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Nonlinear Interaction of Photons and Phonons in Electron-Positron Plasmas

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Abstract

Nonlinear interaction of electromagnetic waves and acoustic modes in an electron-positron plasma is investigated. The plasma of electrons and positrons is quite plastic so that the imposition of electromagnetic (EM) waves causes depression of the plasma and other structural imprints on it through either the nonresonant or resonant interaction. Our theory shows that the nonresonant interaction can lead to the coalescence of photons and collapse of plasma cavity in higher (≥ 2) dimensions. The resonant interaction, in which the group velocity of EM waves is equal to the phase velocity of acoustic waves, is analyzed and a set of basic equations of the system is derived via the reductive perturbation theory. We find new solutions of solitary types: bright solitons, kink solitons, and dark solitons as the solutions to these equations. An implication of the present theory on astrophysical plasma settings is suggested, including the cosmological relativistically hot electron-positron plasma.

I. Introduction

When a plasma becomes so hot that it becomes relativistic, the temperature T of the plasma exceeds the rest mass energy of electrons $mc^2 = 0.5\,\mathrm{MeV}$. In this relativistic regime the processes of electron-positron pair creation and annihilation become important: $2\gamma \leftrightarrows e^+ + e^-$. In relativistic temperatures the electron (and positron) energy ε_e far exceeds the rest mass energy so that electrons and positrons behave kinematically similar to photons and come into equilibrium with nearly equal population. In this case the population of electrons far exceeds that of protons. Such highly relativistic plasmas may be found in the early Universe,[1] in active galactic nuclei (AGN),[2] and in pulsar atmospheres.[3] The pulsar plasma is most likely magnetized, while that of the early Universe may be unmagnetized. The plasma in AGN may or may not be magnetized. The other environment in which relativistic electron-positron plasmas appear is the $e^+ + e^-$ collider. However, it is very transient. Good laboratory examples are found also in semiconductor plasmas of holes and electrons, in which the plasmas are likely to be nonrelativistic. In the present paper we discuss only physical processes of unmagnetized plasmas. Furthermore, our emphasis is on nonlinear processes. Some of the linear processes of such plasmas in a general relativistic formulation may be found in Ref. 4, while some of the magnetized plasmas are in Ref. 5.

The plasma of the early Universe may be relativistic ($10^{-2} < t < 1\,\mathrm{sec}$) or mildly relativistic or nonrelativistic ($1 < t < 10^{13}\,\mathrm{sec}$). Let us consider the epoch of the Universe of approximately $10^{-2} < t < 1\,\mathrm{sec}$ after the big bang (although a slightly wider window of $10^{-3} < t < 10^{1}\,\mathrm{sec}$ may be permissible). In this epoch the temperature of the Universe is low enough that nuclear matter has become familiar hadrons made up from quarks and gluons but high enough that electrons, positrons, and photons are in abundance and in (near) thermal equilibrium. Neutrinos are abundant and may be strongly coupled with other leptons. In the present work, however, we neglect this coupling. In this case the plasma density is from 10^{33} to $10^{28}\,\mathrm{cm}^{-3}$. Before the electron-proton recombination ($t \sim 10^{13}$ sec), the cosmic expansion is such that the cosmic metric a scales as $a \propto t^{1/2}$ and thus the mass density $\rho_m \propto t^{-3/2}$ and the radiation density $\rho_r \propto T^4 \propto t^{-2}$, where $T \propto \hbar\omega \propto a^{-1} \propto t^{-1/2}$. The period of $10^{-2} < t < 10^{13}$ sec is sometimes called the radiation epoch. (More traditionally the radiation epoch is $10^0 < t < 10^{13}$ sec). We may call this epoch the plasma epoch, as the plasma is the main constituent matter form, including photons. During the early part of this epoch $10^{-2} < t < 10^0$ sec the plasma and radiation temperature T is (much) larger than mc^2. Similarly the primordial magnetic field (if any) scales as $B \propto a^{-2}$ so that the plasma beta $\beta = 4\pi nT/B^2$ is invariant (if the dynamo effect is not operative). It is important to note, however, that it was found that even the thermal equilibrium nonmagnetized plasma can sustain low frequency magnetic fluctuations.[6] More details of plasma parameters of this epoch are discussed in Sec. V.

In the present paper we are concerned with collective processes in nonlinear interaction of a relativistic electron positron plasma with photons. For the purpose of illuminating the unique properties of electron-positron plasmas such as plasticity, we emphasize the dominant population of electrons and positrons and neglect ion effects. If ion effects are restored, they would exhibit more traditional phenomena. These less idealized cases are left for a future publication. High frequency photons in such a hot plasma contribute to the equilibrating pressure (P) force where $P \propto T^4$, while there remain low frequency electromagnetic waves.[6] It is these low frequency electromagnetic waves and their nonlinear interaction with the plasma that we are interested in and we investigate in the following. The reason for this is two-fold: (i) its intrinsic nonlinear interaction is rich and reserves a full treatment; (ii) its implication on cosmic evolution is potentially immense. The first point is well recognized by many previous authors.[7,8,9] The second point is not well appreciated yet. The black body radiation from the big bang is observed as the $3°K$ microwave background radiation. Its observed anisotropy is very small and less than 10^{-4}.[1] Although this puts a severe constraint on theory of galaxy formation, it should be noted that the observed highly isotropic distribution of black body radiation is connected with high frequency (i.e., $\hbar\omega \sim T$) photons. No signature of low frequency ($\hbar\omega \ll T$) photons is known and thus constitutes no constraint if their imprints on matter are nonadiabatic. In this regard the nonlinear interaction between the plasma and low frequency photons is important. Perhaps as a result of this, there may emerge a structure in the plasma, which ultimately gives rise to a seed of galaxy formation. Yet we note that such signatures in black body radiation anisotropy are nondetectable, as long as they are of an isothermal nature. We believe that this assertion is critical to cosmology. A full impact of our theory on cosmology cannot be expounded in the present paper.

We thus simplify the problem into that of interaction of the electron positron plasma with low frequency photons with high frequency photons being treated through the pressure term of the ideal photon gas. Once we cast the problem this way, our task becomes a well-defined physics problem of its own. Our present work may be regarded as a physical treatment of such an abstracted plasma and results can stand on their own. In fact our treatments are so simplified that their direct applications to cosmic plasmas and other situations have to be done with considerable caution. For example, we will not discuss collisional effects (and thus kinetic theory) of the plasma at all in the following until Sec. V. Clearly these among others are very important and should be considered in detail in the future. It is, nonetheless, the case that our present model seems to capture most of the essential features of cosmic plasmas.

Our paper is structured as follows: Sec. II discusses the basic model equations of the posed problem in the long wavelength limit, in which the characteristic length of plasma modulation is much longer than that of low frequency electromagnetic radiation. The interaction is nonresonant. It discusses photon packet collapse and structure formation in a plasma. In Sec. III we turn to a resonant interaction between the low frequency photon and the plasma phonon, in which both wavelengths are comparable. A similar structure formation is expected, although the theoretical treatment has to be accordingly modified. Section IV discusses more detail of the result of Sec. III. In Sec. IV we discuss solutions to the equations obtained in Sec. III, including its characteristic steady-state properties, such as shock and soliton formation, and numerical investigation of these. We conclude in Sec. V with discussions of more detailed parameters of the cosmological electron-positron plasma. Kinetic effects are briefly touched upon there.

II. Nonlinear Schrödinger Equation for Relativistic $e^- - e^+$ Plasmas

We model the relativistic electron-positron plasma immersed in a photon gas by a fluid description. Electrons and positrons are described by the two-fluid theory and the long wavelength photon effects are described by the Maxwell equations. Short wavelength photons ($\hbar\omega \sim T$) are treated to contribute only to the pressure term. There are two distinct regimes, one is the frequency of electron-positron collisions ν_c (and the frequency of electron-positron pair creation) is much larger than the plasma frequency, ω_p, and the other is the reverse. In the former case the local thermodynamic equilibrium should be quickly established. In the latter case, on the other hand, collective finite frequency modes survive and interesting nonlinear coupling among electrons, positrons, and photons can take place. In the present article we concentrate on the latter case, as the former is more trivial.

The continuity equations for electrons and positrons take a form

$$\frac{\partial n_\pm}{\partial t} + \nabla \cdot (n_\pm v_\pm) = \nu_c n_\gamma - \nu n_- n_+ \ , \tag{1}$$

where $n_\pm$ are the positron (+) and electron (−) density, $v_\pm$ velocities, n_γ the photon density, ν_c the pair creation frequency, and ν the annihilation rate. Herewith subscripts $\pm$ refer to quantities of positrons and electrons. In a similar fashion the equations of motion for electrons and positrons may be derived. Here, however, we treat the annihilation and creation of particles phenomenologically, as we eventually neglect their effects assuming $\omega_p \gg \nu_c$. Thus we obtain

$$m\,\frac{d\mathbf{v}_\pm}{dt} = \pm e\left(\mathbf{E} + \frac{\mathbf{v}_\pm \times \mathbf{B}}{c}\right) - \frac{1}{n_\pm}\,\boldsymbol{\nabla} P_\pm - m\nu\mathbf{v}_\pm\ , \tag{2}$$

where $P_\pm$ are the positron and electron pressure and the last term on the right-hand side of Eq. (2) is a phenomenological expression.

We can show that the longitudinal electric field and the associated plasma oscillations are completely decoupled from transverse electromagnetic waves in a positron-electron plasma. We can further show that the acoustic oscillations do not accompany charge separation in a positron-electron plasma in contrast to ion-acoustic oscillations in an ordinary plasma. This contributes to the plasticity of the electron-positron plasma; i.e., the relative ease of structure formation in this plasma. Thus for our present purpose of the study of the coupling among electrons, positrons, and photons, Poisson's equation is unnecessary to solve: $\nabla \cdot \mathbf{E}_L = 4\pi e(n_+ - n_-) = 0$ and the longitudinal electric field $\mathbf{E}_L = 0$. The Maxwell equation is then

$$-\nabla^2\mathbf{E} + \frac{1}{c}\,\frac{\partial^2}{\partial t^2}\,\mathbf{E} = -\frac{4\pi}{c^2}\,\frac{\partial \mathbf{J}}{\partial t}\ , \tag{3}$$

where

$$\mathbf{J} = n_+ e\,\mathbf{v}_+ - n_- e\,\mathbf{v}_-\ . \tag{4}$$

The closure of the second order moment equation may be accomplished by assuming that the pressure of the plasma is equal to the photon's, which behaves according to the Rayleigh-Jeans law black-body radiation. Then the pressure is given by

$$P_\pm = n_\pm\,T_\pm = \sigma\,T_\pm^4\ , \tag{5}$$

where Eq. (5) is the Stefan-Boltzmann law with $\sigma \equiv \pi^2/45\hbar^3\,c^3$. Equation (5) yields

$$P_\pm = \frac{1}{\sigma^{1/3}}\,n_\pm^{4/3}\ , \tag{6}$$

i.e., the plasma is a polytropic gas with the adiabatic constant 4/3.

