INSTITUTE FOR FUSION STUDIES

DOE/ET-53088-426 IFSR #426

Resonant MHD Modes with Toroidal Coupling Part I. Tearing Modes

J. W. Connor and *R. J. Hastie*

Culham Laboratory
(UKAEA/Euratom Fusion Association)
Abingdon, Oxfordshire, OX14 3DB, England

J. B. Taylor

Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

March 1990

THE UNIVERSITY OF TEXAS

AUSTIN

Resonant MHD Modes with Toroidal Coupling Part I. Tearing Modes

J.W.Connor and R.J.Hastie
Culham Laboratory
(UKAEA/Euratom Fusion Association)
Abingdon, Oxfordshire, OX14 3DB, England

J.B.Taylor
Institute for Fusion Studies
The University of Texas at Austin
Austin
Texas 78712
U.S.A.

Abstract

In a cylindrical plasma, tearing modes can be calculated by asymptotic matching of ideal mhd solutions across a critical layer. This requires a quantity Δ' which represents the 'discontinuity' in the ideal solution across the layer. In a torus, poloidal harmonics are coupled and there are many critical surfaces for each toroidal mode number, and correspondingly many discontinuities Δ'_m. The ideal mhd solutions do not determine the Δ'_m but only a relation between them–described by an "E-matrix". We discuss the calculation of the E-matrix for a large aspect-ratio tokamak. In a weak-coupling approximation it is tri-diagonal and can be computed from integrals over the uncoupled eigenfunctions or from simple "basis-functions" comprising triplets of coupled poloidal harmonics. This weak coupling approximation fails if Δ'_m is already small for an uncoupled harmonic. An alternative strong-coupling approximation is developed for this case.

Introduction

In the cylindrical limit of a circular cross section tokamak, linear perturbations can be described by independent fourier harmonics with poloidal and toroidal mode numbers m and n. Then if $q(r)$ is the 'safety factor', a singularity of the ideal magneto-hydrodynamic (mhd) equations occurs at $nq(r_0) = m$, where the perturbation is resonant with the field line rotation. There may then be instabilities in which non-ideal effects (such as resistivity) are important only in a critical layer [1] around r_0; elsewhere the perturbation is still described by ideal mhd. These instabilities can be calculated by asymptotic matching of a solution of the full equations in the critical layer to solutions of the marginal ideal mhd equations elsewhere. The mhd solutions satisfy the boundary conditions at the magnetic axis $r = 0$ and at the plasma boundary $r = a$.

The marginal ideal mhd solutions can be represented near $r = r_0$ as

$$\psi_{L,R} = A_{L,R}(\mid x \mid^{\nu-} + \Delta_{L,R} \mid x \mid^{\nu+}) \tag{1}$$

where ψ is the perturbed radial magnetic field, $x = r - r_0$ and L,R refer to left and right of the critical layer. The indices ν relate to the 'large' and 'small' solutions in the sense of Newcomb [2] and

$$\nu_{\pm} = 1/2 \pm (1/4 + D)^{1/2} \tag{2}$$

The Mercier [3] stability criterion is $(D + 1/4) > 0$. At zero pressure $\nu_+ = 1, \nu_- = 0$ and $\Delta_{L,R}$ are then related to the logarithmic derivative of ψ as $\mid x \mid \rightarrow 0$.

The solutions inside a symmetric critical layer have even (tearing) or odd (twisting) parity and asymptotically can be written

$$\psi_{\pm} = \mid X \mid^{\nu-} + \Delta^{\pm}(\omega) \mid X \mid^{\nu+} \tag{3}$$

(where X is a 'stretched' coordinate x/δ with δ the layer width). The quantities $\Delta^{\pm}(\omega)$ depend on the details of the plasma model used in the layer. Matching eq.(3) to eq.(1) yields [4]

$$\Delta^+(\omega)\Delta^-(\omega) - \frac{\delta^{\nu+-\nu-}}{2}(\Delta_R + \Delta_L)(\Delta^+(\omega) + \Delta^-(\omega)) + \delta^{2(\nu+-\nu-)}\Delta_R\Delta_L = 0 \tag{4}$$

If we suppose that $\Delta^+(\omega)$ and $\Delta^-(\omega)$ do not vanish simultaneously (as a function of ω) then there are two distinct solutions to eq.(4) as $\delta \rightarrow 0$. In one of these $\Delta^-(\omega) << \Delta^+(\omega)$ and the eigenvalue is determined by

$$\Delta^-(\omega) = \delta^{\nu+-\nu-}\Delta' \tag{5}$$

with $\Delta' = (\Delta_R + \Delta_L)/2$. This is the 'tearing mode'. In the other solution $\Delta^+(\omega) << \Delta^-(\omega)$ and

$$\Delta^+(\omega) = \delta^{\nu+-\nu-}\Delta'. \tag{6}$$

This is the "twisting mode". Note that Δ' embodies all the information needed from the ideal mhd solution. Henceforth we will refer to Δ' simply as the 'discontinuity' in the mhd solution and if $\Delta' = 0$ we will refer to the solution as "continuous".

