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STABILIZATION OF AXISYMMETRIC MIRROR PLASMAS BY ENERGETIC ION INJECTION

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ABSTRACT

Plasmas in axisymmetric mirror devices can be made stable to MHD interchange modes by injecting energetic ions which contribute significantly to the pressure and spend a sufficiently large fraction of a bounce time in regions of favorable curvature. Pitch-angle scattering adversely affects the method by reducing this fraction. The ions must be sufficiently energetic that pitch-angle scattering is not detrimental for that part of a slowing-down time during which they contribute significantly to the pressure. We have solved the bounce-averaged Fokker-Planck equation, including drag and pitch-angle scattering, and calculated the energetic ion contribution to the stability integral. With specially tailored magnetic fields, the required injection energy and power drain are found to be reasonable.

I. INTRODUCTION

Plasmas in axisymmetric mirror devices are subject to the magnetohydrodynamic (MHD) interchange instability[1]. Quadrupole fields have traditionally been used instead, because plasmas in such minimum-B fields have interchange stability. There are drawbacks to the use of quadrupole fields, however, such as the complexity and expense of the magnets required, and the enhancement of radial losses when quadrupole fields are used to plug a long central-cell region, in a tandem mirror device. Because of the advantages of axisymmetric mirrors, it is desirable to find a method for stabilizing interchange modes in such devices.

We have investigated one such method, the use of energetic ion injection. As with ion injection into tokamaks, partially stripped ions must be used (e.g. He^{+}) so that the ion can be trapped in the magnetic field when it becomes fully stripped. Neutral injection is not feasible because of the high energies required.

When the energetic ions contribute significantly to the pressure and spend a sufficiently large fraction of a bounce time in regions of favorable curvature, they make a stable contribution to an interchange mode. This requires the pitch-angle distribution to be quite narrow, and centered on an optimal value of the pitch angle. The ions must be sufficiently energetic initially that they pitch-angle scatter away from the optimal pitch-angle only after they have slowed down considerably, and are no longer contributing significantly to the pressure.

We have not considered the possible excitation of non-MHD instabilities due to the presence of the injected ions, such as the ion two-stream instability or the Alfven ion instability.

The stability condition is expressed in terms of the fast-ion distribution function in Section 2. The fast-ion contribution is optimized by using a specially tailored magnetic field line model, which is described in Section 3. The kinetic equation for the fast-ion distribution function, which describes the effects of guiding-center motion, pitch-angle scattering and drag, is given in Section 4; the bounce-averaged equation is derived there also. In Section 5, the power drain, required for steady-state injection at marginal stability, is expressed in terms of the rate at which energy is lost from the plasma due to ion-ion collisional scattering into the loss cone. When pitch-angle scattering is neglected, the required power drain can be evaluated analytically, and this is done in Section 6. The effect of pitch-angle scattering is discussed qualitatively in Section 7, and a rough estimate is given for the injection energy required to avoid the adverse effect of pitch-angle scattering. An approximate solution to the bounce-averaged kinetic equation is derived in Section 8, assuming that for the energies which contribute significantly in the stability integral, the fast-ion distribution function remains sharply peaked in pitch angle. Finally, in Section 9, the stability condition is evaluated numerically, using the approximate distribution function derived in Section 8. The power drain is found to be a reasonable fraction of the loss cone scattering loss provided that the injection energy is sufficiently high.

II. STABILITY CONDITION

The stability condition for the MHD interchange mode in a low-β mirror device is[1]

$$\int \frac{d\ell}{B^3 r^2} \bar{\kappa} \cdot \nabla\psi(P_\perp + P_\parallel) > 0 \quad , \tag{1}$$

where the integral is taken along a magnetic field line, $\bar{\kappa}$ is the curvature of the line, $\psi = rA_\phi(r,z)$ is the flux, and

$$P_\perp + P_\parallel = \sum_j \int d^3v m_j(\frac{v_\perp^2}{2} + v_\parallel^2)f_j$$

is the sum of perpendicular and parallel pressures, summed over all particle species, with f_j the distribution function for the j^{th} species. Near the axis of the device (the z-axis), or in a long thin device, Eq. (1) can be written as

$$S \equiv \int_{-1}^{1} \frac{dz}{b(z)} r''(z)(P_\perp + P_\parallel) > 0 \quad , \tag{2}$$

where the magnetic field on the axis is b(z), and the equation of a field line is r = r(z), so that the curvature is given by $r''(z)$. The approximate relation between r(z) and b(z), valid close to the axis, is

$$b(z)\ r^2(z) \ = \ b(1)\ r^2(1) \quad . \tag{3}$$

For the plasma ions and electrons, the distribution functions will be approximated as being isotropic, and will be taken to be Maxwellians. Then

$$P_\perp + P_\parallel = 2n_e(T_e + T_i) + (P_\perp + P_\parallel)_f \quad , \tag{4}$$

where the subscript f denotes the fast-ion contribution,

$$(P_\perp + P_\parallel)_f = 2\pi m_f \int_0^\infty v^4 dv \int_0^{1/b} \frac{bd\lambda}{\xi} (1 - \frac{1}{2}\lambda b) f_f \quad , \tag{5}$$

and m_f is the fast-ion mass. Here the velocity variables used are v, λ, σ where v is the speed, where $\lambda = (\sin^2\chi)/b$, with $\chi = \cos^{-1}(v_\parallel/v)$ the pitch angle, and σ is the sign of $v_\parallel$. We denote $v_\parallel/v$ by ξ, so that

$$\xi = (1 - \lambda b)^{1/2} \quad . \tag{6}$$

The distribution function f_f is assumed to be an even function of σ.

