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A Nonlinear Bounce Kinetic Equation for Trapped Electrons

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Abstract

A nonlinear bounce averaged drift kinetic equation for trapped electrons is derived. This equation enables one to compute the nonlinear response of the trapped electron distribution function in terms of the field-line projection of a potential fluctuation $\langle e^{-inq\theta}\phi_n\rangle_b$. It is useful for both analytical and computational studies of the nonlinear evolution of short wavelength ($n \gg 1$) trapped electron mode-driven turbulence.

I. Introduction

Trapped electron modes have long been considered to be an important cause of transport in a magnetically confined tokamak plasma.[1−3] The mode is destablized by electrons which are trapped in a local magnetic well and execute fast bounce motion. Typically, the trapped electron's bounce motion is the fastest relevent time scale ($\tau_b = \omega_b^{-1}$), and is much faster than the mode oscillations or dynamical interactions($\tau = \omega^{-1}$ or $\triangle\omega^{-1}$) i.e.: $\omega_b \gg \omega$ or $\triangle\omega$, where ω_b is the frequency of the trapped electron's bounce motion, ω is the frequency of the mode oscillation, and $\triangle\omega$ is the decorrelation rate characteristic of dynamical interactions in the presence of turbulence. Due to this fast bounce motion, the trapped electron response is significant in magnitude, and basically hydrodynamic. Particularly, in the collisionless regime, the electrons can no longer be

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treated as laminar, so that nonlinear electron dynamics must be considered in the study of the nonlinear evolution of trapped electron mode driven turbulence.[4] In order to facilitate the analysis of the nonlinear evolution of trapped electron mode driven turbulence, we here derive a nonlinear bounce averaged drift kinetic equation for trapped electrons. The derivation is based on the nonlinear drift kinetic equation of Frieman and Chen,[5] and is carried out by utilizing a multiple scale technique,[6] which is essentially a power series expension of the solution to the nonlinear drift kinetic equation in the small parameter δ ($=\tau_b/\tau$). To zeroth order, the solution simply states that the trapped electrons bounce along the magnetic field lines on the fastest time scale (τ_b). To next order, we project the equation along the magnetic field line, and then proceed with the bounce averaging procedure to eliminate the fast variation due to the bounce motion. In this manner, we can obtain the desired nonlinear bounce averaged drift kinetic equation in terms of the field-line projected potential fields $\langle e^{-inq\theta}\phi_n\rangle_b$. In the calculation, the mode is not presumed to be fully aligned with the field line, i.e. $\langle e^{-inq\theta}\phi_n\rangle_b$ can not be replaced by ϕ_n. As a matter of fact, the modes of interest are localized on the mode rational surface, while the trapped electrons bounce along the field lines. This difference have been shown to have important impact on both linear[1] and nonlinear properties of the trapped electron mode, and the associated turbulence. The derivation is straightforward but tedious, on account of the fact that a partial integration inside a bounce average $\langle e^{-inq\theta}\frac{\partial}{\partial\theta}A\rangle_b \approx inq\langle e^{-inq\theta}A\rangle_b$ is required. This relation cannot be directly obtained from an ordinary partial integration rule, since the function $\frac{e^{-inq\theta}A}{|v_\parallel|}$ does not exist at the ends of the banana orbit. However, a lemma is proposed, proved and utilized to surmount this technical difficulty in the case of high mode number $\mathbf{n} \gg \mathbf{1}$. The application of this equation to the study of nonlinear trapped electron dynamics in drift wave turbulence will be discussed in a future publication.

The next section contains the theoretical formulation and derivation of the nonlinear bounce kinetic equation. A lemma concerning partial integration in a bounce average is rigorously proved. It shows that the equation obtained is only approprate to describe the nonlinear evolution of a short wavelength mode, i.e. $\mathbf{n} \gg \mathbf{1}$. Discussions and conclusions are then presented.