In an axisymmetric torus the individual poloidal harmonics m are coupled together, although toroidal harmonics n remain independent. Consequently, for each toroidal mode number n there can be many critical surfaces (wherever $nq(r) =$ integer). Except in the vicinity of these critical surfaces the perturbation is still given by ideal mhd equations. Our objective is to describe a quantity which summarises all the information required from the ideal mhd equations in a torus, in the same way that Δ' summarises all the information necessary in a cylinder.

In the present paper we consider only the toroidal tearing mode. This will be seen to be a natural extension of the cylinder tearing mode. In part II we will examine the toroidal twisting mode. Unlike the tearing mode, this is an **intrinsically** toroidal perturbation, not directly related to the cylinder twisting mode.

The toroidal tearing mode problem was first addressed by Grimm et al [5] who constructed a 2-D computer code to calculate the 'outer region' solutions. In a later paper Connor et al [6] described the mhd solutions by a set of M poloidal harmonics. Then if there are N resonant surfaces among these harmonics, a set of (M + N) basis-functions (solutions of the marginal mhd equation each consisting of M harmonics) was introduced. These consist of M solutions regular at $r = 0$ and continuous (ie no change in large or small solution) across the critical surfaces, and N solutions constructed by starting with (only) the small solution at each critical surface and continued outwards (continuously) to $r = a$.

The full solution is then written as a linear combination of these (M+N) basis functions, with M coefficients α_i and N coefficients α'_i. By construction this satisfies the boundary condition at $r = 0$. The requirement that it also satisfy the boundary condition at $r = a$ yields M conditions through which the M coefficients α_i can be expressed in terms of the N coefficients α'_i. The introduction of N quantities Δ_m describing the discontinuity in the small component (relative to the large component) at each resonant surface then leads to a solubility condition which can be written

$$| E - \Delta | = 0 \tag{7}$$

where the E-matrix is calculated entirely from the values of the basis functions at the critical surfaces and at $r = a$ (ie from ideal mhd) and $\Delta = \text{diagonal}\{\Delta_m\}$. The dispersion equation for the toroidal tearing mode frequency follows by writing $\Delta_m = \Delta_m^-(\omega)$. Clearly the E-matrix is a quantity, corresponding to Δ' in the cylinder, that summarises all the information needed from the ideal mhd solution in order to determine an eigenvalue of the full problem. (Strictly, the matrix E described here is the inverse of that in ref. 6.)

In the rest of this paper we discuss the calculation of the E-matrix for a large aspect-ratio tokamak. We first find, in section 3, that a systematic treatment in powers of ϵ leads to somewhat different basis-functions to those of ref. 6. These consist only of triplets of harmonics irrespective of how many resonant surfaces there may be. Each triplet consists of a central harmonic with a discontinuity in its small solution at its resonant surface plus small sidebands which are continuous at their resonant surfaces. Furthermore, unlike the basis functions of ref. 6, each triplet individually satisfies the boundary conditions at **both** $r = 0$ and $r = a$. With these basis functions the matrix E has a simple tri-diagonal form. This development is formally a weak-coupling theory with coupling parameter ϵ/Δ_m^0 (where Δ_m^0 is the discontinuity in the m^{th} harmonic in the absence of coupling).

"Strong coupling" occurs, even when $\epsilon << 1$, if one or more of the Δ_m^0 are small; this situation is discussed in section 4. There we find that the most convenient description is in terms of overlap integrals between single harmonics rather than in terms of basis functions. This development also leads to a tri-diagonal E-matrix. An alternative description of the strong coupling situation, using basis functions, is also discussed in section 4. In this case the basis functions are multiplets of coupled harmonics (not generally just triplets) and they do not lead to a tri-diagonal E-matrix.

Finally, in section 5 we mention some consequences of the E-matrix for the toroidal tearing mode frequency and structure. A fuller discussion of this will be given elsewhere.

1 Marginal Ideal MHD Equations

The marginal ideal mhd equations were derived in appendix B of ref. 6 and are summarized here. The coordinate system (r, θ, ϕ) (where ϕ is the toroidal angle, θ is an angle-like poloidal coordinate and r is a flux-surface label) is chosen so that the magnetic field lines are straight and the Jacobian is $J = R^2 r/R_0$. The axisymetric field can be written

$$\boldsymbol{B} = B_0 R_0 [f(r)\nabla\phi \times \nabla r + g(r)\nabla\phi] \tag{8}$$

and the safety factor is $q = rg(r)/R_0 f(r)$.

The linearized marginal mhd equations for a perturbation with toroidal mode number n can be expressed in the form

$$\frac{d}{dr}[(m-nq)y_m] = \sum_k [B^k_m z_k + C^k_m y_k] \tag{9}$$

$$(m-nq)\frac{dz_m}{dr} = \sum_k [D^k_m z_k + E^k_m y_k] \tag{10}$$

where $y = R_0 f \boldsymbol{\xi} \cdot \nabla r$ is effectively the radial component of the displacement $\boldsymbol{\xi}$ and $z = R^2 \delta \boldsymbol{B} \cdot \nabla \phi / B_0$ is effectively the perturbed toroidal magnetic field.

