{\mathrm{R}} = 5/2\langle \widetilde{V_{Ex}} \widetilde{p}\rangle$ if we convert the geometry to slab and assume the isotropic pressure. Then, Ross raised the question of which of these two formulas is correct. Our point, here, is that these two fluxes, Q_1^{R} and Q_2^{R}, are not relevant in the sense that follows and, thus, should not be compared to each other: in Ref. 4, Q^{R} is defined as the energy flux counting for the kinetic energy (which is negligible) and the thermal energy. Notice, however, that the flux for the electric field energy is not included in the definition of Q^{R}. While Q_2^{R} does not include the electrostatic field energy flux, as we explained in the previous paragraph, it is clear that, actually, Q_1^{R} is the total energy flux including the electrostatic field energy.

Thus, it is not quite correct to deduce anomalous "conductive" energy flux $q^{\mathrm{R}} \doteq Q^{\mathrm{R}} - 5/2\Gamma\langle T\rangle$ as defined in Ref. 4 (in our definition, the anomalous heat flux is the Q minus the classical dissipative flux) by using $Q^{\mathrm{R}} = Q_1^{\mathrm{R}}$ to obtain $q_1^{\mathrm{R}} = -\Gamma\langle T\rangle + 3/2\langle n\rangle\langle \widetilde{V_{Ex}} \widetilde{T}\rangle$ in Eq. (22) in Ref. 4. Rather, the remaining portion q^{t} of the total energy flux including the electrostatic field energy subtracting the portion of the flux associated with the anomalous particle flux Γ should be read as $q^{\mathrm{t}} = Q_1^{\mathrm{R}} - 3/2\Gamma\langle T\rangle = 3/2\langle n\rangle\langle \widetilde{V_{Ex}} \widetilde{T}\rangle$.

Yet, another attempt has been made to find the formula for the anomalous heat flux q in Ref. 15: there, even though the author states that the same definition for the q is used, there exists a difference, which is $q = q^{\mathrm{B}} = q^{\mathrm{R}} - 5/2\langle n\rangle\langle \widetilde{V}_x \widetilde{T}\rangle$. See Eq. (A.11) in Ref. 15. The difference arises because the $\boldsymbol{\Gamma}$ in Ref. 15 represents the particle flux $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{B}} = n\boldsymbol{V}$ while the $\boldsymbol{\Gamma}$ used in Ref. 4 is the mean particle flux $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{R}} = \langle n\boldsymbol{V}\rangle$. One can show that if one counts for the heat flux $\boldsymbol{q}_{\mathrm{d}}$ due to the finite-Larmor-radius effect [which is not included in Ref. 15], then, Eq. (A.13) is cancelled out leading to $q^{\mathrm{B}} = 0$ in the slab geometry.

In the energy flux $\boldsymbol{F}_\mathrm{t}$, one can neglect the underlined fluxes associated with $\widetilde{\boldsymbol{B}}$ because they are small compared with $\boldsymbol{Q}_\mathrm{m}$. The remaining underlined fluxes are due to the polarization drifts and since those effects are assumed to be small, they can also be negligible. Thus, as the principal contribution to the total-energy flux one finds

$$\boldsymbol{F}_\mathrm{t} \approx \boldsymbol{Q}_E + \gamma \boldsymbol{Q}_\mathrm{m} + \langle \boldsymbol{q}' \rangle.$$

The effect of the fluctuations on the collisional flux of $\langle \boldsymbol{q}' \rangle$ is important as shown by Kadomtsev and Pogutse[16] and calculated by Hong and Horton.[12] In Ref. 16, the authors take the MHD equation to be $\nabla \cdot \boldsymbol{q}' = 0$ and calculate $\langle \boldsymbol{q}' \rangle$ self-consistently, whereas in Ref. 12, dynamical equations for the $\widetilde{\varphi}$, $\widetilde{A_\|}$, and $\widetilde{p}$ fluctuations are used and the fluxes are computed self-consistently in the quasilinear limit.