II Derivation of Nonlinear Bounce Kinetic Equation

In this section, starting from the nonlinear drift kinetic equation of Frieman and Chen, we derive a nonlinear bounce averaged drift kinetic equation for trapped electrons. We work in a toroidal geometry with circular, concentric magnetic surfaces. We adopt the usual (r, θ, ξ) coordinates corresponding, respectively, to the minor radius, poloidal angle, and toroidal angle. In this coordinates system, the equilibrium magnetic field can be writen as $\vec{B} = B(\vec{e}_\xi + \frac{\epsilon}{q}\vec{e}_\theta)$, where $\epsilon = \frac{r}{R_0}$ is the inverse aspect ratio, $B = B_0(1 - \epsilon\cos\theta)$ is the magnitude of the magnetic field, q is the safety factor, and $\vec{e}_\theta$, $\vec{e}_\xi$ are the unit vectors in the poloidal and toroidal directions, with $\vec{e}_r$ being the unit vector in the radial direction. To the lowest order, the equilibrium distribution function is assumed to be a local Maxwellian with density N and temperature T_e

$$F_e = N(\frac{m_e}{2\pi T_e})^{\frac{3}{2}}\exp\{-\frac{m_e v^2}{2T_e}\} \tag{1}$$

where m_e is electron mass, and v is its velocity.

In the electrostatic approximation, the nonlinear drift kinetic equation is

$$(\frac{\partial}{\partial t} + v_\parallel\nabla_\parallel + \vec{v}_{de}\cdot\vec{\nabla}_\perp + \nu_{eff})h^e + N(h^e,\phi) = -\frac{e}{T_e}F_e(\frac{\partial}{\partial t} + \hat{\omega}_e)\phi \tag{2}$$

and

$$\begin{aligned}
\vec{\nabla}_\perp &= \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} \\
\nabla_\parallel &= \frac{1}{qR_0}(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\xi}) \\
\vec{v}_{de} &= \frac{(\frac{v_\perp^2}{2} + v_\parallel^2)}{R_0|\Omega_e|}(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) \\
\hat{\omega}_e &= \frac{cT_e}{eB}\frac{1}{L_N}[1 + \eta_e(\frac{v^2}{v_e^2} - \frac{3}{2})]\frac{1}{r}\frac{\partial}{\partial\theta} \\
N(h^e,\phi) &= \frac{c}{B}\vec{e}_\parallel\times\vec{\nabla}\phi\cdot\vec{\nabla}h^e \\
&= \frac{c}{B}[(\frac{\partial}{\partial r}\phi)(\frac{1}{r}\frac{\partial}{\partial\theta}h^e) - (\frac{1}{r}\frac{\partial}{\partial\theta}\phi)(\frac{\partial}{\partial r}h^e)]
\end{aligned}$$

where, $v_e^2 = \frac{2T_e}{m_e}$, $\eta_e = \frac{d\ln T_e}{d\ln N}$, $L_n^{-1} = -\frac{d\ln N}{dr}$, $\nu_{eff} = \nu_e(\frac{v_e}{v})^3$ is the effective collision frequency, ν_e is the electron ion collisional deflection frequency for a thermal electron, and

h^e is the nonadiabatic part of the perturbed electron distribution function

$$f^e = \frac{e}{T_e} F_e \phi + h^e \tag{3}$$

Since there is no toroidal coupling, we can choose a fourier representation in ξ for both ϕ and h^e,

$$h^e(r,\theta,\xi) = \sum_n h_n^e(r,\theta) e^{-in\xi} \tag{4a}$$

$$\phi(r,\theta,\xi) = \sum_n \phi_n(r,\theta) e^{-in\xi} \tag{4b}$$

The fourier transform of equation (2) is

$$[\frac{\partial}{\partial t} + \frac{v_\|}{qR_0}(\frac{\partial}{\partial \theta} - inq) + \vec{v}_{de} \cdot \vec{\nabla}_\perp + \nu_{eff}] h_n^e + N_n(h^e, \phi) = -\frac{e}{T_e} F_e (\frac{\partial}{\partial t} + \hat{\omega}_e)\phi_n. \tag{5}$$

with

$$N_n(h^e,\phi) = \frac{c}{B} \sum_{n_1+n_2=n} [(\frac{\partial}{\partial r}\phi_{n_1})(\frac{1}{r}\frac{\partial}{\partial \theta} h_{n_2}^e) - (\frac{1}{r}\frac{\partial}{\partial \theta}\phi_{n_1})(\frac{\partial}{\partial r} h_{n_2}^e)] \tag{6}$$