The stability integral, defined by Eq. (2), is $S = S_p + S_f$, where

$$S_p = 2 \int_{-1}^{1} \frac{dz}{b} r^\parallel n_e (T_e + T_i) \quad , \tag{7}$$

is the plasma contribution, and where the fast-ion contribution is

$$S_f = 2\pi m_f \int_{\lambda_1}^{\lambda_2} d\lambda \int_0^\infty v^4 dv f_f \int_{-z_t}^{z_t} \frac{dz(1 - \frac{1}{2}\lambda b)}{\xi} r^\parallel(z) \quad . \tag{8}$$

We have assumed that f_f is independent of z, and that $f_f = 0$ outside the range of magnetically trapped ions, $\lambda_1 < \lambda < \lambda_2$, where $\lambda_1 = 1/b_{max}$ and $\lambda_2 = 1/b_{min}$, with $b_{max} = b(1)$ and $b_{min} = b(0)$ the maximum and minimum values of b(z) along the field line, and where $\pm z_t$ are the turning points, where $v_\parallel = 0$, i.e. $\lambda b(z_t) = 1$.

III. TAILORED FIELD LINE MODEL

The model for the shape of the magnetic field lines which we use is especially tailored for this problem, to maximize the stabilizing contribution of the fast ions. The field lines are taken to consist of straight line segments:

$$r(s) = \begin{cases} r(0) , & \text{for } 0 \leqslant s \leqslant s_1 \\ r(0) - A_1(s-s_1) , & \text{for } s_1 \leqslant s \leqslant s_2 \\ r(1)[1 + (A_1 - A_2)(1-s)] , & \text{for } s_2 \leqslant s \leqslant 1 \\ r(1) , & \text{for } s \geqslant 1 \end{cases} \qquad (9)$$

where s_1 and s_2 are the break points, and A_1, A_2 are constants. For continuity, we must have

$$r(0) - A_1(s_2 - s_1) = r(1)[1 + (A_1 - A_2)(1 - s_2)] \quad .$$

Equation (3) will be used to relate r(s) and b(s); without loss of generality we may take r(1) = 1, and b(1) = 1. Then $b_{max} = 1$, and $b_{min} = 1/R$, where R is the mirror ratio; hence, $r(0) = R^{1/2}$. The field is thus specified by four parameters, which are taken to be s_1, s_2, R, and r_2, where $r_2 \equiv r(s_2)$. A_1 and A_2 are given, in terms of these four, by

$$A_1 = (R^{1/2} - r_2)/(s_2 - s_1) \quad , \qquad (10)$$

$$A_2 = (1 - s_1)\, A_1/(1 - s_2) - (R^{1/2} - 1)/(1 - s_2) \quad . \qquad (11)$$

The field line for a particular choice of parameters is shown in Figure 1.

The curvature consists of a sum of delta functions:

$$r''(s) = -A_1\delta(s - s_1) + A_2\delta(s - s_2) + (A_1 - A_2)\delta(s-1) \quad . \qquad (12)$$

The integral containing the curvature in Eq. (8), the fast-ion contribution to the stability integral is

$$F(\lambda) \equiv \int_{-z_t}^{z_t} \frac{dz}{\xi} \left(1 - \frac{1}{2}\lambda b\right) r''(z) \quad , \qquad (13)$$

which can be evaluated to give

$$F(\lambda) = \begin{cases} F_1(\lambda), & \text{for } r_2^2 < \lambda < R \\ F_2(\lambda), & \text{for } 1 < \lambda < r_2^2 \end{cases} \quad , \qquad (14)$$

where

$$F_1(\lambda) = -A_1[(1-\lambda/R)^{-1/2} + (1-\lambda/R)^{1/2}] \qquad (15)$$

and

$$F_2(\lambda) = F_1(\lambda) + A_2[(1-\lambda/r_2^2)^{-1/2} + (1-\lambda/r_2^2)^{1/2}] \quad . \qquad (16)$$

Since $F_1 < 0$, the fast-ion contribution is destabilizing if λ is in the range $r_2^2 < \lambda < R$, corresponding to turning points s_t in the range

$s_1 < s_t < s_2$. Such ions have been exposed only to the bad curvature at $s = s_1$. For values of λ in the range $1 < \lambda < r_2^2$, corresponding to $s_2 < s_t < 1$, an ion has seen the good curvature at $s = s_2$; if its turning point is only slightly beyond s_2, its parallel velocity at $s = s_2$ is quite small, and its bounce-averaged curvature drift is large and favorable.

The integral containing the curvature in the plasma contribution, Eq. (7), is easily evaluated also; the limits ± 1 should be interpreted as $\pm(1-\varepsilon)$, so that the good curvature contribution at $z = 1$ is omitted, as it was for the fast ions. Assuming that $n_e(T_e + T_i)$ has approximately the same value at s_1 and s_2, the result is

$$S_p = 4n_e(T_e + T_i)(-A_1R + A_2r_2^2) \quad . \tag{17}$$

Since $R = r^2(0) > r^2(s_2) = r_2^2$ and $A_1 > A_2$ for field lines with $r(s)$ monotonically decreasing from $s = 0$ to $s = 1$, it is obvious that the plasma contribution is destabilizing.

Although the field line shape considered here could only be achieved exactly near the plasma edge, by the use of appropriately shaped conducting surfaces, it could be achieved approximately throughout the plasma in the limit of a long thin device.

IV. PITCH-ANGLE DIFFUSION EQUATION

The drift kinetic equation for the fast-ion distribution function is[2]

$$\frac{\partial f}{\partial t} + (v_\parallel \hat{b} + \bar{v}_d) \cdot \nabla f - \frac{2m_i}{m_f} \nu_s \frac{v_c^3}{v^3} \frac{\xi}{b} \frac{\partial}{\partial \lambda}(\lambda \xi \frac{\partial f}{\sigma \lambda})$$

$$- \frac{\nu_s}{v^2} \frac{\partial}{\partial v}[(v^3 + v_c^3)f] = S\delta(v - v_0)/v_0^2 \quad , \tag{18}$$

where $\hat{b}$ is a unit vector tangent to a field line, and $\bar{v}_d$ is the guiding-center drift velocity. The slowing-down rate due to electron drag is

$$\nu_s = (m_e/m_f)\ z_f^2/\tau_e \quad , \tag{19}$$

where

$$1/\tau_e = \frac{16\pi^{1/2}}{3} \frac{n_e e^4 \ell n\Lambda}{m_e^2 v_e^3} \quad . \tag{20}$$