In the following we consider a situation in which electromagnetic waves propagating only in one direction (x). We assume that the amplitude of the electromagnetic waves is large enough that in the first order equation the electromagnetic force dominates the pressure force, but that it is small enough that the wave is nonrelativistic, $eE/m\omega c < 1$, where ω is the frequency of the radiation and E the electric field of the radiation. This allows us to expand the equation of motion in the following way in the nonrelativistic kinematics. The first order equation in the power of E of the electromagnetic waves is

$$\frac{\partial \mathbf{v}_\pm}{\partial t} = \pm \frac{e\mathbf{E}}{m} , \tag{7}$$

where $\mathbf{E}$ may be assumed to point to the y-direction if the wave is linearly polarized (the option of circular polarization and linear one is not essential and we take the latter for the sake of concreteness). The nonlinear terms $\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}$ and $\mathbf{v}_\pm \times \mathbf{B}/c$ and the pressure term are neglected in this order. The solution of Eq. (7) is

$$\mathbf{v}_\pm^{(1)} = \pm e\mathbf{E}/(-i\omega m) . \tag{8}$$

The perturbed (the first-order) continuity equation derived from Eq. (1) is

$$\frac{\partial n_\pm^{(1)}}{\partial t} = 0 , \tag{9}$$

where we noted that the equilibrium for the zeroth order cancels the right-hand side terms and that $\boldsymbol{\nabla} \cdot \mathbf{v}_\pm^{(1)}$ as obtained from Eq. (8) vanishes. Thus we obtain $n_+^{(1)} = n_-^{(1)} = 0$, which implies no charge separation, as asserted earlier. This can change, of course, if the presence of ions is taken into account, although charge separation remains small as long as ions are a tiny minority.

The second order of the equation of motion Eq. (2) is

$$m \frac{\partial \mathbf{v}_\pm^{(2)}}{\partial t} = -\frac{1}{2} m \boldsymbol{\nabla} \left| \frac{e\mathbf{E}}{\omega m} \right|^2 - \frac{1}{\sigma^{1/3} n_\pm} \boldsymbol{\nabla} n_\pm^{4/3} , \tag{10}$$

where the convective derivative $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ vanishes and the nonlinear term $\pm e\mathbf{v}_{\pm}^{(1)}\times\mathbf{B}/c$ gives rise to the first term on the right-hand side of Eq. (10). We have retained the slow varying component of the nonlinear term in Eq. (10) by taking the absolute value. In the present article we neglect the second harmonic response. Combining Eq. (10) with the second order continuity equation $\partial n_{\pm}^{(2)}/\partial t + n\boldsymbol{\nabla}\cdot v_{\pm}^{(2)} = 0$, we obtain

$$\frac{\partial^2}{\partial t^2}n_{\pm} + \frac{4n}{3}\left(\frac{n_{\pm}}{\sigma}\right)^{1/3}\nabla^2 \ell n\, n_{\pm} = \frac{nm}{2}\nabla^2\left|\frac{e\mathbf{E}}{m\omega}\right|^2 ,$$

where the second term on the left-hand side is linearized to give

$$\frac{\partial^2}{\partial t^2}n_{\pm}^{(2)} - c_s^2\,\nabla^2\, n_{\pm}^{(2)} = \frac{nm}{2}\nabla^2\left|\frac{e\mathbf{E}}{m\omega}\right|^2 , \tag{11}$$

where $c_s^2 = \frac{4}{3}\,(n/\sigma)^{1/3}$. Equation (11) is the acoustic equation driven by the ponderomotive force of photons on the right-hand side.

The Maxwell equation, Eq. (3), is cast into

$$-\nabla^2\mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{4\pi e}{c^2}\frac{\partial}{\partial t}\left(n_+ v_+ - n_- v_-\right) = -\frac{8\pi e^2}{mc^2}\left(n + n^{(2)}\right)\mathbf{E} , \tag{12}$$

where $n^{(2)} \equiv n_+^{(2)} = n_-^{(2)}$ since $n_+^{(1)} = n_-^{(1)} = 0$. By writing $\omega_p^2 = \omega_{pe}^2 + \omega_{pp}^2 = 8\pi n e^2/m$, Eq. (12) becomes

$$-\nabla^2\mathbf{E} + \frac{\omega_p^2}{c^2}\mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{\omega_p^2}{c^2}\frac{n^{(2)}}{n}\mathbf{E} . \tag{13}$$

The term on the right-hand side of Eq. (13) is a third-order quantity, while the terms on the left-hand side are all the first order. However, these terms of the first order cancel each other because of the dispersion relation of electromagnetic waves in the positron-electron plasma $\omega^2 = \omega_p^2 + k^2 c^2$. Let $\mathbf{E} = \mathrm{Re}(\mathbf{a}e^{ikx - i\omega t})$. Equation (13) then reads

$$-\nabla^2 a - 2i\mathbf{k}\cdot\boldsymbol{\nabla}a + \left(k^2 + \frac{\omega_p^2}{c^2} - \frac{\omega^2}{c^2}\right)a - 2i\frac{\omega}{c^2}\frac{\partial a}{\partial t} + \mathcal{O}(\partial_t^2) = -\frac{\omega_p^2}{c^2}\frac{\delta n}{n}a , \tag{14}$$

where $\delta n \equiv n^{(2)}$ and ∇ and $\partial/\partial t$ operate on a in slow scales.

Imposing the linear dispersion relation $\omega^2 = \omega_p^2 + k^2 c^2$, we obtain from Eq. (14)

$$i \frac{\partial a}{\partial t} - i\mathbf{v}_{\mathrm{gr}} \cdot \nabla a + \frac{1}{2} \frac{v_{\mathrm{gr}}}{k} \nabla^2 a + \mathcal{O}(\partial_t^2) = \frac{\omega_p^2}{2\omega} \frac{\delta n}{n} a \,, \tag{15}$$

where $\mathbf{v}_{\mathrm{gr}} = \partial\omega/\partial\mathbf{k}$.

In the present section we investigate the interaction between photons and phonons when they are not in resonance. This means that the group velocity, v_{gr}, of photons in this plasma is not close to the phonon (acoustic) velocity c_s. This can be fulfilled either by $v_{\mathrm{gr}}^2 \gg c_s^2$ or by $v_{\mathrm{gr}}^2 \ll c_s^2$. In the former case such interaction as the beat wave acceleration[10] takes place and will not be discussed any further in this article. Some more discussion of this case may be found in Tajima[11] and in Leboeuf *et al.*[12] In the latter case ($v_{\mathrm{gr}}^2 \ll c_s^2$) $\quad k \ll \omega_p/c$. This condition in a relativistic plasma is much less severe than that in a nonrelativistic plasma, as the relativistic sound velocity $c_s \sim c/\sqrt{3}$ is fairly close to the speed of light. In other words, in relativistic plasma a wave packet of photons with long wavelengths may be relatively easily able to satisfy the nonresonant condition of $v_{\mathrm{gr}}^2 \ll c_s^2$. In between these two nonresonant cases lies, of course, the resonant case $v_{\mathrm{gr}}^2 \approx c_s^2$, which is deferred to the next section for discussion.

The nonresonant case of interest ($v_{\mathrm{gr}}^2 \ll c_s^2$) may arise in a variety of settings. One can imagine a case where most photons propagate in one predominant direction (say, the x-direction) with the mean wavenumber k_x with a much smaller wavenumber spread $\Delta k_x \sim \Delta k_y \ll k_x \ll \omega_p/c$. We refer this to Case (i). Such a wave packet is basically one dimensional. See Fig. 1(a) for a schematic spectral distribution. An extreme alternative to this is a wavepacket spread in three dimensions in a nearly uniform fashion, $\Delta k_x \sim \Delta k_y \sim \Delta k_z \sim |k_x|, |k_y|, |k_z| \ll \omega_p/c$. A schematic spectral distribution of the latter wavepacket is depicted in Fig. 1(b). Such a wavepacket is basically two or three dimensional, depending on the presence of the third dimensional spread. We refer this to Case (ii). Some of the cosmic plasma may fall onto this case ($v_{\mathrm{gr}}^2 \ll c_s^2$), including the "hydrodynamic" cases.

In Case (i) we transform variables x, t to ξ, τ as

$$\xi = \epsilon\left(x - \frac{c^2 k}{\omega} t\right) = \epsilon\,(x - v_{\rm gr} t) \quad , \quad \eta = \epsilon y \quad , \quad \zeta = \epsilon z \quad , \quad \tau = \epsilon^2 t \; , \tag{16}$$

where ϵ is the stretching parameter[13,14] and we take $\epsilon^2 = \frac{\delta n}{n}$. Then Eq. (15) reads

$$i\,\frac{\partial a}{\partial \tau} + \frac{1}{2}\,\frac{v_{\rm gr}}{k}\,\nabla^2 a = \frac{\omega_p^2}{2\omega}\,\frac{\delta n}{n}\,a \; , \tag{17}$$

where $\nabla^2 = \partial_\xi^2 + \partial_\eta^2 + \partial_\zeta^2$. Equation (11) transforms into

$$\left[\epsilon^2\,\frac{\partial^2}{\partial \tau^2} - 2\epsilon\, v_{\rm gr} \cdot \frac{\partial^2}{\partial \xi \partial \tau} + \left(v_{\rm gr}^2 - c_s^2\right)\,\frac{\partial^2}{\partial \xi^2} - c_s^2\,\left(\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right)\right]\delta n = \frac{n e^2}{2m\omega^2}\,\nabla^2\,|\mathbf{E}|^2 \; . \tag{18}$$

The operator on the left-hand side in the brackets of Eq. (18) becomes $-c_s^2\,\nabla^2$ when $v_{\rm gr}^2 \ll c_s^2$. It becomes $v_{\rm gr}^2\,\nabla^2$ when $v_{\rm gr}^2 \gg c_s^2$. If $v_{\rm gr}^2 \simeq c_s^2$, it will be treated separately in Sec. III.