The ordering of each term in the above equation is as follows. Starting from the first term on the left hand side of Eq.(5), each term is proportional to ω, ω_b, ω_{de}, ν_{eff}, $\Delta\omega$, ω, and ω_e^*, respectively. For trapped electrons and physically reasonable turbulence levels, ω_b is the biggest. The remaining parameter may be ordered in different ways. For collisionless trapped electron modes, we have $\omega \geq \omega_{de} > \nu_{eff}$, for collisional trapped electron modes, we have either $\omega > \nu_{eff} > \omega_{de}$ (dynamic) or $\nu_{eff} > \omega > \omega_{de}$ (dissipative). Here the nonlinear term must be treated on equal footing with the linear term in order to include the strong turbulence case. Generally, the nonlinear term is proportional to $\Delta\omega$, where $\Delta\omega$ is the turbulence decorrelation rate. Typically, $\Delta\omega \leq \omega_e^*$ is the case.

We simplify this equation by applying a multiple-time-scale expansion method, i.e., we expand the solution of Eq.(5) in a power series of a small parameter $\delta = \frac{\omega}{\omega_b} (\ll 1)$

$$h_n^e(r,\theta,T) = h_n^{e(0)}(r,\theta,T) + h_n^{e(1)}(r,\theta,T)\delta + h_n^{e(2)}(r,\theta,T)\delta^2 + \cdots \tag{7}$$

where $T \equiv (t, \delta t, \delta^2 t, \cdots)$ represents the (multiple) time scale, and in practice, $h_n^e(r,\theta,T)$ will be approximated by $h_n^{e(0)}(r,\theta,T)$. Substituting Eq.(7) into Eq.(5), to the lowest order we have:

$$(\frac{\partial}{\partial\theta} - inq)h_n^{e(0)}(r,\theta) = 0 \tag{8}$$

The solution of this equation is

$$h_n^{e(0)}(r,\theta) = \tilde{h}_n^e(r)e^{inq\theta} \tag{9}$$

Equation (9) indicates that to the lowest order, the trapped electrons follow to the magnetic field line ($e^{inq\theta}$) on the fast time scale, and the coefficient of this response $\tilde{h}_n^e(r)$ evolves on a slow time scale.

The study of the evolution of $\tilde{h}_n^e(r)$ on the slow time scale is of the most practical interest and therefore is the main subject of this paper. This evolution can be obtained by eliminating the fast bounce motion (bounce average). Proceeding to the next higher order in Eq.(5), i.e.

$$\begin{aligned}&(\frac{\partial}{\partial t} + \vec{v}_{de}\cdot\vec{\nabla}_\perp + \nu_{eff})(\tilde{h}_n^e e^{inq\theta}) + \frac{v_\parallel}{qR_0}(\frac{\partial}{\partial\theta} - inq)h_n^{e(1)} + N_n(h_n^{e(0)},\phi)\\ &= -\frac{e}{T_e}F_e(\frac{\partial}{\partial t} + \hat{\omega}_e)\phi_n\end{aligned} \tag{10}$$

Multiplying this equation on both sides by $e^{-inq\theta}$, proceeding with the bounce average, and using the condition that $h_n^{e(1)}$ is a single-value function of θ, we obtain

$$\begin{aligned}&(\frac{\partial}{\partial t} + \nu_{eff})h_n^e + \langle e^{-inq\theta}\vec{v}_{de}\cdot\vec{\nabla}_\perp(\tilde{h}_n^e e^{inq\theta})\rangle_b + \langle e^{-inq\theta}N_n(h^{e(0)},\phi)\rangle_b\\ &= -\frac{e}{T_e}F_e(\frac{\partial}{\partial t}\langle e^{-inq\theta}\phi_n\rangle_b + \langle e^{-inq\theta}\hat{\omega}_e\phi_n\rangle_b)\end{aligned} \tag{11}$$

Here the bounce average is defined as

$$\langle A\rangle_b = \oint\frac{d\theta}{v_\parallel}A/\oint\frac{d\theta}{v_\parallel}. \tag{12}$$

Before we make any further simplification of this equation, we must prove a useful lemma concerning partial integrations within a bounce average.