The "critical velocity", given by

$$v_c = (\frac{3\pi^{1/2} m_e}{4m_i})^{1/3} v_e \quad , \tag{21}$$

where $v_e \equiv (2T_e/m_e)^{1/2}$, is the velocity below which ion drag exceeds electron drag. The critical fast-ion energy is $E_c = 14.8(m_f/m_H)(m_H/m_i)^{2/3}\ T_e$, where m_H is the proton mass. The pitch-angle scattering term in Eq. (18), which contains the λ-derivatives, becomes comparable to the drag term only for velocities smaller than $v_c' \equiv (m_i/m_f)^{1/3}\ v_c$ (or for distribution functions which are very localized functions of λ). The corresponding fast-ion energy is $E_c' = 14.8(m_f/m_H)^{1/3}\ T_e$. We have assumed that no impurity ions are present in the plasma, and neglected the fast-ion contribution to the density. We are interested in a steady state, so the time derivative will be set equal to zero.

The right-hand side of Eq. (18) is the source of energetic ions at velocity v_0, whose spatial and λ-dependence is contained in the function S. We shall solve Eq. (18) for $v < v_0$, with the boundary condition

$$f \rightarrow \frac{S(\lambda,\bar{x})}{\nu_s(v_0^3 + v_c^3)} \quad , \quad \text{for} \quad v \rightarrow v_0 \quad . \tag{22}$$

The boundary condition corresponding to having no particles in the loss cone is

$$f \rightarrow 0 \quad , \quad \text{for} \quad \lambda \rightarrow 1/b_{max}(= 1) \quad . \tag{23}$$

Since the magnetic field is assumed to have axial symmetry, the guiding-center drift is in the ignorable direction, and makes no contribution to Eq. (18), which becomes

$$v\xi \frac{\partial f}{\partial \ell} - \frac{2m_i}{m_f} \nu_s \frac{v_c^3}{v^3} \frac{\xi}{b} \frac{\partial}{\partial \lambda} (\lambda\xi \frac{\partial f}{\partial \lambda}) - \frac{\nu_s}{v^2} \frac{\partial}{\partial v}[(v^3 + v_c^3)f] = 0 \quad , \quad (24)$$

where ℓ is distance along a field line. Since the appropriate limit here is $\nu_s << L/v$, where L is the distance between the mirrors, the bounce-averaged equation is used:

$$2 \frac{m_i}{m_f} \frac{v_c^3}{v^3} \frac{\partial}{\partial \lambda}[(\oint \frac{d\ell\xi}{b}) \lambda \frac{\partial f}{\partial \lambda}] + (\oint \frac{d\ell}{\xi}) \frac{1}{v^2} \frac{\partial}{\partial v}[(v^3 + v_c^3)f] = 0 \quad , \quad (25)$$

where f is independent of ℓ (to lowest order in $\nu_s v/L$). We have assumed that T_e, and hence v_c, is independent of ℓ. The assumption made earlier, that n_e is independent of ℓ, which implies that ν_s is also independent of ℓ, has also been used. The boundary condition for $v \to v_0$ is the bounce-averaged one:

$$f \to \frac{\oint \frac{d\ell}{\xi} S / \oint \frac{d\ell}{\xi}}{\nu_s(v_0^3 + v_c^3)} \quad . \quad (26)$$

In the following, the dimensionless velocity $u = v/v_c$ will be used. With the substitution

$$f = \frac{G(u,\lambda)}{v_c^3(u^3+1)} \quad . \tag{27}$$

Eq. (25) becomes

$$2(\frac{m_i}{m_f}) \frac{1}{u^3(u^3+1)} \frac{\partial}{\partial\lambda}[I_1(\lambda)\lambda \frac{\partial G}{\partial\lambda}] + \frac{I_2(\lambda)}{u^2} \frac{\partial G}{\partial u} = 0 \quad , \tag{28}$$

where

$$I_1(\lambda) = \int_0^{s_t} \frac{ds\xi}{b} \quad , \tag{29}$$

$$I_2(\lambda) = \int_0^{s_t} \frac{ds}{\xi} \quad , \tag{30}$$

and s_t is the turning point, $\lambda b(s_t) = 1$. The introduction of a time-like variable τ, defined by

$$\tau = \frac{2}{3} \frac{m_i}{m_f} \ell n(\frac{1+u^{-3}}{1+u_0^{-3}}) \quad , \tag{31}$$

where $u_0 \equiv v_0/v_c$, enables us to write Eq. (28) as

$$\frac{\partial G}{\partial\tau} - \frac{1}{I_2(\lambda)} \frac{\partial}{\partial\lambda}[I_1(\lambda)\lambda \frac{\partial G}{\partial\lambda}] = 0 \quad . \tag{32}$$

This equation is to be solved for $\tau > 0$ (i.e. $u < u_0$), with the initial condition

$$G \to \int_0^{s_t} \frac{ds}{\xi} S/(\nu_s \int_0^{s_t} \frac{ds}{\xi}), \quad \text{for} \quad \tau \to 0 \quad , \tag{33}$$

and the boundary condition

$$G \to 0 \quad \text{for} \quad \lambda \to 1 \quad . \tag{34}$$

For simplicity, we assume that the source is a delta function in the pitch-angle variable λ,

$$S = C\delta(\lambda - \lambda_0) \quad , \tag{35}$$

with λ_0 independent of s. The constant C can be related to the rate of fast-ion production, per unit volume:

$$\dot{n}_f \equiv \int d^3v\, S \frac{\delta(v - v_0)}{v_0^2} = 2\pi Cb(1-\lambda_0 b)^{-1/2} \quad .$$

The initial condition, Eq. (33), is then

$$G \to (Q/2\pi)\delta(\lambda-\lambda_0), \quad \text{for} \quad \tau \to 0 \quad , \tag{36}$$

where

$$Q = \frac{\int_0^{s_{t0}} \frac{ds}{b} \dot{n}_f}{\nu_s I_2(\lambda_0)} , \quad (37)$$

where s_{t0} is the turning point for $\lambda = \lambda_0$, and $I_2(\lambda)$ is defined by Eq. (30).