When $v_{\rm gr}^2 \ll c_s^2$, Eq. (18) gives rise to

$$c_s^2\,\nabla^2\,\delta n = -\frac{n_0 e^2}{2m\omega^2}\,\nabla^2\,|\mathbf{E}|^2 \; . \tag{19}$$

If δn and $\mathbf{E}$ are localized; i.e., δn and $|\mathbf{E}|^2 \to 0$ at infinity,

$$\delta n = -\frac{n_0 e^2}{2m\omega^2\,c_s^2}\,|\mathbf{E}|^2 \; . \tag{20}$$

Substituting Eq. (20) into Eq. (17), we obtain

$$i\,\frac{\partial a}{\partial \tau} + \frac{1}{2}\,\frac{v_{\rm gr}}{k}\,\nabla^2 a + \frac{9}{32\pi}\left(\frac{\omega_p^2}{\omega^2}\right)^2\,\frac{1}{c_s^2}\,|a|^2 a = 0 \; . \tag{21}$$

When the plasma is relativistically hot $c_s^2 = \frac{1}{3}\,c^2$, the coefficient of the third term of Eq. (21) becomes $\frac{3}{32\pi}\left(\frac{\omega_p^2}{\omega^2}\right)^2\frac{1}{c_s^2}$. By putting $\mathcal{E} \equiv \left(\frac{9}{32\pi}\right)^{1/2}\frac{\omega_p^2}{\omega^2}\,\frac{1}{c}\,a$, Eq. (21) is cast into

$$i\,\frac{\partial \mathcal{E}}{\partial \tau} + \beta\,\nabla^2 \mathcal{E} + |\mathcal{E}|^2 \mathcal{E} = 0 \; , \tag{22}$$

where $\beta \equiv \frac{c^2}{2\omega}$. Then Eq. (22) can be written in a form[8]

$$i\dot{\mathcal{E}} = \frac{\delta \mathcal{H}}{\delta \mathcal{E}^*} \; , \tag{23}$$

where the Hamiltonian is given by

$$\mathcal{H} = \int \left[\frac{\beta}{\varepsilon^2} |\nabla \times \boldsymbol{\mathcal{E}}|^2 - \frac{|\mathcal{E}|^4}{2} \right] d\boldsymbol{\xi} \; . \tag{24}$$

Note $\boldsymbol{\xi}$ is the stretched coordinates on the moving frame. One can normalize the spatial coordinate $\boldsymbol{\xi}$ such that the coefficient of the first term in Eq. (24) becomes unity. The Hamiltonian is a conservative quantity. Gauge invariance[8,15] implies the conservation of light quanta:

$$N = \int |\mathcal{E}|^2 \, d\boldsymbol{\xi}' \; . \tag{25}$$

The system described by Eq. (22) has been shown[8,15,16] through the virial theorem,[15] direct integration,[16] and numerical integration[17] that the wavepacket tends to collapse in systems with dimensions larger than one, while systems with one dimension do not. In a spherical symmetric system on the moving coordinates $\boldsymbol{\xi}'$ the quantity

$$I = \int \xi_r'^2 \left(|E_r|^2 + |E_\theta|^2 \right) d\xi_r' \; .$$

obeys

$$\frac{\partial^2 I}{\partial t^2} = 8\mathcal{H} - 2 \int |\mathbf{E}|^4 \, d\boldsymbol{\xi}' < 8\mathcal{H} \; .$$

Integrating I twice in time, we obtain

$$I < 4\mathcal{H}t^2 + C_1 t + C_2 \; , \tag{26}$$

where C_1 and C_2 are integral constants. If $\mathcal{H} < 0$, the positivity of I implies that the inequality is valid for $t < t_0$, toward which the system exhibits a singular behavior. The size of the density depression ξ_{r0} behaves[16] as

$$\xi_{r0}(t) \propto (t_0 - t)^{1/2} \; ,$$

and (27)

$$E(t) \propto (t_0 - t)^{-4/3} \; ,$$

exhibiting the collapse of the electromagnetic wavepacket with a density depression in a finite time. However, once the size of the depression becomes of the order of the collisionless skin depth, c/ω_p, the present treatment becomes invalidated. Some of the nonlinear numerical simulations[18] can be a guide to suggest the more detailed evolution of such an entity. Although Ref. 18 handles electrostatic plasma waves, the relevant equations are similar to our case. The high frequency waves collapse, creating a growing plasma density depression. The deepening cavity eventually tries to emit short wavelength acoustic waves of the cavity size. The work in Ref. 18 indicates that when the plasma is isothermal (the temperature of both species are equal) and the (ion) Landau damping is significant, the emitted acoustic waves are quickly absorbed so that the collapsing waves are "burned up." In the present relativistic electron-positron plasma case the Landau damping is supposed to be significant (see Sec. V). Thus the collapsing electromagnetic waves in the plasma cavity would be burned up.

III. Resonant Interaction between Photons and Phonons

In this section we investigate the resonant interaction between photons and phonons in an electron-positron plasma, the case left out in the previous section. That is, the group velocity of (a set of) photons (or electromagnetic waves) is nearly equal to the phase velocity of phonons. See Fig. 1(c). In order to derive the basic equations that govern the resonant interaction of photons and phonons in this plasma, we need a different systematic expansion method than that employed in the previous section. We again use the reductive perturbation theory with the expansion adopted being to specifically incorporate the physics of this resonance.

In the following we take the normalization that the length is measured in terms of c/ω_{pe}, the velocity in c and the density in n_0, the mean density of electrons, where the electron plasma frequency ω_{pe} is defined as $\omega_{pe}^2 = 4\pi n_0 e^2/m$. In these normalized units our equations are derived from Eqs. (1)–(3) as

$$\frac{\partial n_\pm}{\partial t} + \nabla \cdot (n_\pm \mathbf{v}_\pm) = 0 \ , \tag{28}$$

$$\frac{d\mathbf{v}_{\pm\perp}}{dt} = \pm\left[\mathbf{E}_\perp + (\mathbf{v}\times\mathbf{B})_\perp\right] \ , \tag{29}$$

$$\frac{dv_{\pm x}}{dt} = -\frac{1}{n_\pm}\frac{\partial P_\pm}{\partial x} \pm \left[E_x \pm (\mathbf{v}\times\mathbf{B})_x\right] \ , \tag{30}$$

$$\nabla\cdot\mathbf{E} = n_+ - n_- \ , \tag{31}$$

$$\nabla\times\mathbf{B} - \frac{\partial \mathbf{E}_\perp}{\partial t} = n_+\,\mathbf{v}_{+\perp} - n_-\,\mathbf{v}_{-\perp} \ , \tag{32}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E}_\perp = 0 \ , \tag{33}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + v_x\,\frac{\partial}{\partial x}$ and x is the longitudinal direction parallel to the propagation of photons that resonate with the plasma waves. We restrict the photon propagation in the x-direction and the electric linear polarization in the y-direction in this section. Thus Eqs. (28) and (31) can be written as $\partial_t\, n_\pm + \partial_x(n_\pm\, v_{\pm x}) = 0$ and $\partial_x\, E_x = n_+ - n_-$.

We expand the quantities $\mathbf{v}_{\pm\perp}, v_{\pm x}, n_\pm$, and $\mathbf{E}_\perp$ as follows:

$$\mathbf{v}_{\pm\perp} = \epsilon\,\mathbf{v}_{\pm\perp}^{(1)} + \epsilon^{3/2}\,\mathbf{v}_{\pm\perp}^{(\frac{3}{2})} + \cdots \ , \tag{34}$$

$$v_{\pm x} = \epsilon\, v_{\pm x}^{(1)} + \epsilon^2\, v_{\pm x}^{(2)} + \cdots \ , \tag{35}$$

$$n_\pm = 1 + \epsilon\, n_\pm^{(1)} + \epsilon^2\, n_\pm^{(2)} + \cdots \ , \tag{36}$$

$$\mathbf{E}_\perp = \epsilon\,\mathbf{E}_\perp^{(1)} + \epsilon^{3/2}\,\mathbf{E}_\perp^{(\frac{3}{2})} + \cdots \ , \tag{37}$$

where the superscripts in parenthesis indicate the power of ϵ, the expansion parameter. Furthermore, we may write as

$$\mathbf{v}_{\pm\perp}^{(1)} = \tilde{\mathbf{v}}_{\pm\perp 1}^{(1)}\, e^{i(kx-\omega t)} + c.c. \ ,$$

$$\mathbf{E}_\perp^{(1)} = \tilde{\mathbf{E}}_{\perp 1}^{(1)}\, e^{i(kx-\omega t)} + c.c. \ ,$$

$$\mathbf{B}_\perp^{(1)} = \tilde{\mathbf{B}}_{\perp 1}^{(1)}\, e^{i(kx-\omega t)} + c.c. \,,$$

where the subscripts refer to the harmonic component or to the power to the $e^{i(kx-\omega t)}$. On the order of ϵ^1, from Eq. (29) we obtain

$$\tilde{\mathbf{v}}_{\pm\perp}^{(1)} = \pm \frac{i}{\omega}\, \tilde{E}_\perp^{(1)} \,. \tag{38}$$

This is equivalent to Eq. (7).