2 Weak-coupling Theory

In a large aspect ratio tokamak with circular cross section and $\beta \sim \epsilon^2$, the coefficients B^k_m etc are $\sim \epsilon^{|k-m|}$ and, in particular, the coupling between neighbouring harmonics $m, m \pm 1$, is $\sim \epsilon$. In order to develop a systematic expansion correct to $O(\epsilon^2)$ it is necessary to retain diagonal coefficients to order ϵ^2 and off-diagonal coefficients to order ϵ. The coefficients which remain can then all be expressed in terms of the Shafranov shift, the pressure parameter $\alpha = -(2R_0 q^2/B_0^2) dp/dr$ and the shear parameter $s = r d(\log q)/dr$. Finally we then have as our basic equations correct to $0(\epsilon^2)$:

$$r\frac{d}{dr}[(m-nq)y_m] = L^m_m y_m + \sum_{\pm} \left(L^{m\pm1}_m \hat{z}_{m\pm1} + M^{m\pm1}_m y_{m\pm1} \right) \tag{11}$$

$$r(m-nq)\frac{d\hat{z}_m}{dr} = P^m_m y_m + \sum_{\pm} \left(N^{m\pm1}_m \hat{z}_{m\pm1} + P^{m\pm1}_m y_{m\pm1} \right) \tag{12}$$

where we have also introduced $\hat{z}_m = z_m + (M^m_m / L^m_m) y_m$. The coefficients L, M, N, P are given in appendix B of this paper.

If all coupling between different harmonics is ignored, eqs. (11) and (12) reduce to

$$\mathcal{L}_m y_m \equiv (m-nq)\frac{d}{dr}\frac{1}{L^m_m}\frac{d}{dr}(m-nq)y_m - P^m_m y_m = 0 \tag{13}$$

and if we also retained only the zero order (in ϵ) parts of P^m_m and L^m_m this would be the usual cylinder equation

$$\mathcal{L}^{(0)}_m y_m \equiv (m-mq)[\frac{d}{dr} r \frac{d}{dr}(m-nq)y_m - \frac{m^2}{r}(m-nq)y_m - mq\sigma' y_m] = 0 \tag{14}$$

with

$$\sigma = j_{||}/B = \frac{1}{r}\frac{d}{dr}(\frac{r^2}{q}). \tag{15}$$

Now, eq.(14) has Mercier indices $\nu_+ = 1, \nu_- = 0$ but the correct indices are

$$\nu_\pm = 1/2 \pm (1/4 + D)^{1/2}, \qquad D = \frac{2rp'}{s^2 B_0^2}(1-q^2) \tag{16}$$

Consequently, eq.(14) cannot be used as the zero order starting point for a development in powers of ϵ. To avoid singularities in higher order we must include the most singular part of P_m^m (which controls the indices $\nu_\pm$) in the lowest order equation, even though it is formally small in ϵ. Then the lowest order equation becomes

$$\hat{\mathcal{L}}_m^{(0)} y_m \equiv [\mathcal{L}_m^{(0)} - \frac{2rp'}{B_0^2}(1-q^2)] y_m = 0 \tag{17}$$

which has the correct Mercier indices.

As the starting point for our development in powers of ϵ we take a particular solution $y_m^{(0)}(r)$ of eq.(17) which satisfies the boundary conditions at $r = 0$ and $r = a$ and has a discontinuity Δ_m^0 in its small solution across the singularity at $nq(r) = m$. Then, returning to eqs. (11,12), one can see that $y_m^{(0)}$ induces sidebands $m \pm 1$ which are given by

$$\begin{aligned} \hat{\mathcal{L}}_{m\pm1}^{(0)} y_{m\pm1} &= N_{m\pm1}^m \hat{z}_m^{(0)} + P_{m\pm1}^m y_m^{(0)} \\ &+ (m \pm 1 - nq)\frac{d}{dr}[(L_{m\pm1}^{m\pm1})^{-1}(L_{m\pm1}^m \hat{z}_m^{(0)} + M_{m\pm1}^m y_m^{(0)})] \\ \text{where } \hat{z}_m^{(0)} &= (L_m^m)^{-1}\frac{d}{dr}[(m-nq)y_m^{(0)}] \end{aligned} \tag{18}$$

As the first order contribution we take solutions $y_{m\pm1}^{(1)}$ of eq.(18) which satisfy the boundary conditions at $r = 0$ and $r = a$ and which are **continuous** (in the sense described in the introduction) across the singularities at $nq = m \pm 1$.