V. POWER DRAIN

The fast-ion contribution to the stability integral, Eq. (8), can be written in terms of G as

$$S_f = \frac{1}{2} m_f v_0^2 \int_1^R d\lambda\, H(\lambda)\, F(\lambda) \quad , \tag{38}$$

where F(λ) is defined in Eq. (13), where R is the mirror ratio, and where

$$H(\lambda) = \frac{4\pi}{u_o^2} \int_0^{u_0} \frac{u^4 du}{(u^3+1)} G(u,\lambda) \quad . \tag{39}$$

Combining Eqs. (17) and (38), the stability condition, $S_p + S_f > 0$, can be written

$$\frac{1}{2} m_f v_0^2 \int_1^R d\lambda\, H(\lambda)\, F(\lambda) \;>\; 4n_e(T_e + T_i)(A_1R - A_2r_2^2)$$

or, with $\bar{H} \equiv H/Q$,

$$\frac{1}{2} m_f\, v_0^2\, Q \;>\; \frac{4n_e(T_e + T_i)(A_1R - A_2r_2^2)}{\int_1^R d\lambda\, \bar{H}(\lambda)\, F(\lambda)} \quad . \tag{40}$$

The injection power density, integrated over a flux tube, is

$$P = \frac{1}{2} m_f v_0^2 \int \frac{ds}{b} \dot{n}_f \quad .$$

The required power drain from a fusion reactor can be written in terms of the plasma energy density multiplied by the ion-ion 90^o-scattering rate, which is approximately the rate at which plasma ions are scattered into the loss cone. The effect of an electrostatic potential is neglected here, for simplicity. We thus define the dimensionless power drain as

$$P \equiv \frac{1}{2} m_f v_0^2 \int \frac{ds}{b} \dot{n}_f / [\frac{3}{2} n_e (T_e + T_i)/\tau_i] \quad , \tag{41}$$

where

$$1/\tau_i = (m_e/2m_i)^{1/2} (T_e/T_i)^{3/2} \cdot (1/\tau_e) \quad ,$$

with the electron-ion collision time τ_e defined by Eq. (20). The power drain needed to marginally satisfy the stability condition, Eq. (40), is given by

$$P = \frac{8}{3} \frac{(A_1 R - A_2 r_2^2) I_2(\lambda_0)}{\int_1^R d\lambda \bar{H}(\lambda) F(\lambda)} \nu_s \tau_i \quad , \tag{42}$$

where, using the definitions of ν_s and τ_i,

$$\nu_s \tau_i = (2m_e/m_i)^{1/2}(m_i/m_f)z_f^2(T_i/T_e)^{3/2} \quad . \qquad (43)$$

Because of the factor z_f^2, it is most advantageous to inject light ions, whose fully stripped values of z_f are as small as possible. Helium injection is therefore best, although lithium is almost as good. (Since trapping in the magnetic field requires a change in the charge state of the ion, hydrogen injection is ruled out.) We note that although the factor $(T_i/T_e)^{3/2}$ tends to be unfavorable in mirror devices, where T_i/T_e may be larger than unity, the slowing down of the fast ions heats mainly the electrons, which tends to prevent T_i/T_e from being too large.

VI. FAST-ION STABILIZATION NEGLECTING PITCH-ANGLE SCATTERING

If we consider velocities $u \gg (m_i/m_f)^{1/3}$ in Eq. (28), the pitch-angle scattering term is negligible and we have G = const. (independent of u), given by the initial condition, Eq. (36). Assuming $u_0 \equiv v_0/v_c \gg 1$, Eq. (39) may be evaluated approximately to give $H = 2\pi G$, and Eq. (40) becomes

$$\frac{1}{2} m_f v_0^2 Q > \frac{4n_e(T_e + T_i)(A_1R - A_2r_2^2)}{F(\lambda_0)} \quad . \tag{44}$$

Equation (44) requires that the fast ions contribute significantly to the pressure, so that $\frac{1}{2} m_f v_0^2 Q$ is comparable with $n_e(T_e+T_i)$, and that they see a bounce-averaged favorable curvature, so that $F(\lambda_0)$ is positive. (Recall that $-A_1R + A_2r_2^2$ is always negative.)

For example, using the parameters $s_1 = 0.05$, $s_2 = 0.2$, $R = 4$, $r_2^2 = 2$, we have $-A_1R + A_2r_2^2 = -8.8$. The function $F(\lambda)$ for these parameters is shown in Figure 2. By choosing $\lambda_0 = 1.8$, we have $F(\lambda_0) = 3.5$. Thus, Eq. (44) becomes

$$\frac{1}{2} m_f v_0^2 Q > 10.1\, n_e(T_e + T_i)$$

From the definition of Q, Eq. (37), and the fact that the steady-state fast-ion density is given approximately by

$$n_f \simeq \int \frac{ds}{b} \dot{n}_f/\nu_s = QI_2(\lambda_0)$$

one finds for this example (using $I_2(\lambda_0) \simeq 1$) that the steady-state fast-ion energy density must exceed the plasma thermal energy density by roughly a factor of 6. If the value $\lambda_0 = 1.9$ were used instead of 1.8, then $F(\lambda_0) = 7.7$, and this factor need only be about 3.

The dimensionless power drain, Eq. (42), using the parameters corresponding to Figure 2, with $\lambda_0 = 1.8$, is $P = 0.22(m_H/m_i)^{1/2}(m_i/m_f)z_f^2(T_i/T_e)^{3/2}$. With α-particles in a deuterium plasma with $T_e/T_i = 2$, for example, we have $P = 0.11$. Hence, if direct conversion of the energy lost by loss cone scattering were used, about 11% of it could be used for the ion injection. If the value $\lambda_0 = 1.9$ were used, this becomes 4.5%.