Lemma: If a function A(r, θ) is well behaved (nonsingular) at both ends of a banana orbit, then, for sufficiently large mode number $\mathbf{n} \gg 1$, we have

$$\langle \mathbf{e^{-inq\theta}\frac{\partial}{\partial\theta}A}\rangle_{\mathbf{b}} \approx \mathbf{inq}\langle \mathbf{e^{-inq\theta}A}\rangle_{\mathbf{b}} \tag{13}$$

Proof:

$$\begin{aligned}
I =& \oint \frac{d\theta}{v_\parallel} e^{-inq\theta}\frac{\partial}{\partial\theta}A \\
=&2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{|v_\parallel|} e^{-inq\theta}\frac{\partial}{\partial\theta}A \\
=&2\int_{-\theta_0}^{\theta_0} d\theta\frac{\partial}{\partial\theta}(\frac{e^{-inq\theta}}{|v_\parallel|}A) - 2\int_{-\theta_0}^{\theta_0} d\theta A\frac{\partial}{\partial\theta}(\frac{e^{-inq\theta}}{|v_\parallel|})
\end{aligned}$$

Here θ_0 is the poloidal angle at which $v_\parallel$ vanishes. The first integral on the right hand side of the above equation does not exist since the function $\frac{e^{-inq\theta}}{|v_\parallel|}A$ is singular at both θ_0 and $-\theta_0$. Nevertheless, we can consider the limit:

$$I = 2\lim_{\theta_0^- \to \theta_0}[(\frac{e^{-inq\theta}}{|v_\parallel|}A)_{\theta_0^-} - (\frac{e^{-inq\theta}}{|v_\parallel|}A)_{-\theta_0^-} - \int_{-\theta_0^-}^{\theta_0^-} d\theta A\frac{\partial}{\partial\theta}(\frac{e^{-inq\theta}}{|v_\parallel|})] \tag{14}$$

Where θ_0^- is defined by $\theta_0^- = \theta_0 - \epsilon$ and $\epsilon > 0$. θ_0^- approaches θ_0 as ϵ approaches 0 from above. The strategy here is to separate the divergent part from the last term in the above equation, so that exact cancellation with the first two terms can occur. At this stage, an explicit expression for $|v_\parallel|$ in terms of θ is desirable. We define a pitch angle variable κ such that $\kappa^2 = [\frac{1}{2}v^2 - \mu B_0(1-\epsilon)]/2\epsilon B_0$, where μ is the magnetic moment. For marginally trapped particles $\kappa \approx 1$, while for deeply trapped particles $\kappa \ll 1$. Since both v^2 and μ are the constants of motion for trapped particles, κ is also a constant of motion. In terms of κ, we have $|v_\parallel| = 2\sqrt{\mu B_0 \epsilon}(\kappa^2 - \sin^2\frac{\theta}{2})^{\frac{1}{2}}$, where θ_0 is defined by $\kappa = \sin\frac{\theta_0}{2}$. Now

$$\begin{aligned}
&\int_{-\theta_0^-}^{\theta_0^-} d\theta A\frac{\partial}{\partial\theta}(\frac{e^{-inq\theta}}{|v_\parallel|}) \\
=&\int_{-\theta_0^-}^{\theta_0^-} d\theta A\frac{e^{-inq\theta}}{|v_\parallel|}(-inq - \frac{\partial}{\partial\theta}\ln|v_\parallel|)
\end{aligned} \tag{15}$$

with

$$\frac{\partial}{\partial\theta}\ln|v_\parallel| = -\frac{1}{4}\frac{\sin\theta}{\kappa^2 - \sin^2\frac{\theta}{2}}$$

Let's divide the integration range $(-\theta_0^-, \theta_0^-)$ into three subranges: I, $(-\theta_0^-, -\theta_0^- + \eta)$; II, $(-\theta_0^- + \eta, \theta_0^- - \eta)$; III, $(\theta_0^- - \eta, \theta_0^-)$, where η is determined by

$$|nq| = |\frac{\partial}{\partial\theta}\ln|v_\parallel||_{\theta_0^- - \eta} \tag{16}$$