We also require the $0(\epsilon^2)$ contribution to y_m. This is given by

$$\begin{aligned} \hat{\mathcal{L}}_m^{(0)} y_m^{(2)} &= [P_m^{m(2)} + P_m^{m(0)}(L_m^{m(0)})^{-1}] y_m^{(0)} + \frac{(m-nq)}{L_m^{m(0)}}\frac{d}{dr}\left(\frac{L_m^{m(2)}}{L_m^{m(0)}}\right)\frac{d}{dr}(m-nq)y_m^{(0)} \\ &+ (m-nq)\frac{d}{dr}[(L_m^{m(0)})^{-1}\sum_\pm (L_m^{m\pm1}\hat{z}_{m\pm1}^{(1)} + M_m^{m\pm1} y_{m\pm1}^{(1)})] \\ &+ \sum_\pm (N_m^{m\pm1}\hat{z}_{m\pm1}^{(1)} + P_m^{m\pm1} y_{m\pm1}^{(1)}) \end{aligned} \tag{19}$$

and we again take the continuous solution satisfying both boundary conditions.

Thus we have constructed a particular solution to the mhd equations consisting of a triplet of harmonics $(y_{m-1}^{(1)}, (y_m^{(0)} + y_m^{(2)}), y_{m+1}^{(1)})$ all of which satisfy both boundary conditions. The central harmonic has a discontinuity in its small component at its singular surface and is correct to order ϵ^2. The sidebands are continuous at their singular surfaces and correct to order ϵ. (It is important that this triplet satisfies both boundary conditions. If, like the basis functions of ref. 6, it satisfied only that at $r = 0$ then instead of $y_{m\pm1}^{(1)}$ being $O(\epsilon)$ it would be $O(m\epsilon)$ and the theory could not accommodate large m).

The triplet described above may be computed by integrating the coupled equations for three adjacent coupled harmonics (ignoring coupling to all other harmonics) from $r = 0$ to $r = a$. A discontinuity Δ^0_m in the central harmonic at its critical surface, and the three initial boundary conditions at $r = 0$, are iterated until all three harmonics satisfy the boundary conditions at $r = a$. A computer code for constructing triplets in this way has been written and will be reported on elsewhere.

Note that a triplet does **not** itself describe a general tearing mode; this requires a solution with a discontinuity Δ_m in every harmonic. Nevertheless a set of these triplets forms a basis for constructing a general tearing mode.

3 The E-matrix

A general tearing mode can be constructed from a superposition of the triplets defined in the preceding section. As we will need to identify both harmonic and triplet we now write the triplet as

$$Y_m = \left\{ y^{(1)}_{m,m-1}, y^{(0)}_{m,m} + y^{(2)}_{mm}, y^{(1)}_{m,m+1} \right\}, \tag{20}$$

(the first subscript identifies the triplet, the second the harmonic). In the superposition

$$Y = \sum_m \alpha_m Y_m \tag{21}$$

three triplets contribute to any harmonic m and the discontinuity Δ_m in its small component at the singular surface ($nq(r_m) = m$) is given by

$$\Delta_m \left[\alpha_{m-1} C^{(1)}_{m-1,m} + \alpha_m \left(C^{(0)}_{m,m} + C^{(2)}_{mm} \right) + \alpha_{m+1} C^{(1)}_{m+1,m} \right] = \Delta^0_m \alpha_m C^{(0)}_{m,m} \tag{22}$$

where the $C_{j,m}$ are the coefficients of the large component of the corresponding $y_{j,m}$ at the resonance r_m. This recurrence relation leads to a solubility condition

$$\mid F - \Delta^{-1} \mid = 0 \tag{23}$$

where $\Delta = \text{diagonal}\{\Delta_m\}$ and F is tridiagonal. Bearing in mind that F is calculated only as an expansion in ϵ, eq.(23) can equally be written

$$\mid E - \Delta \mid = 0 \tag{24}$$

where E is also tri-diagonal. The elements of E are given by

$$E_{mm} = \Delta^0_m \left(\frac{C^{(0)}_{m,m} - C^{(2)}_{m,m}}{C^{(0)}_{m,m}} \right) \tag{25}$$

$$E_{m,m\pm1} = -\frac{\Delta_m^0 C_{m\pm1,m}^{(1)}}{C_{m,m}^{(0)}} \tag{26}$$

These elements can be expressed entirely in terms of the zero order function $y_m^{(0)}$. If eq.(18) is multiplied by $y_{m\pm1}^{(0)}$ and the left hand side integrated by parts, one obtains an expression for $C_{m\pm1,m}^{(1)}$

$$\frac{C_{m\pm1,m}^{(1)}}{C_{m,m}^{(0)}} = \frac{I}{\Delta_{m\pm1}^0} \tag{27}$$

where I is a bi-linear integral of $y_{m\pm1}^{(0)}$ and $y_m^{(0)}$ (of order ϵ) defined in appendix B. A similar prodecure applied to eq.(19) allows one to express $C_{mm}^{(2)}$ in terms of integrals of $y_m^{(0)}$ and $y_{m\pm1}^{(0)}$. Another derivation of the E-matrix is given in appendix A.

4 Strong-coupling Theory

The expressions for $C^{(1)}$ and $C^{(2)}$ in the preceding section highlight the fact that the coupling parameter is really ϵ/Δ_m^0. Consequently, even when $\epsilon << 1$, the coupling becomes large if Δ_m^0 is small. A different treatment is then required.