VII. THE EFFECT OF PITCH-ANGLE SCATTERING

Pitch-angle scattering adversely affects the stabilization method by spreading out the fast-ion distribution so that it includes values of λ for which $F(\lambda)$ is negative or zero. For example, in Figure 2, $F(\lambda)$ is only significantly stabilizing in the range $1.5 < \lambda < 2.0$. When the λ value of an ion scatters to values greater than 2.0, the ion sees only bad curvature; for $\lambda < 1.5$, the averaged good and bad curvature tend to cancel. For $\lambda < 1.0$, of course, the ion has scattered into the loss cone and makes no further contribution.

In order to minimize the effect of pitch-angle scattering, the injection velocity v_0 must exceed the velocity $(m_i/m_f)^{1/3}\ v_c$ by a sufficient amount, which can be estimated as follows.

By solving Eq. (31) for u, assuming that $u \gg 1$ and $u_0 \gg 1$, we have, approximately,

$$u = u_0\left(1 + \frac{3}{2}\frac{m_f}{m_i}u_0^3\tau\right)^{-1/3}$$

The integral in Eq. (39) can also be written approximately as

$$H(\lambda) = 2\pi\frac{m_f}{m_i}u_0^3\int_0^\infty \frac{d\tau\ G(u,\lambda)}{\left(1 + \frac{3}{2}\frac{m_f}{m_i}u_0^3\ \tau\right)^{5/3}} \quad . \tag{45}$$

Only values of $\tau \lesssim (m_i/m_f)\ u_0^{-3}$ contribute significantly to the integral. For all such values, we want the spread in λ, due to pitch-angle diffusion, to be small, i.e. $\Delta \lesssim 0.25$. From Eq. (32), we have the rough estimate

$$\Delta\lambda \simeq (2\tau)^{1/2} \quad . \tag{46}$$

Thus, it is necessary that

$$[2(m_i/m_f)\ u_0^{-3}]^{1/2} \lesssim 0.25$$

which leads to the condition on the injection velocity:

$$v_0 \gtrsim 3.2(m_i/m_f)^{1/3}\ v_c \quad . \tag{47}$$

The required injection energy, $E_0 \equiv \frac{1}{2}\, m_f\, v_0^2$, is therefore required to exceed by a factor of ten the energy E_c' defined by

$$E_c' \equiv \frac{1}{2}\, m_f[(m_i/m_f)^{1/3}\ v_c]^2 = 14.8(m_f/m_H)^{1/3}\ T_e \quad , \tag{48}$$

where m_H is the proton mass. For example, at an electron temperature of 50 kev, the required injection energy for helium ions is E_0 = 12 Mev. Since this conclusion is based on the rough approximation given by Eq. (46), a more accurate calculation is needed.

VIII. SOLUTION OF THE PITCH-ANGLE DIFFUSION EQUATION

The rough considerations of the previous section have shown that $\tau \lesssim 0.03$ is required to avoid the adverse effects of pitch-angle scattering. An approximate method for solving Eq. (32), for small values of τ, can therefore be applied. We first write the equation as

$$\frac{\partial G}{\partial \tau} = -A(\lambda)\,\frac{\partial G}{\partial \lambda} + D(\lambda)\,\frac{\partial^2 G}{\partial \lambda^2} \quad , \tag{49}$$

where

$$A(\lambda) = -\frac{d}{d\lambda}\,(\lambda I_1)/I_2 \quad , \tag{50}$$

and

$$D(\lambda) = \lambda I_1/I_2 \quad , \tag{51}$$

and $I_1(\lambda)$, $I_2(\lambda)$ are given by Eqs. (29) and (30).

We solve Eq. (49) by first finding a solution which satisfies the initial condition, Eq. (36), and then adding a second solution to satisfy the boundary condition, Eq. (34). The initial condition is

$$G \to (Q/2\pi)\,\delta(\lambda - \lambda_0), \quad \text{for} \quad \tau \to 0 \quad , \tag{52}$$

where Q is defined by Eq. (37).

A solution of Eq. (49) is sought in the form

$$G_1(\lambda,\tau) = \exp[\phi(\lambda,\tau)] \quad , \tag{53}$$

where ϕ is a solution of

$$\frac{\partial\phi}{\partial\tau} = -A(\lambda)\,\frac{\partial\phi}{\partial\lambda} + D(\lambda)\left[\,\frac{\partial^2\phi}{\partial\lambda^2} + \left(\frac{\partial\phi}{\partial\lambda}\right)^2\right] \quad . \tag{54}$$

By substituting

$$\phi(\lambda,\tau) = -w(\lambda)/\tau - \frac{1}{2}\,\ell n\tau + y(\lambda,\tau) \quad , \tag{55}$$

where $y(\lambda,\tau)$ is assumed to have a power series,

$$y(\lambda,\tau) = y_0(\lambda) + y_1(\lambda)\tau + \ldots \quad , \tag{56}$$

we may equate terms of the same order in τ, for $\tau \to 0$, on either side of Eq. (54). Then it is easily shown that

$$w(\lambda) = \frac{1}{4}\left[\,\int_{\lambda_0}^{\lambda} \frac{d\lambda}{D^{1/2}(\lambda)} + C\right]^2 \quad , \tag{57}$$

and

$$y_0(\lambda) = -\frac{1}{4}\,\ell n[\lambda I_1(\lambda)I_2(\lambda)] + K \quad . \tag{58}$$

The integration constants, C and K, are chosen to satisfy the initial condition, Eq. (52). Thus, with the approximation $y(\lambda,\tau) \approx y_0(\lambda)$, we have

$$G_1(\lambda,\tau) = (Q/4\pi^{3/2})[D(\lambda_0)\tau]^{-1/2}\left[\frac{\lambda_0 I_1(\lambda_0)I_2(\lambda_0)}{\lambda\, I_1(\lambda)I_2(\lambda)}\right]^{1/4}\exp\left(-\frac{w_1(\lambda)}{\tau}\right) \quad (59)$$

where $w_1(\lambda)$ is given by Eq. (57) with C = 0.