In the total-energy density $\mathcal{E}_\mathrm{t}$ the kinetic energy part due to the $\boldsymbol{E}\times\boldsymbol{B}$ drifts and the diamagnetic drifts may be smaller than the electrostatic energy in the case when $T_i \approx T_\mathrm{e}$ and $k_\perp \rho < 1$, which is the regime where one can expand $\boldsymbol{V}_\perp$ as in Eq. (2.1). Here, $k_\perp^{-1}$ is the scale length of the fluctuation and ρ is the gyroradius. Also, one can work out that the ratio between the fluctuating-magnetic-field energy and the parallel kinetic energy is that

$$\left| \frac{m\langle n \rangle \langle V_\|^2 \rangle /2}{\langle \widetilde{B}^2 \rangle / 8\pi} \right| \sim \left(\frac{ck_\perp}{\omega_\mathrm{p}} \right)^2,$$

where c/ω_p is the collisionless skin depth. Thus, for the η_e modes[1] with $k_\perp^{-1} \sim c/\omega_\mathrm{p}$, the magnetic field energy and the electron parallel kinetic energy are comparable. For the resistive MHD mode with $k_\perp^{-1} \gg c/\omega_\mathrm{p}$, the magnetic energy density dominates the parallel kinetic energy.

Another important point is that while the volume average of the magnetic particle flux $\overline{\Gamma_{\mathrm{m},x}^{(\mathrm{e})}}$ vanishes, the volume average of the heat flux $\overline{Q_{\mathrm{m},x}^{(\mathrm{e})}}$ does not in the slab geometry. The reason is that as far as the particle flux is concerned, the electrons are just moving along the oscillating magnetic fields and the volume average of the flux vanishes. As for the heat transport, however, the electrons see the temperature gradient across the flux surface during their streaming along the oscillating field lines:

$$
\begin{aligned}
Q_{\mathrm{m}}^{(\mathrm{e})} &\approx -\frac{3\langle T_{\mathrm{e}}\rangle}{2e}\left\langle \widetilde{j}_{\parallel}\frac{\widetilde{\boldsymbol{B}}}{B}\right\rangle, \\
\overline{Q_{\mathrm{m},x}^{(\mathrm{e})}} &\approx \frac{3}{2}\left(\frac{c}{4\pi eB}\right)\overline{\left\langle \widetilde{B}_x\widetilde{B}_y\right\rangle\frac{\partial\langle T_{\mathrm{e}}\rangle}{\partial x}},
\end{aligned}
\tag{2.14}
$$

or, in the quasilinear approximation that the correlation length of $\widetilde{\boldsymbol{B}}$ is much shorter than the macroscopic scale,

$$
Q_{\mathrm{m},x}^{(\mathrm{e})} \approx -\left(-\frac{3c}{8\pi eB}\left\langle \widetilde{B}_x\widetilde{B}_y\right\rangle\right)\frac{\partial\langle T_{\mathrm{e}}\rangle}{\partial x},
$$

thus, $-(3c/8\pi enB)\langle\widetilde{B}_x\widetilde{B}_y\rangle$ plays the role of the anomalous thermal conductivity χ_{a}. Mathematically, the form of $-\langle\widetilde{B}_x\widetilde{B}_y\rangle = \langle\left(\partial\widetilde{A_{\parallel}}/\partial x\right)\left(\partial\widetilde{A_{\parallel}}/\partial y\right)\rangle$ is analogous to the Reynolds stress tensor $\langle\widetilde{V}_x\widetilde{V}_y\rangle = -\langle(\partial\psi/\partial x)(\partial\psi/\partial y)\rangle$ for the two-dimensional neutral fluid flow, where ψ is the steaming function. In general, $\overline{Q_{\mathrm{m},x}^{(\mathrm{e})}} \neq 0$. However, in very special cases, for example, when $\widetilde{A_{\parallel}}$ is either even or odd in x, $\overline{Q_{\mathrm{m},x}^{(\mathrm{e})}}$ would vanish.

For the ions and the electrostatic fluctuations, the total energy conservation law is the same as Eq. (2.12) except that $\widetilde{\boldsymbol{B}} = 0$ in the ion equation.