In the subranges I and III, we have

$$|nq| < |\frac{\partial}{\partial\theta}\ln|v_{\parallel}||$$

In the subrange II, we have

$$|nq| > |\frac{\partial}{\partial\theta}\ln|v_{\parallel}||$$

Since $|nq| = |m| \gg 1$, we expect $\eta \ll 1$. By solving Eq. (16), we obtain $\eta \approx \frac{1}{2|nq|}$, which relates boundary layer size to the mode number of the fluctuations. Eq. (15) then becomes

$$\begin{aligned}\int_{-\theta_0^-}^{\theta_0^-} d\theta A\frac{\partial}{\partial\theta}(\frac{e^{-inq\theta}}{|v_{\parallel}|}) = -inq\int_{-\theta_0^-}^{\theta_0^-} d\theta A\frac{e^{-inq\theta}}{|v_{\parallel}|}\\ -(\int_{-\theta_0^-+\frac{1}{2|nq|}}^{\theta_0^- - \frac{1}{2|nq|}} + \int_{\theta_0^- - \frac{1}{2|nq|}}^{\theta_0^-} + \int_{-\theta_0^-}^{-\theta_0^- + \frac{1}{2|nq|}})d\theta A\frac{e^{-inq\theta}}{|v_{\parallel}|}\frac{\partial}{\partial\theta}\ln|v_{\parallel}|\end{aligned} \tag{17}$$

The last two terms on the right hand side of Eq. (17) diverge in the limit of $\theta_0^- \to \theta_0$, and can be determined to be

$$\int_{\theta_0^- - \frac{1}{2|nq|}}^{\theta_0^-} d\theta A\frac{e^{-inq\theta}}{|v_{\parallel}|}\frac{\partial}{\partial\theta}\ln|v_{\parallel}| = -(A\frac{e^{-inq\theta}}{|v_{\parallel}|})_{\theta_0^-} + \sqrt{2|nq|}\frac{(Ae^{-inq\theta})_{\theta_0^-}}{\sqrt{\mu B_0\epsilon\sin\theta_0}} \tag{18}$$

$$\int_{-\theta_0^-}^{-\theta_0^- + \frac{1}{2|nq|}} d\theta A\frac{e^{-inq\theta}}{|v_{\parallel}|}\frac{\partial}{\partial\theta}\ln|v_{\parallel}| = (A\frac{e^{-inq\theta}}{|v_{\parallel}|})_{-\theta_0^-} - \sqrt{2|nq|}\frac{(Ae^{-inq\theta})_{-\theta_0^-}}{\sqrt{\mu B_0\epsilon\sin\theta_0}} \tag{19}$$

Substituting Eqs. (17), (18), (19) into Eq. (14), we finally obtain

$$\begin{aligned}I =& inq\int_{-\theta_0}^{\theta_0} d\theta\frac{Ae^{-inq\theta}}{|v_{\parallel}|}\\ &+\int_{-\theta_0+\frac{1}{2|nq|}}^{\theta_0-\frac{1}{2|nq|}} d\theta\frac{Ae^{-inq\theta}}{|v_{\parallel}|}\frac{\partial}{\partial\theta}\ln|v_{\parallel}|\\ &+\sqrt{2|nq|}\frac{(Ae^{-inq\theta})_{\theta_0} - (Ae^{-inq\theta})_{-\theta_0^-}}{\sqrt{\mu B_0\epsilon\sin\theta_0}}\end{aligned} \tag{20}$$

The above equation clearly indicates that for sufficiently large mode number **n**, the last two terms are negligible, so that

$$\oint\frac{d\theta}{|v_{\parallel}|}e^{-inq\theta}\frac{\partial}{\partial\theta}A \approx inq\oint\frac{d\theta}{|v_{\parallel}|}Ae^{-inq\theta} \tag{21}$$

which in turn leads to relation (13). The accuracy of making this approximation is determined by $\frac{1}{\sqrt{nq}}$. Hence, the approximation is accurate only for high mode number modes. Below, this lemma will serve to simplify those bounce averages in Eq.(11).