The boundary condition, Eq. (34), can be satisfied by adding a second solution G_2 which is equal to $-G_1$, as given by Eq. (59), but with w_1 replaced by

$$w_2(\lambda) = \frac{1}{4}\left[\int_1^{\lambda}\frac{d\lambda}{D^{1/2}(\lambda)} + \int_1^{\lambda_0}\frac{d\lambda}{D^{1/2}(\lambda)}\right]^2 \quad , \quad (60)$$

which corresponds to a different choice of the integration constant in Eq. (57). Thus, the approximate solution which satisfies both initial and boundary conditions is

$$G(\lambda,\tau) = (Q/4\pi^{3/2})[D(\lambda_0)\tau]^{-1/2}\left[\frac{\lambda_0 I_1(\lambda_0)I_2(\lambda_0)}{\lambda\, I_1(\lambda)I_2(\lambda)}\right]^{1/4}\left[\exp\left(-\frac{w_1(\lambda)}{\tau}\right) - \exp\left(-\frac{w_2(\lambda)}{\tau}\right)\right] \quad . \quad (61)$$

Figure 3 shows this function for a few small values of τ, for $\lambda_0 = 1.8$, $u_0 = 4.0$, and the same magnetic field parameters as in Figure 1.

IX. NUMERICAL EVALUATION OF THE STABILITY CONDITION

The integrals in Eqs. (38) and (39) have been carried out numerically, in order to evaluate the right-hand side of the stability condition, Eq. (40), with the effect of pitch-angle scattering included. The results can be expressed in terms of the dimensionless power drain, Eq. (42), needed to marginally satisfy the stability condition.

The quantity P_0, defined in terms of the power drain P, by

$$P = P_0(m_H/m_i)^{1/2} \; z_f^2(T_i/T_e)^{3/2} \quad , \tag{62}$$

is shown in Figure 4, as a function of v_0/v_c', where

$$v_c' = (m_i/m_f)^{1/3} \, v_c \quad , \tag{63}$$

and v_c is given by Eq. (21). The same field parameters as in Figure 1 were used, and the mass ratio was $m_f/m_i = 2$. Three different values of the injection pitch-angle parameter λ_0 were used. The asymptotic values of P_0, for $v_0/v_c' >> 1$, agree well with the results obtained by replacing G by its initial condition, as described in Section 6. These results were as follows: for $\lambda_0 = 1.8$, $P_0 = 0.11$; for $\lambda_0 = 1.9$, $P_0 = 0.045$. For $\lambda_0 = 2.0$, with the value $F(\lambda_0) = 40.0$, which is used in the numerical evaluation procedure, the analytical expression, Eq. (44), gives $P_0 = 0.005$.

Figure 4 shows that the value $v_0/v_c' \simeq 5$ is adequate, to reduce the power drain to a value comparable with its $v_0/v_c' \gg 1$ asymptotic value. The rough considerations of Section 7 had predicted that $v_0/v_c' = 3.2$ would be needed, for $\lambda_0 = 1.8$. The required injection energy is

$$E_0 = (v_0/v_c')^2 E_c'$$

where E_c' is given by Eq. (48). For helium injection, for example, $E_c' = 23.5T_e$. At $T_e = 50$ Kev, for example, $E_c' = 1.17$ Mev. Then $v_0/v_c' = 5$ requires an injection energy $E_0 = 29$ Mev. Such energies may be within the range of ion accelerators presently under consideration for tokamak heating.

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FIGURE CAPTIONS

1. Magnetic field line shape, for the model used in this paper, Eq. (9) where the parameters have been chosen to be $s_1 = 0.05$, $s_2 = 0.2$, $A_1 = 3.9$, and $A_2 = 3.4$. The mirror ratio is $R = 4$.

2. The bounce-averaged curvature factor $F(\lambda)$, defined by Eq. (13), for the same field parameters used in Figure 1.

3. The fast-ion distribution function $G(u,\lambda)$, defined by Eq. (61), as a function of λ, for $\lambda_0 = 1.8$ and the same field parameters as in Figures 1 and 2. The value of u_0 used is 4.0, and five different values of the dimensionless velocity u were used: (a) $u = 3.8$, (b) $u = 3.6$, (c) $u = 2.9$, (d) $u = 2.5$, and (e) $u = 2.1$.

4. Power drain P_0, defined by Eq. (62), as a function of v_0/v_c' [where v_c' is defined by Eq. (63)], for three different values of λ_0(1.8, 1.9, and 2.0), and for $m_f/m_i = 2$, and the same field parameters as in Figures 1 and 2.