Although Eq. (2.12) expresses an important property of the energy conservation, it does not provide information about how the correlations between the fluctuations are balanced through the production, the dissipation, and the transfer of the fluctuation energies. For this purpose, one must consider the variance $\langle\widetilde{p}^2\rangle/\langle p\rangle$ of the pressure fluctuation rather than $\langle p\rangle$ itself in the energy equation. Thus, after multiplying Eq. (2.7) with $2\widetilde{p}/3\langle p\rangle$ and taking the surface average, we have

$$
\frac{1}{\langle p\rangle}\frac{\partial}{\partial t}\tfrac{1}{2}\langle\widetilde{p}^2\rangle + \frac{1}{\langle p\rangle}\langle\widetilde{p}\boldsymbol{V}\cdot\boldsymbol{\nabla}p\rangle = -\frac{\gamma}{\langle p\rangle}\langle\widetilde{p}p\boldsymbol{\nabla}\cdot\boldsymbol{V}\rangle - \frac{2}{3\langle p\rangle}\langle\widetilde{p}(\boldsymbol{\nabla}\cdot\boldsymbol{q} + \boldsymbol{\mathit{\Pi}} : \boldsymbol{\nabla}\boldsymbol{V} - \mathcal{Q})\rangle. \tag{2.15}
$$

Now, we change Eq. (2.15) into the conservative form. In doing so, we will neglect the terms associated with $\partial\langle p\rangle/\partial t$ and $\nabla\langle p\rangle$ because they are small. This can be justified by comparing the magnitudes of the terms by using Eq. (2.11). Then, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{2\gamma}\frac{\langle\widetilde{p}^2\rangle}{\langle p\rangle}\right) + \nabla\cdot\left[\frac{1}{2\gamma\langle p\rangle}\langle\widetilde{p}^2 \boldsymbol{V}_E\rangle + \langle\widetilde{p}\boldsymbol{V}_{\rm p}\rangle + \frac{2}{3\gamma\langle p\rangle}\langle\widetilde{p}\boldsymbol{q}'\rangle\right] \\
&\approx \underset{\text{(a)}}{-\frac{2}{3\gamma}\boldsymbol{Q}_E\cdot\frac{\nabla\langle p\rangle}{\langle p\rangle}} \underset{\text{(b)}}{-\frac{q}{T}\boldsymbol{V}_{\rm g}\cdot\langle\widetilde{p}\nabla\varphi\rangle} \underset{\text{(c)}}{-\langle\widetilde{p}\nabla\cdot\boldsymbol{V}_{\parallel}\rangle} \underset{\text{(d)}}{+\frac{\boldsymbol{V}_{\rm g}}{\langle n\rangle}\cdot\langle\widetilde{p}\nabla\widetilde{n}\rangle} \\
&\quad \underset{\text{(e)}}{+\langle\boldsymbol{V}_{\rm p}\cdot\nabla\widetilde{p}\rangle} \underset{\text{(f)}}{+\frac{2}{3\gamma\langle p\rangle}\langle\boldsymbol{q}'\cdot\nabla\widetilde{p}\rangle} \underset{\text{(g)}}{-\frac{2}{3\gamma\langle p\rangle}\left\langle\widetilde{p}\left(\boldsymbol{\Pi}:\nabla\boldsymbol{V}_{\perp}^{(1)} - \mathcal{Q}\right)\right\rangle},
\end{aligned}
\tag{2.16}
$$

where we use the intermediate relation by multiplying Eq. (2.10) with $\widetilde{p}$,

$$
\begin{aligned}
-\gamma\frac{\boldsymbol{V}_{\rm g}}{\langle p\rangle}\cdot\left(\nabla\langle\widetilde{p}^2\rangle/2 + \langle n\rangle\left\langle\widetilde{p}\nabla\widetilde{T}\right\rangle\right) &\approx \gamma\frac{\boldsymbol{V}_{\rm g}}{\langle T\rangle}\cdot\left\langle\widetilde{T}\nabla\widetilde{p}\right\rangle \\
&\approx \gamma\frac{\boldsymbol{V}_{\rm g}}{\langle n\rangle}\cdot\langle\widetilde{p}\nabla\widetilde{n}\rangle,
\end{aligned}
$$

$\boldsymbol{V}_{\rm g}$ being assumed to be perpendicular to the mean gradients, and the integration by parts is performed.

In Eq. (2.16) the term (a) represents the production of energy due to the interaction of the mean pressure gradient and the $\boldsymbol{E}\times\boldsymbol{B}$ heat flux, the terms (b) and (c) are the source terms responsible for the transfer between the fluctuation energies. Thus, they are canceled out after we add up the fluctuation energies [Eq. (2.3) plus Eq. (2.6) and Eq. (2.16)]. The term (d) is due to the nonuniformity of the magnetic field. If we consider the evolution of the total fluctuation energy, then, the term (e) combined with the $-q\langle n\boldsymbol{E}\cdot\boldsymbol{V}_{\rm p}\rangle$ leads to the change of $\frac{1}{2}m\langle n{V_{\perp}^{(1)}}^2\rangle$ and the term (g) will be negligible. [See Eq. (2.13).] If $\boldsymbol{q}' = -\kappa\nabla p$, then, the term (f) becomes to be proportional to $-\langle|\nabla\widetilde{p}|^2\rangle$ representing the loss of the fluctuation energy due to the dissipation.