From Eqs. (32) and (38) we obtain

$$\mathbf{k} \times \tilde{\mathbf{B}}_\perp^{(1)} + \omega\, \tilde{\mathbf{E}}_\perp^{(1)} = \frac{2}{\omega}\, \tilde{\mathbf{E}}_\perp^{(1)} \,, \tag{39}$$

and from Eq. (33) we obtain

$$\tilde{\mathbf{B}}_\perp^{(1)} = \frac{k_x}{\omega}\, \widehat{x} \times \tilde{\mathbf{E}}_\perp^{(1)} \,, \tag{40}$$

where $\mathbf{k} = k_x\, \widehat{x}$. From Eq. (39) and (40) we arrive at the dispersion relation

$$\omega^2 = 2 + k_x^2 \,. \tag{41}$$

Hence the group velocity λ is

$$\lambda = \frac{\partial \omega}{\partial k_x} = \frac{k_x}{\omega} \,. \tag{42}$$

In executing the derivatives, we employ the reductive perturbation theory.[13,14] In the problem of the present section we stretch the coordinates as

$$\xi = \epsilon^{1/2}(x - \lambda t) \qquad , \qquad \tau = \epsilon^{3/2} t \,, \tag{43}$$

$\mathcal{O}\left(\frac{\partial}{\partial \xi}\right) = k_x \mathcal{O}(\epsilon^{1/2})$, and $\mathcal{O}\left(\frac{\partial}{\partial \tau}\right) = \omega\, \mathcal{O}(\epsilon^{3/2})$. On the order of $\epsilon^{3/2}$, Eq. (29) then yields

$$-\lambda \frac{\partial}{\partial \xi}\, \tilde{\mathbf{v}}_{\pm\perp 1}^{(1)} - i\omega \mathbf{v}_{\pm\perp 1}^{(\frac{3}{2})} = \pm \tilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} \,, \tag{44}$$

Eqs. (32) and (33) give rise to

$$ik \times \widetilde{\mathbf{B}}_{\perp 1}^{(\frac{3}{2})} + i\omega\, \widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} + \lambda \widehat{x} \times \frac{\partial}{\partial \xi} \widetilde{\mathbf{B}}_{\perp 1}^{(1)} + \lambda \frac{\partial}{\partial \xi} \widetilde{\mathbf{E}}_{\perp 1}^{(1)}$$

$$= \frac{1}{-i\omega} \left(2\widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} - \lambda \frac{1}{i\omega} \frac{\partial}{\partial \xi} \widetilde{\mathbf{E}}_{\perp 1}^{(1)} \right) , \tag{45}$$

$$- i\omega\, \widetilde{\mathbf{B}}_{\perp 1}^{(\frac{3}{2})} + ik_x\, \widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} - \lambda \frac{\partial}{\partial \xi} \mathbf{B}_{\perp 1}^{(1)} + \widehat{x} \times \frac{\partial}{\partial \xi} \mathbf{E}_{\perp 1}^{(1)} = 0 . \tag{46}$$

Hence

$$\mathbf{v}_1^{\pm(\frac{3}{2})} = \frac{\pm i}{\omega} \left(\widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} + \frac{i\lambda}{\omega} \frac{\partial}{\partial \xi} \widetilde{\mathbf{E}}_1^{(1)} \right) . \tag{47}$$

It follows from Eq. (32) that

$$i\, \mathbf{k} \times \widetilde{\mathbf{B}}_{\perp}^{(\frac{3}{2})} + i \frac{k_x^2}{\omega} \widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} + \frac{\lambda}{\omega} \mathbf{k} \times \frac{\partial}{\partial \xi} \widetilde{\mathbf{B}}_{\perp}^{(1)} + \frac{k_x}{\omega} \frac{\partial}{\partial \xi} \mathbf{E}_{\perp}^{(1)} = 0 , \tag{48}$$

and from Eq. (33) that

$$-i\omega\, \widetilde{\mathbf{B}}_{\perp 1}^{(\frac{3}{2})} + ik \times \widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} - \lambda \frac{\partial}{\partial \xi} \mathbf{B}_{\perp 1}^{(1)} + \widehat{x} \times \frac{\partial}{\partial \xi} \widetilde{\mathbf{E}}_{\perp 1}^{(1)} = 0 . \tag{49}$$

Combining Eqs. (48) and (49) and making use of the zeroth order dispersion relation (42), we obtain

$$\mathbf{B}_{\perp 1}^{(\frac{3}{2})} = \frac{k_x \widehat{x}}{\omega} \times \widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})} - \frac{2i}{\omega^3} \widehat{x} \times \frac{\partial}{\partial \xi} \mathbf{E}_{\perp 1}^{(1)} . \tag{50}$$

On the order ϵ^2 we collect terms arising from the momentum equation (29) as

$$- i\omega\, \widetilde{\mathbf{v}}_{\pm\perp 1}^{(2)} + ik_x\, v_x^{(1)}\, \widetilde{\mathbf{v}}_{\pm\perp 1}^{(1)} - \lambda \frac{\partial}{\partial \xi} \widetilde{\mathbf{v}}_{\pm\perp 1}^{(\frac{3}{2})} = \pm \left(\widetilde{\mathbf{E}}_{\perp 1}^{(2)} + v_x^{(1)}\, \widehat{x} \times \mathbf{B}_{\perp 1}^{(1)} \right) , \tag{51}$$

where $d/dt = -i\omega + \epsilon^{3/2}\, \partial/\partial\tau - \lambda\epsilon^{1/2}\, \partial/\partial\xi + \epsilon\, v_x^{(1)}(ik_x + \epsilon^{1/}\lambda\, \partial/\partial\xi)$. This leads to

$$\widetilde{\mathbf{v}}_{\pm\perp 1}^{(2)} = \frac{i}{\omega} \left(\pm\, \widetilde{\mathbf{E}}_{\perp 1}^{(2)} + \lambda \frac{\partial}{\partial \xi} \mathbf{v}_{\pm\perp 1}^{(\frac{3}{2})} \right) . \tag{52}$$

The Maxwell's equations (32) and (33) in $\mathcal{O}(\epsilon^2)$ yield

$$ik_x \widehat{x} \times \widetilde{\mathbf{B}}_{\perp 1}^{(2)} + \widehat{x} \times \frac{\partial}{\partial \xi} \mathbf{B}_{\perp 1}^{(\frac{3}{2})} + i\omega\, \widetilde{\mathbf{E}}_{\perp}^{(2)} + \lambda \frac{\partial}{\partial \xi} \widetilde{\mathbf{E}}_{\perp 1}^{(\frac{3}{2})}$$

$$= n^{(1)} \left(\mathbf{v}_{+\perp 1}^{(1)} - \mathbf{v}_{-\perp 1}^{(1)} \right) + \left(\mathbf{v}_{+\perp 1}^{(2)} - \mathbf{v}_{-\perp 1}^{(2)} \right) , \tag{53}$$

and

$$-i\omega\,\tilde{\mathbf{B}}^{(2)}_{\perp 1} + ik_x\hat{x}\times\tilde{\mathbf{E}}^{(2)}_{\perp 1} - \lambda\frac{\partial}{\partial\xi}\tilde{\mathbf{B}}^{(\frac{3}{2})}_{\perp 1} + \hat{x}\times\frac{\partial}{\partial\xi}\tilde{\mathbf{E}}^{(\frac{3}{2})}_{\perp} = 0\ . \tag{54}$$

Using Eqs. (47) and (54) to eliminate $\mathbf{v}^{(\frac{3}{2})}_{\pm\perp 1}$ and $\mathbf{v}^{(2)}_{\pm\perp 1}$, we obtain by combination of Eqs. (53) $\times\omega$ and (54) $\times\mathbf{k}$

$$(i\omega^2 - ik_x^2 - 2i)\,\tilde{\mathbf{E}}^{(2)}_{\perp 1} + (\omega - \lambda k_x)\hat{x}\times\frac{\partial}{\partial\xi}\tilde{\mathbf{B}}^{(\frac{3}{2})}_{\perp 1} + (\lambda\omega - k_x)\frac{\partial}{\partial\xi}\tilde{\mathbf{E}}^{(\frac{3}{2})}_{\perp 1} + \frac{2\lambda}{\omega}\frac{\partial}{\partial\xi}\mathbf{E}^{(\frac{3}{2})}_{\perp 1}$$

$$= 2in^{(1)}\,\tilde{\mathbf{E}}^{(1)}_{\perp 1} - \frac{2\lambda}{\omega}\frac{i\lambda}{\omega}\frac{\partial^2}{\partial\xi^2}\tilde{\mathbf{E}}^{(1)}_{\perp 1}\ . \tag{55}$$

Equation (55) can be simplified by using Eq. (50), yielding the coefficients of $\frac{\partial}{\partial\xi}\mathbf{E}^{(\frac{3}{2})}_{\perp 1}$ as well as that of $\tilde{\mathbf{E}}^{(2)}_{\perp 1}$ to vanish because of the dispersion relation $\omega^2 = k_x^2 + 2$. The term $\frac{\partial}{\partial\xi}\tilde{\mathbf{B}}^{(\frac{3}{2})}_{\perp 1}$ was replaced by the relation (50). After these algebras, we obtain from Eq. (55)

$$\frac{1}{\omega^2}\frac{\partial^2}{\partial\xi^2}\tilde{\mathbf{E}}^{(1)}_{\perp 1} = n^{(1)}\,\mathbf{E}^{(1)}_{\perp 1}\ . \tag{56}$$

Equation (56) is one of our basic equations governing the nonlinear behavior of the coupling between the e^-e^+ plasma and radiation; i.e., $n^{(1)}$ and $\mathbf{E}^{(1)}$.

For relativistically hot electron-positron plasmas with photons interacting with leptons opaquely, the equation of state can be taken as that of ultrarelativistic particles or equivalently that of photons. Thus the pressure of the positrons or electrons Eq. (6) is written as

$$P_\pm = \beta(n_\pm)^\gamma \simeq \beta\left[1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \cdots\right]^\gamma\ , \tag{57}$$

with the adiabatic gas constant $\gamma = 4/3$. Then

$$P^{(0)}_\pm = \beta \quad , \quad P_{\pm(1)} = \beta\,\gamma\,n^{(1)} \tag{58}$$

$$P^{(2)}_\pm = \beta\left[\gamma\,n_\pm(2) + \frac{1}{2}\gamma(\gamma-1)(n^{(1)}_\pm)^2\right]\ , \tag{59}$$

and β is equal to the normalized $\sigma^{-1/3}$. If the nonrelativistic gas constant is used, the exponent γ of Eq. (57) takes the value of 5/3, the coefficient γ and $\frac{1}{2}\gamma(\gamma-1)$ in Eqs. (58) and (59) take the values of 5/3 and 5/9. In our present normalization, the field E and B are measured in the unit of $(mc\omega_{pe}/e)$ and the pressure is measured in the unit of $\frac{1}{4\pi}(mc\omega_{pe}/e)^2 = n_0\, mc^2$. Therefore, the radiation pressure is measured in the unit of the electron rest mass energy density. Introducing these expressions into the equation of longitudinal motion (30), we have in $\mathcal{O}(\epsilon^{3/2})$

$$-\lambda \frac{\partial}{\partial\xi} n_{\pm}^{(1)} + \frac{\partial}{\partial\xi} v_{\pm x}^{(1)} = 0 \,, \tag{60}$$

and

$$-\lambda \frac{\partial}{\partial\xi} v_{\pm x}^{(1)} + \beta\gamma \frac{\partial}{\partial\xi} n_{\pm}^{(1)} = 0 \,. \tag{61}$$

From Eqs. (60) and (61) we obtain

$$\lambda^2 = \gamma\beta \,. \tag{62}$$

In the present section we consider a specific case in which the group velocity of the electromagnetic wave is equal to the phase velocity of the longitudinal wave. We refer the reader to Fig. 1(c). The longitudinal wave is basically the acoustic wave.