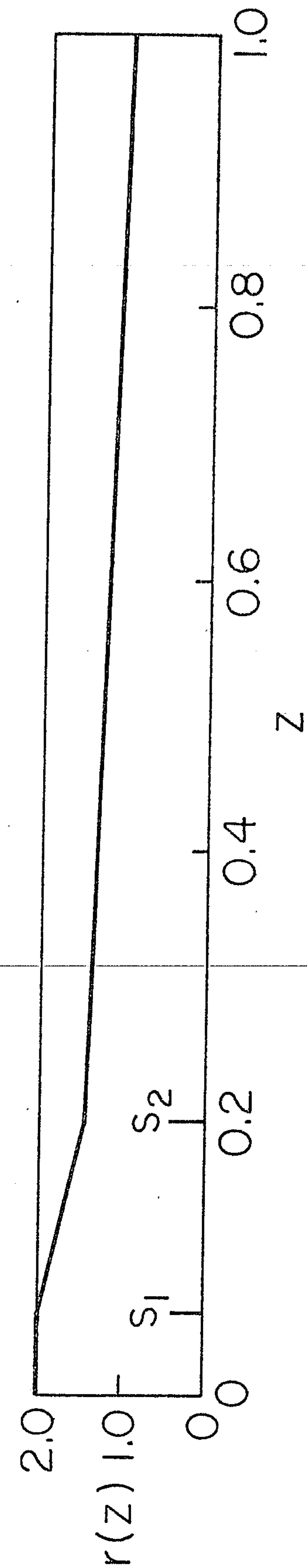

FIG. 1

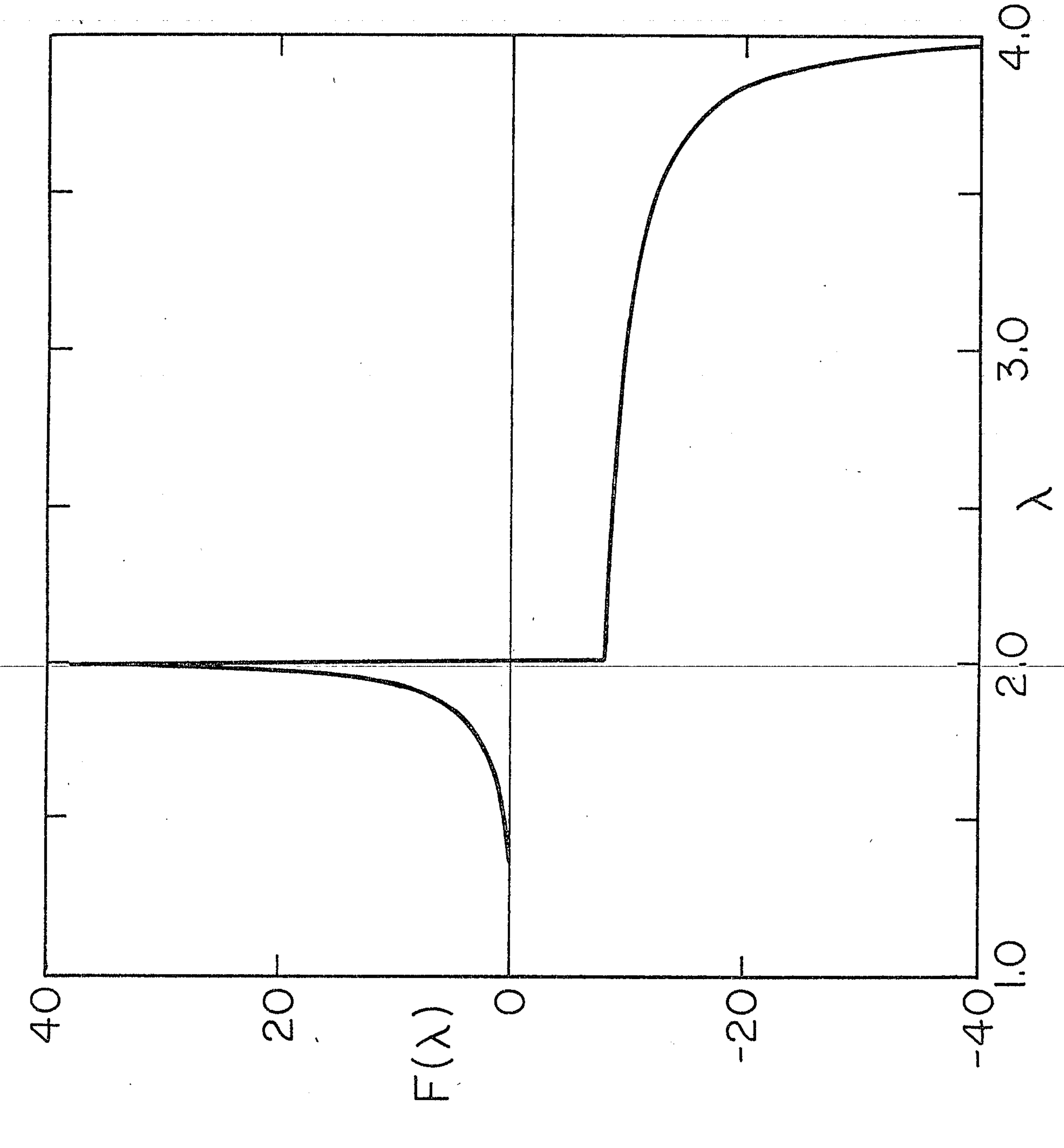

FIG. 2