We write the underlying equations in the form

$$\mathcal{L}_m \psi_m = \sum_{\pm} K_m^{m\pm1} \psi_{m\pm1} \tag{28}$$

where $\mathcal{L}$ and K are defined through eqs.(11) and (12) and $\psi_m = (m - nq)y_m$ is the m^{th} harmonic of the perturbed radial magnetic field. We again start with the uncoupled function ψ_m^0, satisfying $\mathcal{L}_m \psi_m = 0$ and the boundary conditions with a discontinuity Δ_m^0 in its small component at its singularity. We then introduce a set of **radial** harmonics of the uncoupled equation, such that

$$(\mathcal{L}_m^{(0)} + \lambda_m^p)\psi_m^p = 0 \tag{29}$$

and ψ_m^p has the **same** discontinuity Δ_m^0 and satisfies the same boundary conditions as ψ_m^0. Assuming that the ψ_m^p (which are orthogonal in p) form a complete set we can expand ψ_m in them and obtain

$$\psi_m = \sum_{p,\pm} \frac{\psi_m^p < \psi_m^p \mid K_m^{m\pm1} \mid \psi_{m+1} >}{(\bar{\lambda}_m^p + \bar{\Delta}_m)(C_m^p)^2} \tag{30}$$

where $\bar{\Delta}_m = \Delta_m - \Delta_m^0$, C_m^p is a coefficient of the large component of ψ_m^p at its singularity and

$$\bar{\lambda}_m^p = \frac{\lambda_m^p < (\psi_m^p)^2 >}{(C_m^p)^2} \tag{31}$$

Now $|K_m^{m\pm1}|$ is a small quantity, of order ϵ, while the $\bar{\lambda}_m^p (p \neq 0)$ are of order unity. Consequently for each m we need only retain the $p = 0$ term in eq.(30) (unless $\bar{\Delta}_m$ were also of order unity, but then ψ_m is itself negligible). Then we have, in the lowest approximation,

$$\psi_m = \alpha_m \psi_m^0. \tag{32}$$

Inserting this in eq.(30) again yields a three term recurrence relation

$$\alpha_m(\Delta_m - \Delta_m^0) = \sum_{\pm} I_{m,m\pm1}\alpha_{m\pm1} \tag{33}$$

and a solubility condition

$$|E - \Delta| = 0 \tag{34}$$

with

$$E_{m,m\pm1} = I_{m,m\pm1} = < \psi_m^0 \mid K_m^{m\pm1} \mid \psi_{m\pm1}^0 > . \tag{35}$$

$$E_{m,m} = \Delta_m^0$$

Thus, the E-matrix for strong coupling is again tri-diagonal and similar to that in weak-coupling theory. Indeed, as shown in appendix A, the off diagonal elements of the strong and weak coupling forms of E-matrix are identical to $0(\epsilon)$. Nevertheless the two forms are formally based on quite different approximations. The weak-coupling result depends on (ϵ/Δ_m^0) being small and includes order $(\epsilon/\Delta_m^0)^2$ contributions, whereas the strong coupling result depends on (ϵ/λ_m^p) being small and includes only lowest order contributions. One expects the strong-coupling form to be more generally applicable.

Although we have constructed the strong coupling E-matrix from overlap integrals, it can also be computed from basis-functions $\psi^j(r)$. Thus one sets-up a set of basis-functions $\psi^j(r)$, each of which is a multiplet containing N coupled harmonics (where N is the number of critical surfaces) launched from $r = 0$. In each multiplet one harmonic (only) has a discontinuity Δ_j' at its critical surface r_j, the others are continuous. For each multiplet basis-function the discontinuity, and the boundary conditions at $r = 0$, are iterated until all harmonics satisfy the boundary condition at $r = a$. When a tearing mode is constructed from these multiplet basis functions with coefficients α_j, the discontinuity Δ_m' at the m^{th} singularity is given by

$$\Delta_m \sum_{j=1}^{N} \alpha_j \psi_m^j(r_m) = \Delta_m' \alpha_m \psi_m^m(r_m) \tag{36}$$

with a solubility condition

$$| F - \Delta^{-1} | = 0 \tag{37}$$

In this case F need not be tridiagonal, since each multiplet basis-function may contain several "strong" harmonics if several Δ_m^0 are small.

Returning to eq.(34), it is clear that a self consistent solution exists when $(\Delta_m - \Delta_m^0)$ is order ϵ for all relevant m. However a self consistent solution also exists if $(\Delta_m - \Delta_m^0)$ is order ϵ^2 for a single m and 0(1) for the remainder. This special case can be computed using just three basis-functions, each of which contains the same triplet of coupled harmonics, centered on the harmonic with near vanishing $\bar{\Delta}_m$. In each triplet one (only) harmonic has a discontinuity at its critical surface. This discontinuity and the boundary conditions at r=0 are iterated until the triplet satisfies the boundary conditions at r=a.