In the following we take the ultrarelativistic case in particular for concreteness sake. The acoustic phase velocity in the ultrarelativistic temperature limit is $c_s = c/\sqrt{3}$, as derived from the equation of state (57). Thus let λ be specified as

$$\lambda = \frac{1}{\sqrt{3}} \,. \tag{63}$$

This sets the normalization of the constant such that

$$\beta = \frac{1}{4} \qquad \text{and} \qquad \gamma = \frac{4}{3} \,. \tag{64}$$

We also note that

$$v_{\pm x}^{(1)} = \lambda\, n_{\pm}^{(1)} \,, \tag{65}$$

and

$$n_{+}^{(1)} = n_{-}^{(1)} \qquad \text{and} \qquad E_{x0}^{(1)} = 0 \; , \tag{66}$$

as found in Sec. II already.

In the order ϵ^2 there is no term. In the order $\epsilon^{5/2}$ for $\ell = 0$ (the harmonic number for k_x) we have from Eq. (28) and (29)

$$-\lambda \frac{\partial}{\partial \xi} n^{(2)} + \frac{\partial}{\partial \xi} v_x^{(2)} + \frac{\partial}{\partial \tau} n^{(1)} + n^{(1)} \frac{\partial}{\partial \xi} v_x^{(1)} + v_x^{(1)} \frac{\partial}{\partial \xi} n^{(1)} = 0 \; , \tag{67}$$

and

$$-\lambda \frac{\partial}{\partial \xi} v_x^{(2)} + \frac{\partial}{\partial \tau} v_x^{(1)} + v_x^{(1)} \frac{\partial}{\partial \xi} v_x^{(1)} + \frac{4}{3} \beta \frac{\partial}{\partial \xi} n^{(2)}$$
$$+ \frac{2}{q} \beta \frac{\partial}{\partial \xi} (n^{(1)})^2 - \frac{4}{3} \beta n^{(1)} \frac{\partial}{\partial \xi} n^{(1)} = \pm (v \times B)_x^{(2)} \; , \tag{68}$$

where $n_{\pm 0}^{(2)}$ is denoted as $n^{(2)}$ here and the right-hand side of Eq. (68) can be cast into $\pm (v \times B)_x^{(2)} = -\frac{2}{\omega^4} \frac{\partial}{\partial \xi} |\widetilde{E}_{\perp 1}^{(1)}|^2$ by utilizing the relation $v_{\pm \perp 1}^{(1)} \times B_{\perp -1}^{(\frac{3}{2})} = \pm \frac{i}{\omega} \widetilde{E}_{\perp 1}^{(1)} \times \left(\frac{k_x}{\omega} \widehat{x} \times E_{\perp -1}^{(\frac{3}{2})} + \frac{2i}{\omega^3} \widehat{x} \times \frac{\partial}{\partial \xi} E_{\perp -1}^{(1)} \right)$ and a similar relation for $v_{\pm \perp -1}^{(\frac{3}{2})} \times B_{\perp 1}^{(1)}$. Equations (67) and (68) yield

$$\lambda \frac{\partial n^{(1)}}{\partial \tau} + \lambda \left(n^{(1)} \frac{\partial}{\partial \xi} v_x^{(1)} + v_x^{(1)} \frac{\partial}{\partial \xi} n^{(1)} \right) + \frac{\partial}{\partial \tau} v_x^{(1)}$$
$$+ v_x^{(1)} \frac{\partial}{\partial \xi} v_x^{(1)} - \frac{2}{9} n^{(1)} \frac{\partial}{\partial \xi} n^{(1)} = -\frac{1}{\omega^4} \frac{\partial}{\partial \xi} |E_{\perp}^{(1)}|^2 \; . \tag{69}$$

Noting that the wavenumber k_x at which the group velocity of the photon becomes equal to the phase velocity of the phonon is $k_x = 1$ and that the photon frequency ω at $k_x = 1$ is $\omega^2 = k_x^2 + 2 = 3$, Eq. (69) can be written as

$$\frac{\partial v_x^{(1)}}{\partial \tau} + \frac{7}{6} v_x^{(1)} \frac{\partial}{\partial \xi} v^{(1)} = -\frac{1}{9} \frac{\partial}{\partial \xi} |\widetilde{E}_{\perp 1}^{(1)}|^2 \; , \tag{70}$$

where $|E_{\perp}^{(1)}|^2 = 2 |\widetilde{E}_{\perp}^{(1)}|^2$ as $E_{\perp}^{(1)} = \widetilde{E}_{\perp 1}^{(1)} e^{i(kx - \omega t)} + c.c.$ From Eq. (56), on the other hand, we obtain with the aid of $n_{\pm 0}^{(1)} = v_{\pm x0}^{(1)} / \lambda = \sqrt{3}\, v_{\pm x0}^{(1)}$

$$\frac{\partial^2}{\partial \xi^2} \widetilde{E}_{\perp 1}^{(1)} = 3\sqrt{3}\, v_{\pm x0}^{(1)} \widetilde{E}_{\perp 1}^{(1)} \; . \tag{71}$$

Introducing v and $\widetilde{E}$ by the relations

$$v = \frac{7}{6}\, v^{(1)}_{\pm x0} \,, \tag{72}$$

$$\widetilde{E} = \left(\frac{7}{54}\right)^{1/2} \widetilde{E}^{(1)}_{\perp 1} \,, \tag{73}$$

and $\xi' = \left(\frac{7}{18}\sqrt{3}\right)^{1/2}\xi$, $\tau' = \left(\frac{7}{18}\sqrt{3}\right)^{1/2}\tau$, Eqs. (70) and (71) are now written as

$$\frac{\partial v}{\partial \tau} + v\,\frac{\partial v}{\partial \xi} = -\frac{\partial}{\partial \xi}\,|\widetilde{E}|^2 \,, \tag{74}$$

$$\frac{\partial^2 \widetilde{E}}{\partial \xi^2} = v\,\widetilde{E} \,, \tag{75}$$

where the primes of ξ' and τ' have been removed in Eqs. (74) and (75) and herewith. Equations (74) and (75) are our basic equations for the photons and phonons in resonance ($\partial\omega/\partial k = c_s$) in an electron-positron plasma. When the viscosity effect comes in at the order $\mathcal{O}(\mu) \sim \epsilon^{1/2}$, Eq. (74) becomes

$$\frac{\partial v}{\partial \tau} + v\,\frac{\partial v}{\partial \xi} = \mu\,\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial}{\partial \xi}\,|\widetilde{E}|^2 \,. \tag{74'}$$

IV. Stationary Structure — Shocks and Solitons

We look for particular solutions to the problem of one-dimensional resonant interaction of electron-positron plasma with electromagnetic waves described by Eqs. (74) and (75). In this section we write $\widetilde{E}$ as E. We also restrict ourselves to a class of solutions that depend only on $\zeta \equiv \xi - V\tau$, where V is the phase velocity. In the following the derivative with respect to ζ is expressed by a prime.

Equations (74) and (75) are cast into

$$-Vv' + \left(\frac{v^2}{2}\right)' = -(|E|^2)' \,, \tag{76}$$

$$E'' = vE \,. \tag{77}$$

Integration of Eq. (76) yields

$$P = -Vv + \frac{v^2}{2} + |E|^2 \ , \tag{78}$$

where P is a constant and is essentially energy except for an additive constant. The velocity v can be written from Eq. (78) as

$$v = V \pm \sqrt{2}\sqrt{\frac{1}{2}V^2 + P - |E|^2} \ . \tag{79}$$

It is required for reality of v that the "total energy" Q be positive

$$Q = \frac{V^2}{2} + P > 0 \ . \tag{80}$$

Equation (79) takes the form

$$v = V \pm \sqrt{2}\,(Q - |E|^2)^{1/2} \ . \tag{81}$$

With Eq. (81) Eqs. (77) yields

$$E'' = \left\{V \pm \sqrt{2}\,(Q - |E|^2)^{1/2}\right\} E \ . \tag{82}$$

Some discussion is presented in Appendix whether the initial and boundary conditions to Eqs. (74) and (75) are well or ill-posed.

$\langle$Real $E\rangle$

Let E be real and we introduce a pseudo-potential (Sagdeev potential) Φ by the relation

$$-\frac{\partial \Phi}{\partial E} = \left\{V \pm \sqrt{2}\,(Q - E^2)^{1/2}\right\} E \ , \tag{83}$$

so that the "equation of motion" for E is

$$\frac{\partial^2 E}{\partial \zeta^2} = -\frac{\partial \Phi}{\partial E} \ . \tag{84}$$

Integrating Eq. (83), we obtain

$$\Phi = -\frac{V}{2}E^2 \pm \frac{\sqrt{2}}{3}(Q - E^2)^{3/2} \mp \frac{\sqrt{2}}{3}Q^{3/2} . \tag{85}$$

Integration of (84) with respect to ζ yields the "energy conservation"

$$\frac{1}{2}(E')^2 + \Phi = W , \tag{86}$$

where W is the total energy.

[**Case A**] $(V > 0)$

From Eq. (83) it follows that when $V > 0$ the plus sign in Eq. (83) gives $\partial\Phi/\partial E = 0$ only at $E = 0$, while the minus sign yields two roots $E = 0$ and $E^2 = P$, which requires $P > 0$. From this the minus sign should be taken. The three extrema of Φ are $E = 0$, $E = \pm\sqrt{P}$. See Fig. 2. The potential maximum Φ_M takes at $E = \pm\sqrt{P}$ with $\Phi_M = -\frac{V}{2}P - \frac{1}{6}V^3 + \frac{\sqrt{2}}{3}\left(\frac{V^2}{2} + P\right)^{3/2}$.