(a) First, let's examine the last term in Eq.(11). By direct application of relation (13), we have

$$\langle e^{-inq\theta}\hat{\omega}_e\phi_n\rangle_b = i\omega_e^T\langle e^{-inq\theta}\phi_n\rangle_b \tag{22}$$

with

$$\omega_e^T = \omega_e^*[1+\eta_e(\frac{v^2}{v_e^2}-\frac{3}{2})]$$
$$\omega_e^* = \frac{nq}{r}\frac{cT_e}{eB_0L_n}$$

(b) For the second term in Eq. (11), it follows that

$$\langle e^{-inq\theta}\vec{v}_{de}\cdot\vec{\nabla}_\perp(\tilde{h}_n^e e^{inq\theta})\rangle_b = i\frac{nq}{r}\frac{1}{R_0|\omega_e|}\langle(v_\parallel^2+\frac{v_\perp^2}{2})(\cos\theta+\hat{s}\theta\sin\theta)\rangle_b \tag{23}$$

After a lengthy but straightforward calculation, the bounce average on the right hand side of the Eq. (23) is found to be

$$\begin{aligned}
&\langle(v_\parallel^2+\frac{v_\perp^2}{2})(\cos\theta+\hat{s}\theta\sin\theta)\rangle_b\\
=&\frac{1}{2}\{\frac{4}{3}\mu\epsilon B_0[1-\kappa^2+(2\kappa^2-1)\frac{E(\kappa)}{K(\kappa)}\\
&+\frac{16}{9}\mu\epsilon B_0\hat{s}[(\kappa^2-1)(3\kappa^2-2)+(4\kappa^2-2)\frac{E(\kappa)}{K(\kappa)}]\\
&+v^2(2\frac{E(\kappa)}{K(\kappa)}-1)+4v^2\hat{s}[(\kappa^2-1)+\frac{E(\kappa)}{K(\kappa)}]\}
\end{aligned} \tag{24}$$

where $K(\kappa)$ and $E(\kappa)$ are the complete elliptical functions of the first and second kind, i.e.

$$K(\kappa) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa t^2)}}$$
$$E(\kappa) = \int_0^1 dt\frac{\sqrt{1-\kappa t^2}}{\sqrt{1-t^2}}$$

In cases of practical interest , however, the complicated, albeit complete expression above is not needed, since stability of the trapped electron mode is determined primarily by deeply trapped electrons. Therefore, a simplified expression can be obtained for $\kappa \ll 1$, i.e.

$$\langle (v_\parallel^2 + \frac{v_\perp^2}{2})(\cos\theta + \hat{s}\theta\sin\theta)\rangle_b \approx \frac{v^2}{2} \tag{25}$$

and

$$\langle e^{-inq\theta}\vec{v}_{de}\cdot\vec{\nabla}_\perp(\tilde{h}_n^e e^{inq\theta})\rangle_b \approx i\omega_R \frac{v^2}{v_e^2}\tilde{h}_n^e \tag{26}$$

with

$$\omega_R = \frac{nq}{r}\frac{cT_e}{eB_0R_0}$$

.

(c) For the nonlinear term, we have

$$\begin{aligned}
&\langle e^{-inq\theta}N_n(h^{e(0)},\phi)\rangle_b \\
&= \frac{c}{B}\sum_{n_1+n_2=n}\{\frac{in_2q}{r}\langle e^{-in_1q\theta}\frac{\partial}{\partial r}\phi_{n_1}\rangle_b\tilde{h}_{n_2}^e - \frac{1}{r}\langle e^{-in_1q\theta}\frac{\partial}{\partial\theta}\phi_{n_1}\rangle_b\frac{\partial}{\partial r}\tilde{h}_{n_2}^e \\
&\quad - \frac{in_2q}{r}\frac{d\ln q}{dr}\langle\theta e^{-in_1q\theta}\frac{\partial}{\partial\theta}\phi_{n_1}\rangle_b\tilde{h}_{n_2}^e\} \\
&= i\frac{c}{B}\sum_{n_1+n_2=n}\{\frac{n_2q}{r}\frac{\partial}{\partial r}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\tilde{h}_{n_2}^e - \frac{n_1q}{r}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\frac{\partial}{\partial r}\tilde{h}_{n_2}^e \\
&\quad + \frac{n_2q}{r}\frac{d\ln q}{dr}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\tilde{h}_{n_2}^e\}
\end{aligned} \tag{27}$$