5 Summary and Conclusions

A. The E-matrix

In a cylinder, tearing modes can be calculated through an asymptotic matching procedure in which the full plasma physics is needed only in a critical layer where $nq(r) = m$. Elsewhere the plasma is described by marginal ideal mhd. At the critical surface the ideal mhd solution allows a discontinuity Δ' in its small component which is to be equated to the corresponding $\Delta^-(\omega)$ from the solution to the equations with full physics in the layer. This gives the tearing mode dispersion equation $\Delta^-(\omega) = \Delta'$. Thus Δ' contains all the information needed from the ideal mhd 'external' part of the problem.

In a torus the various poloidal harmonics m are coupled, so that for each toroidal number n there are many critical surfaces $nq =$ integer and correspondingly many discontinuities Δ_m in the ideal mhd solutions. Each of the Δ_m must be matched to its corresponding $\Delta_m(\omega)$. However, the mhd equations do not specify any particular set of Δ_m, **only a relation between them.** This relation is given by the E-matrix discussed in this paper. Just as the single quantity Δ' contains all the ideal mhd information needed for the dispersion equation in a cylinder, so the E-matrix contains all the information needed in a torus.

We have discussed the structure and calculation of the E-matrix in two large aspect ratio limits. In the weak-coupling theory $\epsilon << \Delta_m^0$ [Δ_m^0 is the discontinuity for an uncoupled harmonic] and the coupling between adjacent harmonics is $0(\epsilon/\Delta_m^0)$. The E-matrix is tridiagonal and can be calculated by perturbation or by computing a set of triplet basis-functions. In each triplet the central harmonic has a discontinuity at its critical surface but the side bands are continuous. Note that only triplets of coupled harmonics need be computed irrespective of the number of critical surfaces. The number of triplets required

equals the number of critical surfaces.

However, even at large aspect ratio ($\epsilon << 1$) the weak-coupling approximation breaks down when Δ_m^0 is small. An alternative strong-coupling approximation requires that $\epsilon \ll \lambda_m^p$, where λ_m^p is an eigenvalue associated with a higher **radial** mode of the uncoupled m^{th} poloidal harmonic–and is of order unity. This approximation again leads to a tri-diagonal form of E-matrix when this is computed from overlap integrals of uncoupled harmonics. It may also be computed from multiplet basis functions but it is then not necessarily in tri-diagonal form. A special case of the strong-coupling approximation involves only three harmonics and can be computed from triplet basis-functions as in the weak-coupling approximation.

B The Dispersion Equation

An important feature of the asymptotic approach to tearing modes is that the E-matrix calculated from ideal mhd alone, can be used in conjunction with many different models for the critical layers. Each such applications must be discussed individually, but some general observations may be made.

The dispersion equation is obtained by substituting $\Delta_m^-(\omega)$ from the layer model for the Δ_m in the E-matrix.

$$D(\omega) = \mid E - \Delta_m^-(\omega) \mid = 0. \tag{38}$$

Now, in both weak and strong coupling approximations, the off-diagonal elements of the E-matrix are proportional to a small coupling parameter. Consequently the toroidal dispersion equation (38) will have solutions only when at least one of the diagonal elements is small. There seem to be two generic cases: either $(\Delta_m - \Delta_m^0)$ may be $0(\epsilon)$ for several harmonics, or it may be $0(\epsilon^2)$ for one harmonic and $0(1)$ for the remainder. In either event, the frequency of a toroidal tearing mode will be close to that of an uncoupled harmonic. Thus the toroidal tearing mode is a natural extension of the cylinder tearing mode. [However this is not the case for the twisting mode which we discuss in part II. The toroidal twisting modes are **intrinsically** toroidal and are not an extension of the cylinder modes – to which they are essentially unrelated.]

As far as mode structure is concerned, if $(\Delta_m^0 - \Delta_m^-(\omega))$ is small for only one harmonic then the toroidal eigenmode will comprise only that principal harmonic and weak side-bands. If $(\Delta_m^0 - \Delta_m^-(\omega))$ is simultaneously small for several harmonics, then the toroidal eigenmodes will consist of strong admixture of these harmonics as well as many weaker sidebands.

Two important models for plasma within the critical layer are the low-β resistive mhd and the Rutherford[7] non-linear resistive mhd models. For these models, $\Delta_m^-(\omega)$ is proportional to some power of ω and vanishes at marginal stability for all harmonics simultaneously. In

the cylindrical case marginal stability is then described by $\Delta' = 0$ and stability by $\Delta' > 0$. In the same way marginal stability of toroidal tearing modes, for these models, is given by $|E| = 0$ and stability by $\lambda_E < 0$ where λ_E is the greatest eigenvalue of the E-matrix.

Another important layer model is that of a kinetic plasma with diamagnetic drifts.[8] Then $\Delta_m^-(\omega)$ will be **large** (in the reciprocal layer width δ) unless ω is close to the local diamagnetic frequency ω_*. In this case the side bands will be small both in the toroidicity ϵ and the layer width δ.

Appendix A. Formal Theory.