Case I: $0 < W < \Phi_M$, periodic solution.

Case II: $W = \Phi_M$ (homoclinic), kink or antikink soliton solution.

Case III: $W > \Phi_M$, free particle (unbounded).

Case IV: $W < \Phi_M$, $E < -\sqrt{P}$, one-side unbounded.

Case V: $W < \Phi_M$, $E > \sqrt{P}$, one-side unbounded.

Thus bounded solutions of interest take place in Cases I and II. In particular, Case II yields kink or antikink solitary solution. For $\zeta \to \infty$, $E \to \sqrt{P}$ and for $\zeta \to -\infty$, $E \to -\sqrt{P}$ for the kink and vise versa for the antikink. From Eq. (81) $v \to 0$ as $|\zeta| \to \infty$.

[**Case B**] $(V < 0)$

In this case for a reason similar to Case A we must take the plus sign in Eq. (83). The potential is now $\Phi = -\frac{V}{2}E^2 + \frac{\sqrt{2}}{3}(Q - E^2)^{3/2} - \frac{\sqrt{2}}{3}Q^{3/2}$. Hence when $\frac{5}{4}V^2 - 2P \gtrless 0$, $\Phi(E = \pm\sqrt{Q}) \gtrless 0$. This situation is depicted in Fig. 3 I, II, and III. Note that $|E|^2 < Q$ for real v from Eq. (81).

Case I: $\frac{5}{4}V^2 - 2P > 0$

A solitary wave exists for $W = 0$. For $0 < W < \infty$ a periodic wave (approximately with two periods) solution and for $W < 0$ a periodic wave (approximately one period) solution exist. See Fig. 3(I). The electric field E goes to zero and $v \to V + \sqrt{V^2 + P} > 0$ as $|\zeta| \to \infty$.

Case II: $\frac{5}{4}V^2 - 2P < 0$

Asymptotic solutions; i.e. E becomes uniform as $|\zeta| \to \infty$, are not possible.

⟨Complex E⟩

To solve Eqs. (74) and (75) for more general cases of complex electric field E, we introduce the expression

$$E = \rho^{1/2}(\xi) \exp\left(\frac{i}{2} \int \sigma(\xi)\, d\xi\right) , \tag{87}$$

where ρ determines the amplitude squared and σ the phase of E. Substituting Eq. (87) into Eq. (75), we obtain

$$(\rho\sigma)' = 0 ,$$

$$\sigma\sigma' = -2v' + \left[\rho^{-1/2}\left(\rho^{-1/2}\rho'\right)'\right]' ,$$

namely

$$\rho\sigma = F(\tau) , \tag{88}$$

$$\frac{1}{2}\sigma^2 = -2v + \rho^{-1/2}\left(\rho^{-1/2}\rho'\right)' + G(\tau) , \tag{89}$$

where the primes denote the derivatives with respect to ξ. Let us choose $G = 0$ and $F(\tau) = F_0 =$ constant so that

$$\sigma = F_0 \rho^{-1} , \tag{90}$$

$$\frac{F_0^2}{2}\frac{1}{\rho^2} = -2v + \rho^{-1/2}\left(\rho^{-1/2}\rho'\right)' . \tag{91}$$

From Eq. (91), we obtain

$$\frac{1}{2}F_0^2\frac{1}{\rho^2} = -2\left[V \pm \sqrt{2}\left(\frac{1}{2}V^2 + P - \rho\right)^{1/2}\right] + \rho^{-1/2}\left(\rho^{-1/2}\rho'\right)' . \tag{92}$$

By writing $R \equiv \sqrt{\rho} > 0$, Eq. (92) can be written as

$$R'' = -\frac{\partial\tilde{\Phi}}{\partial R} , \tag{93}$$

where

$$\tilde{\Phi} = \Phi + \frac{1}{8}\frac{F_0^2}{R^2} . \tag{94}$$

[Case A] $(V > 0)$

The potential $\tilde{\Phi}$ looks like Fig. 4(a), if F_0^2 is sufficiently small. The maximum point $R_{\max}$ is given by $\partial\tilde{\Phi}/\partial R = 0$; i.e.

$$V - \sqrt{2}\left(\frac{V^2}{2} + P - R^2\right)^{1/2} + \frac{F_0^2}{4R^4} = 0 . \tag{95}$$

The soliton solution appears where the total energy is equal to $\tilde{\Phi}(R = R_{\max})$. In this case the electric field E in Eq. (87) suggests that it holds its maximum $\rho_{\max} = R_{\max}$ as $|\xi| \to \infty$ and its minimum $\rho_{\min} = R_{\min}^2$ at $|\xi| < \infty$ [see Fig. 4(b)]. This is a dark solitary wave.

[Case B] $(V < 0)$

In this case the potential $\tilde{\Phi}$ looks like Fig. 4(c). There is no homoclinic point and thus no soliton solution, unless $F_0 = 0$, that is $\sigma = 0$. If $\sigma =$ const. then $\rho =$ const. so that there are no plane wave solutions for which $v =$ const. If $F_0 = 0$, we have a solitary wave solution that was already given as a particular solution in Fig. 3.

⟨Numerical Integration⟩

We carried out numerical time integration of Eqs. (74) and (75). Starting from an arbitrary initial condition, we obtained various results, including some wavebreaking resulting in angular profiles. These are too complicated to present here. Instead we present a few cases of runs starting from the equilibria discussed earlier in this section. The discussion on well-posed and ill-posed boundary conditions is given in the Appendix.

[Case A I] (real E)

This case corresponds to [Case A] of real E in the first part of this section, the positive V, and $0 < W < \Phi_M$, the mode being trapped in the potential Φ and thus bounded. This is a periodic solution. We take the periodic boundary conditions for v and E. It should be noted that Eq. (75) is a homogeneous differential equation with respect to E and does not fix the amplitude of E by itself. In order to determine the amplitude of E, therefore, it is necessary to invoke the energy conservation. The electric field amplitude at $v = v_{\text{max}}$ is determined as

$$E(v_{\text{max}}) = \left[Q - \frac{1}{2}\left(v_{\text{max}} - V\right)^2\right]^{1/2} . \tag{96}$$

The initial condition is determined by the steady-state solution of Eqs. (74) and (75)

$$E'' = \left[V - \sqrt{2}\,(Q - E^2)^{1/2}\right] E \tag{97}$$

and

$$v = V - \sqrt{2}\,(Q - E^2)^{1/2} \,, \tag{98}$$

with the boundary condition of $E' = 0$ at the edge of the computation box. An example of $V = 1$ and $P = 5$ is shown in Fig. 5. The time step and spatial grid size are 0.001 and 0.05, respectively. A numerical viscosity of 0.01 is included for computational purpose. We see that the periodic waves of v and E propagate to the right with the roughly prescribed velocity without hardly any change of the profile.

[**Case A II**] (real E)

This case corresponds to the homoclinic (soliton) solution of Case A for real E. This is a kink soliton for E. The initial condition is determined by solving Eqs. (97) and (98) with $E(\zeta = 0) = 0$ and

$$E'(\zeta = 0) = \left[-\frac{VP}{2} - \frac{\sqrt{2}}{3}\left(\frac{V^2}{2}\right)^{3/2} + \frac{\sqrt{2}}{3}\left(\frac{V^2}{2} + P\right)^{3/2} \right]^{1/2} .$$

The symmetry of $E(-\zeta) = -E(\zeta)$ and $v(-\zeta) = v(\zeta)$ is imposed for solving Eqs. (97) and (98). With this initial kink soliton profile, we solve Eqs. (74) and (75) in time with the boundary conditions $E(0) = -\sqrt{P}$, $E(L) = \sqrt{P}$, and $v(0) = v(L) = 0$, where 0 and L are the left- and right-most coordinates of the system. An example of $V = 0.5$ and $P = 0.1$ for this case is illustrated in Fig. 6. The kink soliton propagates to the right. Towards the end, the distortion of v profile appears, which is attributable to our boundary handling.

[**Case B I**] (real E)

This case belongs to Case B as discussed in the first half of this section, $V < 0$. For the total energy in the range $0 < W < \infty$ (Case B I) the solution is periodic with two independent periods, with the shorter one arising from the small hump and the longer one from the overall well. The initial condition is once again determined by Eqs. (83) and (84) with boundary conditions $E = 0$. The periodic boundary conditions for v and E are employed. The normalization of E is again that of Eq. (96). An example is given in Fig. 7 for the parameters of $V = -5$ and $P = 1$. One discerns the non-sinusoidal oscillations due to double periods.

[**Case B II**]

This is the homoclinic (solitary wave) case of Case B of real E with $W = 0$. The initial condition for Eqs. (74) and (75) is given by the solution of Eqs. (97) and (98) with

$5V^2 - 8P \geq 0$ under the boundary conditions $E = E_0$ and $E' = 0$ at the center, where E_0 is a zero of $\Phi = 0$ between $\sqrt{P}$ and $\sqrt{Q}$. The boundary conditions are chosen to be $v(0) = v(L) = V + (V^2 + P)^{1/2}$ and

$$E(v_{\min}) = \left[Q - \frac{1}{2}(v_{\min} - V)^2\right]^{1/2} , \tag{99}$$

and $E'(v_{\min}) = 0$. An example with parameters of $V = -2$ and $P = 1$ is shown in Fig. 8. Note that the velocity v changes its sign from positive at the asymptotic points to negative at the peak of the soliton. It is observed to move to the left approximately with velocity $V(< 0)$.