Since the fluctuation scale length is much shorter than the equilibrium scale length, the last term in the above equation, which is proportional to $\frac{d\ln q}{d\ln r}$, can be neglected. Therefore

$$\begin{aligned}
&\langle e^{-inq\theta}N_n(h^{e(0)},\phi)\rangle_b \\
&= i\frac{c}{B}\sum_{n_1+n_2=n}\{\frac{n_2q}{r}\frac{\partial}{\partial r}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\tilde{h}_{n_2}^e - \frac{n_1q}{r}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\frac{\partial}{\partial r}\tilde{h}_{n_2}^e
\end{aligned} \tag{28}$$

Substituting the results from (a), (b), (c) into Eq. (11), we finally obtain the desired nonlinear bounce averaged drift kinetic equation for trapped electrons:

$$\begin{aligned}
&(\frac{\partial}{\partial t} + i\omega_{de} + \nu_{eff})\tilde{h}_n^e = -\frac{e}{T_e}F_e(\frac{\partial}{\partial t} + i\omega_e^T)\langle e^{-inq\theta}\phi_n\rangle_b \\
&- i\frac{c}{B}\sum_{n_1+n_2=n}\{\frac{n_2q}{r}\frac{\partial}{\partial r}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\tilde{h}_{n_2}^e - \frac{n_1q}{r}\langle e^{-in_1q\theta}\phi_{n_1}\rangle_b\frac{\partial}{\partial r}\tilde{h}_{n_2}^e\}
\end{aligned} \tag{29}$$

This equation describes the nonlinear evolution of the perturbed distribution function of trapped electrons on the slow time scale of mode variation. It shows that this nonlinear evolution is completely determined by the knowledge of a single quantity $\langle e^{-inq\theta}\phi_n\rangle_b$, and that the result is not sensitive to detailed imformation about ϕ_n. This simple form of the nonlinear bounce kinetic equation renders an analytical study of the trapped electron's nonlinear dynamics tractable. Also, it is potentially useful for computer simulation study of the nonlinear evolution of trapped electron mode turbulence. In the derivation, the fact that the bounce motion of the trapped electrons and the mode variation are on different time scales have been utilized. This enables us to separate the slow oscillations of the mode and dynamical interactions from the fast bounce motion, and eliminate the latter by a bounce average.

The linear part of the equation is the same as that used by Catto and Tsang in Ref.(1). In that reference, the authors have shown that the response to the bounce averaged field ($\langle e^{-inq\theta}\phi_n\rangle$) introduces several effects in the linear stability theory of the trapped electron mode. First, it introduces linear couplings among modes with different **m** but same **n**. However, this coupling effect does not qualitatively effect the growth rate. The more subtle effect introduced by the bounce average is the radial localization of trapped electron's responce to a narrow region of width $\frac{1}{k_\theta \hat{s}}$ around mode rational surface. For a slab-like mode, this results in a strong reduction in the linear growth rate by a factor of $\sqrt{\frac{L_n}{L_S}}$. Nonlinearly, one can see from Eq. (29) that this radial localization effect demonstrates that the nonlinear interaction of trapped electrons weak, since the radial localization effect not only reduces the overlap of different modes but also spatially restricts the nonlinear interaction to a very narrow region around mode rational surface. A comprehensive discussion of trapped electron dynamics and its impact on trapped electron mode turbulence will be presented in a future publication.

We note that a related equation has been previously[7] used in their study of the two-point theory of trapped particle mode turbulence. However, the equation obtained in this paper is much more general than the previous equation in that it can be used in any mode representation rather than just in ballooning representation. Thus, previous result is a special case of that presented here. Also, in this work we have shown that the equation is appropriate only for description of the nonlinear evolution of a high **n** mode, i.e. accurate only to an order of $o(\frac{1}{\sqrt{nq}})$. Finally, we want to point it out that as long as finite ion larmor radius effects are not important, Eq. (29) can also be used to describe the nonlinear dynamics of trapped ions as well.

In conclusion, we have derived a simple, tractable *nonlinear bounce averaged drift kinetic equation* for trapped electrons valid in the limit of large mode number $\mathbf{n} \gg \mathbf{1}$. The equation, with its physical content and simplicity, is quiet suitable for the study of the nonlinear evolution of trapped electron mode turbulence analytically or through computer simulation.

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