The weak–and strong–coupling approximations can be considered as formal perturbation schemes. The basic coupled mhd equations can be written schematically as

$$\left(\mathcal{L}_m + \hat{\Delta}_m\right)\psi_m = \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1} \tag{A.1}$$

where $\hat{\Delta}_m$ represents both the magnitude of the discontinuity at the singular surface and an appropriate operator [for a pressureless plasma this operator is simply $\delta(r - r_m)$]. Then the triplet introduced in section 2 satisfies

$$(\mathcal{L}_m + \Delta_m^T)\psi_m^T = \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1}^T \tag{A.2a}$$

$$\mathcal{L}_{m\pm1}\psi_{m\pm1}^T = K_{m\pm1}^m\psi_m^T \tag{A.2b}$$

[Note carefully the difference between the eqs.(A.2) for a triplet, which refer only to a **single** m, and the eqs.(A.1), which refer to all m.].

If we formally solve eq.(A.2b) for $\psi_{m\pm1}^T$ we can obtain an **uncoupled** equation for the central harmonic of the m^{th} triplet, ie.,

$$\left\{\mathcal{L}_m + \Delta_m^T - \sum_{\pm}\left(K_m^{m\pm1}\frac{1}{\mathcal{L}_{m\pm1}}K_{m\pm1}^m\right)\right\}\psi_m^T = 0 \tag{A.3}$$

It is convenient to denote this as

$$H_m\psi_m^T = \left(\mathcal{L}_m + \Delta_m^T - G_m\right)\psi_m^T = 0 \tag{A.4}$$

then we can write the full set of eqs.(A.1) (for all m) as

$$H_m\psi_m = (\Delta_m^T - \Delta_m - G_m)\psi_m + \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1} \tag{A.5}$$

The right hand side of eq.(A.5) is a perturbation of $0(\epsilon^2)$ (the off-diagonal term $\sim \epsilon$ is equivalent to a diagonal term $\sim \epsilon^2$). Thus we can write

$$\psi_m = \alpha_m \psi_m^T + \tilde{\psi}_m \tag{A.6}$$

where

$$H_m \tilde{\psi}_m = (\Delta_m^T - \Delta_m - G_m)\alpha_m \psi_m^T + \sum_{\pm} K_m^{m\pm1} \alpha_{m\pm1} \psi_{m\pm1}^T \tag{A.7}$$

The left side of this equation can be annihilated in the usual way. Then

$$\left(\Delta_m^T - \Delta_m - < \psi_m^T | G_m | \psi_m^T >\right) \alpha_m + \sum_{\pm} < \psi_m^T | K_m^{m\pm1} | \psi_{m\pm1}^T > \alpha_{m\pm1} = 0 \tag{A.8}$$

and the solubility condition is

$$| E - \Delta | = 0 \tag{A.9}$$

with

$$E_{mm} = \Delta_m^T - < \psi_m^T \mid \sum K_m^{m\pm1} \frac{1}{\mathcal{L}_{m\pm1}} K_{m\pm1}^m \mid \psi_m^T >$$

$$E_{m,m\pm1} = < \psi_m^T \mid K_m^{m\pm1} \mid \psi_{m\pm1}^T > \tag{A.10}$$

which is equivalent to the weak coupling results derived in section 3.

The strong coupling approximation can be considered in a similar way. The basic equations are again represented by eq.(A.1). Now, however, we assume that several poloidal harmonics have Δ_m small in the cylinder limit and a zero order solution is therefore constructed from a linear combination of these M 'degenerate' harmonics. Thus ψ_m is written

$$\psi_m = \alpha_m \psi_m^0 + \tilde{\psi}_m \tag{A.11}$$

The coefficients α_m are undetermined in lowest order. When eq.(A.11) is inserted in eq.(A.1) the solubility condition for $\tilde{\psi}_m$ in first order yields

$$(\Delta_m^0 - \Delta_m)\alpha_m + \sum_{\pm} < \psi_m^0 | K_m^{m\pm1} | \psi_{m\pm1}^0 > \alpha_{m\pm1} = 0 \tag{A.12}$$

with solubility condition

$$|E - \Delta_m| = 0 \tag{A.13}$$

where $E_{m,m} = \Delta_m^0$ and $E_{m,m\pm1} = < \psi_m^0 | K_m^{m\pm1} | \psi_{m\pm1}^0 >$, which is equivalent to the results of section 4.

Note that in this strong-coupling approximation, the elements of E are described in terms of ψ_m^0 and the off-diagonal elements are $0(\epsilon)$ while the diagonal elements are $0(1)$. In the weak-coupling approximation the off-diagonal elements are also $0(\epsilon)$, and to this order are identical with those of strong-coupling, but the diagonal elements contain contributions of both $0(1)$ and $0(\epsilon^2/\Delta_{m\pm1})$.