Therefore, we obtain the energy balance for the electromagnetic fluctuations and the electrons by adding Eqs. (2.3), (2.6), and (2.16) and by using Eqs. (2.4) and (2.13)

$$\begin{aligned}
&\frac{\partial \mathcal{E}_{\mathrm{f}}}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{F}_{\mathrm{f}} \\
&\quad = -\frac{2}{3}\left(\frac{1}{\gamma}\boldsymbol{Q}_E + \boldsymbol{Q}_{\mathrm{m}}\right)\boldsymbol{\cdot}\frac{\boldsymbol{\nabla}\langle p\rangle}{\langle p\rangle} + \underline{\frac{\boldsymbol{V}_{\mathrm{g}}}{\langle n\rangle}\boldsymbol{\cdot}\langle\widetilde{p}\boldsymbol{\nabla}\widetilde{n}\rangle} \\
&\qquad + \langle\boldsymbol{\Pi} : \boldsymbol{\nabla}\boldsymbol{V}\rangle + \frac{2}{3\gamma\langle p\rangle}\langle\boldsymbol{q}'\boldsymbol{\cdot}\boldsymbol{\nabla}\widetilde{p}\rangle + \left\langle\boldsymbol{R}\boldsymbol{\cdot}\boldsymbol{V}^{(1)}\right\rangle,
\end{aligned} \tag{2.17}$$

where the total fluctuation energy $\mathcal{E}_{\mathrm{f}}$ and the total fluctuation energy flux $\boldsymbol{F}_{\mathrm{f}}$ are defined as

$$\mathcal{E}_{\mathrm{f}} \doteq -\tfrac{1}{2}e\langle\widetilde{n}\widetilde{\varphi}\rangle + \frac{\langle\widetilde{B}^2\rangle}{8\pi} + \tfrac{1}{2}m\langle n\rangle\left\langle V^{(1)^2}\right\rangle + \frac{1}{2\gamma}\frac{\langle\widetilde{p}^2\rangle}{\langle p\rangle},$$

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{f}} \doteq& -e\langle n\rangle\left\langle\widetilde{\varphi}V_{\parallel}\left(\frac{\widetilde{\boldsymbol{B}}}{B}\right)\right\rangle + \frac{c}{4\pi}\left\langle(\boldsymbol{E}+\boldsymbol{\nabla}\varphi)\times\widetilde{\boldsymbol{B}}\right\rangle + \tfrac{1}{2}m\langle n\rangle\left\langle V^{(1)^2}\boldsymbol{V}^{(1)}\right\rangle \\
&+ \frac{1}{2\gamma\langle p\rangle}\langle\widetilde{p}^2\boldsymbol{V}_E\rangle + \langle\widetilde{p}\boldsymbol{V}_{\mathrm{p}}\rangle + \frac{2}{3\gamma\langle p\rangle}\langle\widetilde{p}\boldsymbol{q}'\rangle + \left\langle\boldsymbol{\Pi}\boldsymbol{\cdot}\boldsymbol{V}^{(1)}\right\rangle.
\end{aligned}$$

By comparing $\boldsymbol{F}_{\mathrm{f}}$ with $\boldsymbol{F}_{\mathrm{t}}$ one can immediately see that $|\boldsymbol{F}_{\mathrm{f}}| \ll |\boldsymbol{F}_{\mathrm{t}}|$. After integrating Eq. (2.17) over the entire plasma volume we notice that in the steady state the dissipation loss of the fluctuation energy due to the viscosity and the classical thermal conductivity is balanced by the production of the energy arising from the anomalous energy flow across the nonuniform pressure and the particle magnetic-drift motions owing to the curved magnetic field. Also, notice that since $\widetilde{n_e} \approx -(\langle n\rangle e/T_i)\widetilde{\varphi}$, one can further examine the underlined term in Eq. (2.17) as

$$\frac{\boldsymbol{V}_{\mathrm{g}}}{\langle n\rangle}\boldsymbol{\cdot}\langle\widetilde{p}\boldsymbol{\nabla}\widetilde{n}\rangle = -\frac{e}{T_i}\boldsymbol{V}_{\mathrm{g}}\boldsymbol{\cdot}\langle\widetilde{p}\boldsymbol{\nabla}\widetilde{\varphi}\rangle$$

and can conclude that it cancels the driving term $(-\langle p\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{V}_E\rangle)$ in the thermal-energy equation (2.11) if $T_i = T_{\mathrm{e}}$. Once again, it demonstrates that the energy transfer between the fluctuations and the thermal energy occurs.