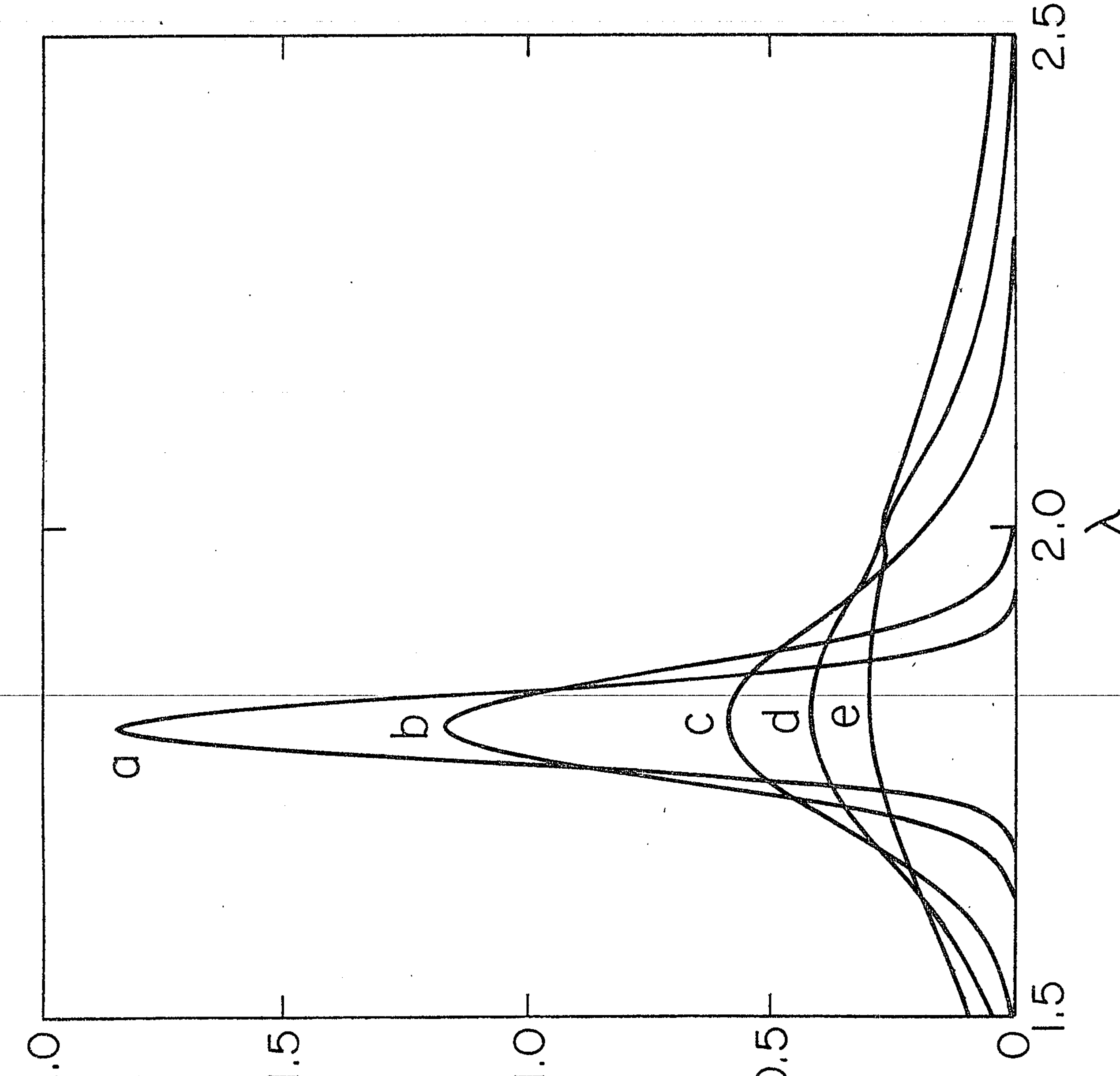

FIG. 3

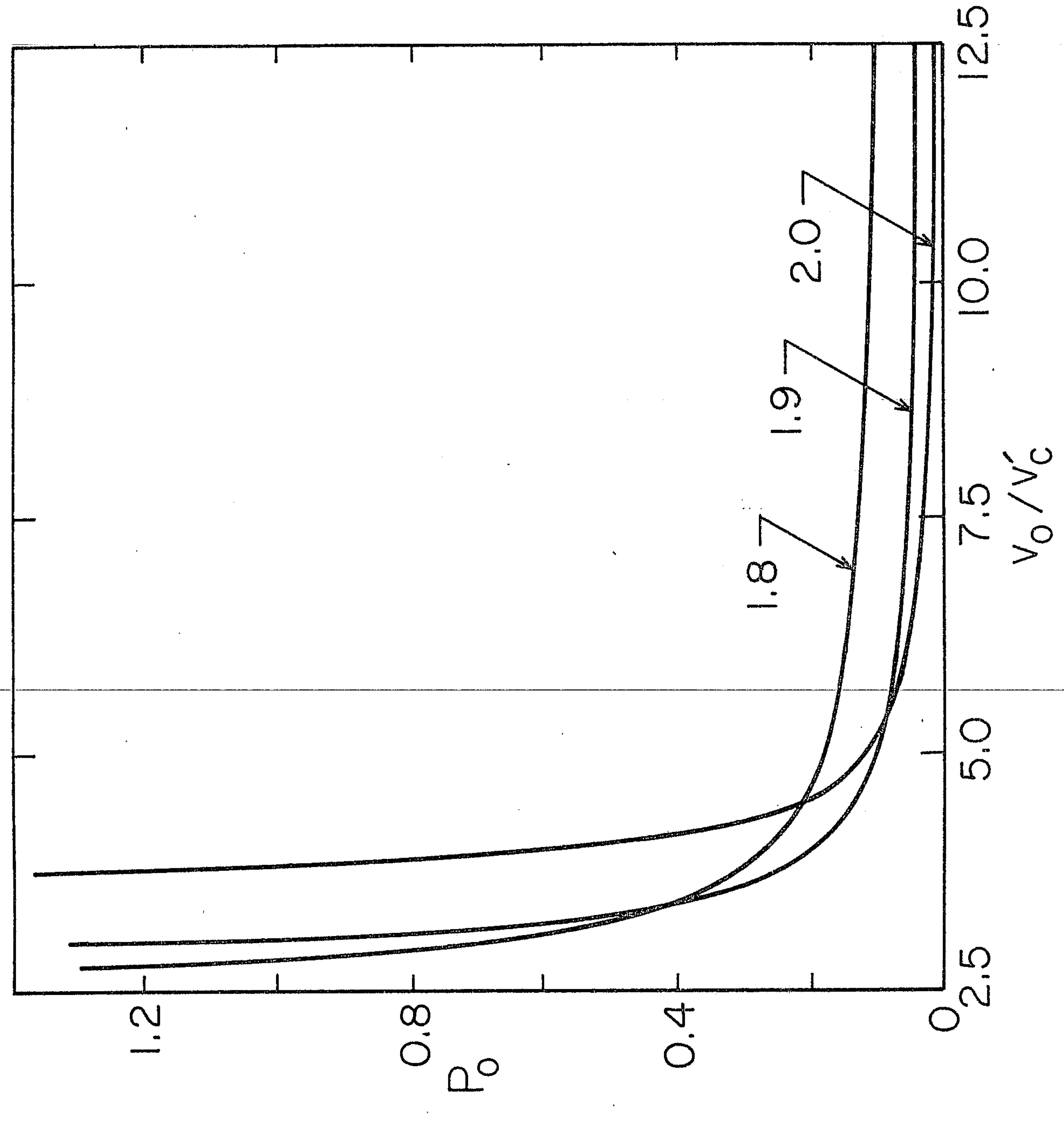

FIG. 4