Appendix B. Some definitions

The coefficients L, M, N, P of eqs.(11) and (12) are given by the following expressions.

$$L_m^m = m^2[1 + \frac{3}{2}\Delta_s'^2 - \frac{\Delta_s}{R} + \frac{r^2}{R^2}(\frac{n^2}{m^2} - \frac{3}{4})] \tag{B.1}$$

$$L_m^{m\pm1} = m(m \pm 1)\Delta_s' \tag{B.2}$$

$$M_m^m = N_m^m \equiv 0 \tag{B.3}$$

$$M_m^{m\pm1} = \pm m\{\frac{\alpha}{2}(m - nq) + (m \pm 1 - nq)[\frac{r}{R} - \Delta_s'(1 - s)]\} \tag{B.4}$$

$$N_m^{m\pm1} = \pm(m \pm 1)\left\{\frac{\alpha}{2}(m \pm 1 - nq) + (m - nq)[\frac{r}{R} - \Delta_s'(1 - s)]\right\} \tag{B.5}$$

$$\begin{aligned} P_m^m \;=\; & (m - nq)^2 + (r\sigma')q\frac{(m - nq)}{m} + \frac{2rp'}{B_0^2}(1 - q^2) \\ & -\frac{2rp'}{B_0^2}\frac{(m - nq)}{m}(2 - s) \\ & -q\frac{(m - nq)}{m}r\frac{d}{dr}\left\{\frac{r^2}{R^2q^3}(2 - s) + \frac{s}{q}(\frac{3}{4}\frac{r^2}{R^2} - \frac{3}{2}\Delta_s'^2 + \frac{\Delta_s}{R})\right. \\ & \left.-\frac{1}{q}(3\frac{r^2}{R^2} + 2\frac{r}{R}\Delta_s' - 2\Delta_s'^2 + 2\frac{\Delta_s}{R})\right\} \\ & +\frac{(m - nq)^2}{m^2}\left\{\frac{n}{m}r\frac{d}{dr}[r\frac{d}{dr}(\frac{r^2}{R^2q})] - [\frac{d}{dr}(\frac{r^2}{R^2q})]^2 - r\frac{d}{dr}(\frac{rp'}{B_0^2})\right. \\ & \left.+m^2[\frac{3}{2}\Delta_s'^2 + \frac{7}{4}\frac{r^2}{R^2} + 3\frac{r}{R}\Delta_s']\right\} \end{aligned} \tag{B.6}$$

$$P_m^{m\pm1} = \frac{\alpha}{2}(1 + s) + (m - nq)(m \pm 1 - nq)(\Delta_s' + \frac{r}{R}) \tag{B.7}$$

where $\alpha = -2Rp'q^2/B_0^2, s = rq'/q$ and Δ_s is the Shafranov shift of the magnetic axis.

The coupling integral I which appears in eq.(27) is a generalisation to finite pressure equilibria of the coupling integrals discussed by Edery et al[9]. It is defined by

$$\begin{aligned} I \;=\; & \mathcal{P}\int_0^a dr\left\{m(m+1)\frac{\psi_m\psi_{m+1}}{r}\left[\Delta_s' + \frac{r}{R} + \frac{\alpha(1+s)}{2(m-nq)(m+1-nq)}\right]\right. \\ & -r\Delta_s'\frac{d\psi_m}{dr}\frac{d\psi_{m+1}}{dr} + m\psi_m\frac{d\psi_{m+1}}{dr}\left[\Delta_s'(s-1) + \frac{r}{R} + \frac{\alpha}{2}\frac{(m+1-nq)}{(m-nq)}\right] \\ & \left.-(m+1)\psi_{m+1}\frac{d\psi_m}{dr}\left[\Delta_s'(s-1) + \frac{r}{R} + \frac{\alpha}{2}\frac{(m-nq)}{(m+1-nq)}\right]\right\} \end{aligned} \tag{B.8}$$

where the integration is to be interpreted as a principal part integration at each singular surface in the sense that

$$\mathcal{P}\int_0^a X dr \equiv \int_0^{r_m-\delta} X dr + \int_{r_m+\delta}^{r_{m+1}-\delta'} X dr + \int_{r_{m+1}+\delta'}^a X dr \tag{B.9}$$

in the limit $\delta, \delta' \to 0$.

Acknowledgement

Part of this work was supported by U.S. Department of Energy Contract No. DEFG05-80ET-53088.

References

[1] H.P. Furth, J. Killeen and M.N. Rosenbluth, *Phys. Fluids* **6** 459 (1963)

[2] W.A. Newcomb, *Ann. Phys.* NY **10** 232 (1960)

[3] C. Mercier *Nucl. Fusion* **1** 47 (1960)

[4] A.H. Glasser, S.C. Jardin and G. Tesauro, *Phys. Fluids* **27** 1225 (1984)

[5] R.C. Grimm, R.L. Dewar, J. Manickam, S.C. Jardin, A.H. Glasser, and M.S. Chance, in *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 9th Int. Conf., Baltimore 1982) Vol. 3 (IAEA Vienna, 1983) p35

[6] J.W. Connor, S.C. Cowley, R.J. Hastie, T.C. Hender, A. Hood and T.J. Martin, *Phys. Fluids* **31** 577 (1988)

[7] P.H. Rutherford, *Phys. Fluids* **16** 1903 (1973)

[8] S.C. Cowley, R.M. Kulsrud and T.S. Hahm, *Phys Fluids* **29** 3230 (1986)

[9] D. Edery, J.L. Soulé, R. Pellat, M. Frey and J.P. Somon, in *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 8th Int. Conf., Brussels 1980) Vol. 1 (IAEA, Vienna 1981) p269