[Complex E]

We numerically investigate the homoclinic (dark soliton) case of the complex E solution of type Eq. (87). Once again the initial value is chosen as the equilibrium value

$$E = R\, e^{i/2\,\theta} ,$$

$$\theta = \int_0^{\xi} d\xi\, F_0/R^2 ,$$

$$R'' = \left\{V - \left[2(Q - R^2)\right]^{1/2}\right\} R + \frac{1}{4}\frac{F_0^2}{R^3} ,$$

$$v = V - \left[2(Q - R^2)\right]^{1/2} ,$$

with $R(\xi = 0) = R_{\min}$ and $R'(\xi = 0) = 0$, where $R_{\min}$ is found from the $\tilde{\Phi}$ curve. The symmetry is imposed as $v(-\xi) = v(\xi)$, $\mathrm{Re}(E(-\xi)) = \mathrm{Re}(E(\xi))$, and $\mathrm{Im}(E(-\xi)) = -\mathrm{Im}(E(\xi))$. The temporal Eqs. (74) and (75) are solved by

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial \xi}\left(\frac{v^2}{2} + R^2\right) = 0 , \tag{100}$$

$$\frac{\partial^2 R}{\partial \xi^2} = v\, R + \frac{1}{4}\frac{F_0^2}{R^3} , \tag{101}$$

with the periodic boundary conditions for v and R and the normalization of

$$R(v_{\max}) = \left[Q - \frac{1}{2}(v_{\max} - V)^2\right]^{1/2} . \tag{102}$$

An example of a dark soliton with $V = -5$, $P = 1$, and $F_0 = 0.1$ is shown in Fig. 9.

All these solutions that started from the equilibria we discussed early in this section seem to show stable propagation in time (within our integration period), although in some cases a certain degree of degradation of the original profile of v and E is observed. In particular, the kink soliton (and, therefore, perhaps a trapezoidal soliton; i.e. a pair of kink and antikink solitons) for Case A is stably observed; the soliton in Case B is also stably observed; and the dark soliton in the complex E case is also observed to be a stable entity over the period of runs we performed.

V. Summary and Discussion

We have investigated nonlinear interaction of electromagnetic waves and acoustic modes in an electron-positron plasma. The ponderomotive force of electromagnetic waves acts on the electron-positron plasma density. The density depression of the plasma created by the EM waves in turn acts on the propagation and diffraction of the EM waves: it tends to trap the EM waves. Thus the self-trapping of EM waves and self-evacuation of the plasma result. When the typical wavelength λ of the EM waves is much greater than the collisionless skin depth c/ω_{pe}, the packet of the EM waves is virtually stationary with respect to the sound propagation (which is $c/\sqrt{3}$ in a relativistic plasma). Such a situation was called the nonresonant interaction of EM waves and acoustic waves. This process may happen explosively in two or three dimensional cases, as the phenomenon accelerates, as the density perturbation grows and further intensifies the process. Thus the nonlinear coupling of EM waves and plasma in this case can give rise to significant structure formation and the plasma density profile can be easily sculptured by the "self-attractive force" of EM waves.

When the group velocity of the EM waves and the phase velocity of the acoustic waves match, a resonant interaction (and possibly amplification) of acoustic waves by the EM waves becomes possible. This happens when the typical wavelength of the EM wavepacket is equal to the collisionless skin depth for the relativistic e^-e^+ plasma case. The coupling between the EM waves and acoustic waves has been analyzed, using the reductive perturbation theory. Our analysis led to coupled equations for the acoustic wave field v, and the EM field E, Eqs. (74) and (75). Equation (74) appears in many literatures of EM-plasma coupling. Equation (75), however, appears to be a new type of equation in this connection. This is a homogeneous equation. These two equations conserve energy. The initial and boundary value problem of this system is discussed in the Appendix. We are able to obtain stationary (or propagating) solutions to the system of Eqs. (74) and (75). They take such forms as bright solitons, shocks, trapezoidal solitons, and dark solitons, as well as periodic nonlinear wave trains. To our best knowledge, these solitons obtained for the system of Eqs. (74) and (75) are a new discovery. These solutions represent, once again, a possible significant structure formation in an electron-positron plasma. Thus in either nonresonant or resonant cases we find possible mechanisms of structure formation in the electron-positron plasma. This plasma can be said to be more plastic than the usual electron-ion plasma. This is because the former does not develop charge separation and thus no restoring force, while the latter invariably develops charge separation and this tends to saturate the above process.

Let us discuss the cosmological relativistically hot electron-positron plasma in particular. Several plasma densities characteristic of each particular physical process are examined. For the plasma to exhibit collective processes such as plasma oscillations, it is necessary to have the mean distance of plasma particles much less than the characteristic mean free path[19]:

$$\frac{1}{n^{1/3}} < (n\sigma_{\mathrm{KN}})^{-1} , \tag{103}$$

where n is the electron density and σ is the cross-sections of electron-photon collisions. Since in the relativistic plasma photons and leptons behave similarly kinematically speaking, this cross-section can be a measure of photon-photon, electron-electron etc. collisions as well. When $T \gg mc^2$, the cross-section should be, instead of the Thompson cross-section, the Klein-Nishina formula:

$$\sigma_{\mathrm{KN}} = \frac{3}{8}\left(\frac{mc^2}{\hbar\omega}\right)\sigma_{\mathrm{T}} \qquad (\text{for } \hbar\omega \gg mc^2) , \tag{104}$$

while

$$\sigma_{\mathrm{KN}} = \sigma_{\mathrm{T}} \qquad (\text{for } \hbar\omega \ll mc^2) , \tag{105}$$

where the Thompson cross-section $\sigma_{\mathrm{T}} = \frac{8\pi}{3}(e^2/mc^2)^2$. The typical energy of photons $\hbar\omega$ is, of course, T. Then the critical density n_p below which Eq. (103) is realized is

$$n_p = \pi^{-3/2}\left(\frac{\hbar\omega}{mc^2}\right)^{3/2} r_e^{-3} \sim 7\times 10^{36}\left(\frac{\hbar\omega}{mc^2}\right)^{3/2} \mathrm{cm}^{-3} , \tag{106}$$

where $r_e = e^2/mc^2$ the classical electron radius. Thus Eq. (103) is fulfilled in our plasma, where n is typically $10^{28} - 10^{33}\,\mathrm{cm}^{-3}$. For more kinetic theory, see such as Ref. 20. Similarly the plasma collisionless skin depth c/ω_{pe} for collective behavior should be shorter than the mean free path

$$\frac{c}{\omega_{pe}} < (n\sigma_{\mathrm{KN}})^{-1} . \tag{107}$$

[Note, however, that for modes with wavelengths λ much greater than $(n\sigma_{\mathrm{KN}})^{-1}$, the plasma may be regarded as a usual fluid.] More discussion on the collisionless nature of the plasma and kinetic effects is provided later. The critical density below which Eq. (107) is fulfilled is

$$n_c = \frac{8}{\pi} r_e^{-3}\left(\frac{\hbar\omega}{mc^2}\right)^2 = 5\times 10^{37}\left(\frac{\hbar\omega}{mc^2}\right)^2 \mathrm{cm}^{-3} . \tag{108}$$

The condition that the mean distance of particles is much smaller than the typical collective length (the collisionless skin depth)

$$\frac{1}{n^{1/3}} < \frac{c}{\omega_{pe}} , \tag{109}$$

yields the threshold density of

$$n_t = \frac{1}{(4\pi)^3} r_e^{-3} = 2 \times 10^{34} \mathrm{cm}^{-3} , \qquad (110)$$

below which Eq. (109) is fulfilled. Thus in our plasma the condition $n^{-1/3} < c/\omega_{pe} < (n\sigma_{\mathrm{KN}})^{-1}$ holds.

In our plasma the (average) photon energy $\hbar\omega \sim T \gg mc^2$. In our classical treatment of plasma collective mode to be relevant, we require

$$T \sim \hbar\omega \gg \hbar\omega_p . \qquad (111)$$

The density at which $\hbar\omega_p = \mathrm{mc}^2$ is given by

$$n_q = \frac{1}{4\pi} r_e^{-1} \lambda_c^{-2} = 2 \times 10^{32} \mathrm{cm}^{-3} , \qquad (112)$$

where $\lambda_c = \hbar/mc$ the Compton wavelength. Therefore most of the time we have

$$T \sim \hbar\omega \gg mc^2 > \hbar\omega_p \qquad (113)$$

in our particular epoch of $10^{-2} < t < 1\,\mathrm{sec}$ after the big bang.

So far in our investigation we have neglected collision and kinetic effects. The viscosity μ in a collisional fluid is inversely proportional to the collision frequency. When the fluid becomes less collisional, the collisional viscosity begins to lose simple meaning. In a less collisional plasma the mixing of particles, so to speak, gives rise to large viscosity. In the collisionless limit we now have interpenetrating "fluid elements"; i.e. collisionless plasma particles. Each particle preserves its memory for a long time with a straight orbit as the first approximation. Because of this, the particles now suffer Landau damping, which can play a role of effective collision. The kinetic equation under consideration is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \left[-\frac{1}{n} \nabla \mathbf{P} + q \frac{\mathbf{v} \times \mathbf{B}}{c} + q\mathbf{E} \right] \cdot \frac{\partial f}{\partial \mathbf{p}} \cong 0 , \qquad (114)$$

where the collision term on the right-hand side is neglected and the pressure term can incorporate the photon interaction through $P = \sigma^{-1/3} n^{4/3}$. An approximate dispersion relation for the phonon in $e^- - e^+$ plasmas may be given by

$$1 + \frac{4}{3} W \left(\frac{\omega}{kc} \right) = 0 , \tag{115}$$

which exhibits large Landau damping, where W is the W-function.[19] Because of the significant Landau damping for phonons with wavelength larger than c/ω_{pe}, the most important coupling would be the nonresonant (Sec. II) interaction.[18] Another distinct kinetic effect (or effect of particle nature) is the Compton scattering of photons (EM waves). The Compton scattering by plasmons becomes important when $\hbar\omega \sim \hbar\omega_p$ for densities of $n > n_q$, which can happen very early in the plasma epoch. The discussion of such topics is beyond the scope of the present paper.

In the description of fluid behavior the Reynolds number sometimes plays an important role. For wavelengths much larger than $(n\sigma_{\rm KN})^{-1}$, the plasma behaves like a usual fluid and the Reynolds number may be expressed as $\mathcal{R} = \lambda^2 \nu / \mu$, which can be much larger than unity, where ν is effective collision frequency either replaced by the Landau damping rate or other collisionless mechanisms such as the chaotic orbit effect. On the other hand, for wavelengths $\lambda \sim c/\omega_{pe} \ll (n\sigma_{\rm KN})^{-1}$, the plasma is collisionless and nearly dissipationless.