3. CONCLUSION

The transfer processes between the particle kinetic energy, the electromagnetic field energy and the thermal energy, in general, is known to occur and is perceived to be important in the understanding of the evolution of the energies in a magnetized, nonuniform plasma. In this work, we have studied the balances between the various fluctuation energies and the thermal energy. In particular, the emphasis was put on the change of the total fluctuation energy. Certainly, this work is not the first to address these issues for the drift-wave problems. There are numerous works discussing these matters. For example, we refer the readers to the review in Ref. 6 and the comments in Ref. 4. However, in many cases, the importance of the over all energy balance problem has not been stressed or the intermediate steps necessary for the authors to reach their conclusions have not been stated. Thus, here, we have taken the energy balance and the transfer rate as a single subject and have examined the processes in detail and in some generality to explain the mechanisms more thoroughly.

As a model for the plasma dynamics we have used the two-component fluid equations developed by Braginskii and have assumed that the particles respond to the fields adiabatically. We have shown that:

1. The evolution of the thermal energy is determined by the toroidal pumping, convection due to the $\boldsymbol{E}\times\boldsymbol{B}$ drift and the parallel flow along the fluctuating magnetic field, and collisional heating and cooling mechanisms. [See Eq. (2.11).]

2. In the total energy conservation [Eq. (2.12)], while the usual $5/2$ of the flux $\langle p\rangle\langle\left(\widetilde{\boldsymbol{B}}/B\right)\widetilde{V}_{\|}\rangle$ due to the fluctuating magnetic field contributes to the total energy flux across the magnetic surface, only $3/2$ of the flux of $\langle\widetilde{p}\widetilde{\boldsymbol{V}_E}\rangle$ due to the $\boldsymbol{E}\times\boldsymbol{B}$ advection contributes. This is because the electrostatic-field energy flux $q\langle n\boldsymbol{V}_{\rm d}\widetilde{\varphi}\rangle$ due to the diamagnetic drift across the magnetic surface cancels out one of the $\langle\widetilde{p}\widetilde{\boldsymbol{V}_E}\rangle$'s. Thus, in the electrostatic limit, the energy flux consists of $\frac{3}{2}\langle\widetilde{p}\widetilde{\boldsymbol{V}_E}\rangle$ and other classical fluxes due to the dissipations.

3. For the fluctuation energy balance [Eq. (2.17)], we identified the production terms due to the heat convection through the inhomogeneous plasma across the magnetic surface, transfer terms between the various fluctuations, and the dissipations. The nonuniformity of the magnetic field comes into play through the finite-Larmor-radius heat flux and serves either as a source or as a sink of the fluctuation energy depending on the location. In the steady state, globally, the production terms will balance the dissipative loss leading to the finite saturation level of the fluctuations.

As we stated in the Introduction, the balance equation [Eq. (2.17)] for the fluctuation energy can be used to predict upper bounds for the flux. However, it only generates one-point constraint. One may argue that upper bounds obtained only from one-point constraints would not be good enough mainly because one-point constraints lack information about the mode propagation, which is essential in the study of the plasma turbulence, in general. In other words, the predicted bounds may be too large to be useful. Thus, one needs two-point constraints. Nonetherless, it would, still, be useful to compute upper bounds from one-point constraints in the case when the plasma is strongly turbulent, where convection in eddies or vortices dominates the propagation of the fluctuation.

Finally, the magnetic fluctuations play a important role in the anomalous energy transport of the electrons. Thus, it is important to measure the magnetic fluctuations in addition to the electrostatic potential, especially in the short wavelength spectrum as suggested from the study of the η_e modes. Some attempts have been made in this direction.[17] However, the electromagnetic spectrum at the short wavelengths is yet to be resolved.

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