The impact of the present theory on astrophysical settings is expected, as the theory predicts a fairly stable stationary structure carved out in a relativistic plasma. Detailed examinations of the application of this theory to cosmological plasmas, AGN plasmas, pulsar plasmas, etc., however, are too much to be contained in this short article. Instead, it should be left to future astrophysical publications. For semiconductor electron-hole plasmas the process of combination into positronium has to be taken into consideration along with lattice ions and bound electronic responses.

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Appendix

We examine well-posed initial and boundary conditions to the system of Eqs. (74) and (75), where the spatial interval is $a \leq \xi \leq b$. To this end, by means of Eq. (75), we express v by E as

$$v = E_{\xi\xi}/E \ ,$$

which is introduced into Eq. (74) to give

$$(E_{\xi\xi}/E)_\tau = -\left(\frac{v^2}{2} + |E|^2\right)_\xi \ ,$$

i.e.

$$E_{\xi\xi\tau}\,E - E_{\xi\xi}\,E_\tau = -\left(\frac{v^2}{2} + |E|^2\right)_{,\xi} E^2 \equiv F$$

or

$$(E_{\xi\tau}\,E - E_\xi\,E_\tau)_\xi = F \ . \tag{75$'$}$$

Integrating this equation from a to ξ, we have

$$E_{\xi\tau}\,E - E_\xi\,E_\tau = \int_a^\xi F\,d\,\xi + [E_{\xi\tau}\,E - E_\xi\,E_\tau]_{\xi=a} \ , \tag{A-1}$$

which is formally solved for E_τ as

$$E_\tau = C\,E + E\int_a^\xi (G/E^2)d\xi + D\,E\int_a^\xi d\xi/E^2 \ , \tag{A-2}$$

where C is a function of τ and

$$G \equiv \int_a^\xi F\,d\xi \ , \tag{A-3}$$

$$D = [E_{\xi\tau}\,E - E_\xi\,E_\tau]_{\xi=a} \ . \tag{A-4}$$

If E at $\xi = a$ is given and does not identically vanish, namely $E(a,\tau) = f(\tau) \neq 0$, then

$$\frac{df}{d\tau} = C\,f \ , \qquad \text{i.e. } C = f'/f \ .$$

Therefore, when E and E_ξ are given at $\xi = a$ as functions of τ, E_τ is determined by

$$E_\tau = (f'/f)E + E\int_a^\xi (G/E^2)d\xi + (g'\,f - g\,f')E\int_a^\xi d\xi/E^2 \tag{A-5}$$

where $E_\xi(a,\tau) \equiv g$. We thus find that the initial and boundary conditions can be given in $a \leq \xi \leq b$ as

Case (I):

$\tau = 0$: $E(\xi, 0)$ is given

$\xi = a$: $E(a,\tau) \neq 0$ (does not identically vanish), $E_\xi(a,\tau)$ are given.

A simple boundary condition may be that both f and g are constant. In this case C and D vanish, and (A-5) takes the form

$$E_\tau = E\int_a^\xi (G/E^2)d\xi\ . \tag{A-5'}$$

In this case, the boundary "$\xi = a$" may be $-\infty$, so that E approaches asymptotically the constant value f as $\xi \to -\infty$. (Then $E_\xi \to 0$ i.e. $g = 0$).

If $E(a,\tau) = 0$ for $\forall\tau (f = 0)$, but $E(b,\tau)$ does not identically vanish so that $E(b,\tau) = h(\tau)$, we have

$$h' = C\,h + h\int_a^b (G/E^2)d\xi\ ,$$

hence

$$C = h'/h - \int_a^b (G/E^2)d\xi\ .$$

(D vanishes regardless of g, hence it is not necessary to specify g). Consequently, the time evolution is given by

$$\begin{aligned} E_\tau &= \left[h'/h - \int_a^b (G/E^2)d\xi\right]E + E\int_a^\xi (G/E^2)d\xi \\ &= (h'/h)E - \left(\int_\xi^b (G/E^2)d\xi\right)E\ . \end{aligned} \tag{A-5''}$$

If h is constant, Eq. (A-5$''$) reduces to

$$E_\tau = -\left(\int_\xi^b (G/E^2)d\xi\right) E\ .$$

We thus find

Case (II):

$\tau = 0:$ $E(\xi, 0)$ is given

$\xi = a:$ $E(a,\tau) = 0$ and $\xi = b:$ $E(b,\tau) = h(\tau) \neq 0$ is given.

($a \to -\infty$, $b \to +\infty$ may be assumed.) If both $E(a,\tau)$ and $E(b,\tau)$ vanish identically, C is not determined, hence the time evolution of E at $\tau = 0$, $(E_{,\tau})_{\tau=0}$, is not uniquely determined by the initial value $E(\xi, 0)$. There may be infinitely many solutions satisfying the same initial value. Unique solutions will be possible under special initial conditions only, which are examined as follows. From Eq. (75$'$) and the boundary condition it follows immediately

$$\int_a^b F\, d\xi = 0, \qquad \text{i.e.}\ \int_a^b \left(\frac{1}{2}v^2 + |E|^2\right)_{,\xi} E^2\, d\xi = 0\ , \tag{A-6}$$

where $v = E_{\xi\xi}/E$. When (A-6) holds initially, it perpetuates, provided $E_{,\tau}$ is given by (A-2) (because (A-2) yields (75$'$).) Hence (A-6) is a necessary condition in order that the boundary condition is satisfied. In the special case that E is real and v is a function of E, F becomes perfectly differential. Then in the limit $E \to 0$ as $a \to -\infty$, $b \to -\infty$, (A-6) is satisfied automatically, and it leads to the solitary waves of Fig. 3(I). The solitary waves comprise two independent parameters (V and P), which can be selected freely. (This differs from the KdV soliton). Hence by fixing these two parameters and a phase constant we have a solitary wave, which is given uniquely. If the initial value is given by the solitary wave at $\tau = 0$, the initial form moves with constant velocity V as τ evolves. That is, the solitary wave is a unique solution for the initial value. In view of Eq. (A-2), for the solitary wave we have the relation $E_{,\tau} = -V E_{,\xi}$, which makes it possible to uniquely determine C. However, if the initial state is disturbed slightly so that it is no longer a solitary wave, (though (A-6) is valid), then for the perturbed initial value, C is not uniquely determined. Therefore a continuous dependence of solutions on the initial value (given by the solitary wave) is lost. In this regard, it may be said that the solitary wave is not evolutionary, hence it is physically irrelevant. Here we note that this difficulty of non-uniqueness of solution originates not in the condition (A-6), but in the evolutional property of (A-2) (or (75′)) associated with the boundary condition. (In fact, (A-6) is required also in Case II in the limit $a \to -\infty$, $b \to \infty$, but C (solution) is unique provided (A-6) holds initially.) The periodic boundary condition also leads to the same difficulty. Therefore we conjecture that

Case (III):

$$E(a,\tau) = E(b,\tau) = 0 \qquad (a, b, \text{ may be } \pm\infty)$$

or the periodic boundary condition is ill-posed.

Finally, we examine the case of infinite region of

$$-\infty < \xi < \infty \qquad \text{and} \qquad \lim_{|\xi|\to\infty} E = \text{indefinite} .$$

In this case as a physically relevant boundary condition we assume the outgoing wave at $\xi = +\infty$ or $\xi = -\infty$, that is

$$E \to E_0 \exp i\, k(\xi - Vt) \quad \text{as} \quad \begin{array}{l} \xi \to +\infty \text{ for } V > 0 \\ \xi \to -\infty \text{ for } V < 0 , \end{array}$$

where $E_0 (\neq 0)$ is a constant. From Eq. (75) we see

$$v \to -k^2 < 0 \qquad\qquad \text{as } \xi \to \pm\infty \ (V \gtrless 0) .$$

Under this condition D at $\xi = +\infty$ or $\xi = -\infty$ vanishes. Hence (A-1) becomes

$$E_{\xi\tau}\, E - E_\xi\, E_\tau = \int_{-\infty}^{\xi} F\, d\xi \text{ or } = -\int_{\xi}^{\infty} F\, d\xi \ ;$$

consequently one has

$$E_\tau = C\,E + E\int_{-\infty}^{\xi}(G/E^2)d\xi \quad \left(\text{or } -\int_{\xi}^{\infty} F\,d\xi\right) .$$

$$\left(G = \int_{-\infty}^{\xi} F\,d\xi \qquad \text{or } -\int_{\xi}^{\infty} F\,d\xi\right)$$

Then, by means of the boundary condition at $\xi = -\infty$ or $+\infty$, C is given by

$$C = -ik\,V \quad .$$

We thus obtain the evolution equation for E. Therefore, we find the well-posed initial and boundary conditions.

Case (IV):

$\tau = 0$; $E(\xi, 0)$ is given

the boundary at $\xi \to +\infty$ or $-\infty$: the outgoing wave

$$E \to E_0\,e^{-ik(\xi - V\tau)} \qquad \text{for } V > 0 \qquad \text{or } < 0, \quad \text{respectively.}$$

The kink solution of real E (Case A, $V > 0$) is a special solution (F is perfect differential), but it belongs to (I) with $a \to -\infty$, $E \to f \cdot (\text{constant}) \neq 0$. Hence, for perturbed states, solutions are determined uniquely. Namely, in contrast to the solitary waves in Case (III) the kink solution is evolutional and physically relevant. The dark solitary wave of complex E (Case A, $V > 0$) belongs to IV. If the outgoing wave at $\xi = +\infty$ ($V > 0$) is assumed, in general at $\xi = -\infty$, the solution will is comprised of the incoming and outgoing waves. Only in a special case that solitary waves propagate at $\xi = -\infty$, the outgoing wave (namely the reflection wave) does not exist. That is, the (nonlinear) potential v in Eq. (75) is reflectionless. We thus find that the difficulty of the non-uniqueness of solution arises in Case (III) only.

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Figure Captions

1. The wavenumber space locations of photon wavepackets (a) and (b) for the nonresonant case $v_{\rm gr}^2 \ll c_s^2$. (c) The dispersion relations; the resonant photon (indicated by a dot) with the phonon branch $\omega = c_s k$.

2. The pseudopotential Φ [Eq. (85)] for Case A with real E as a function of the electric field E. The maxima at $E = \pm\sqrt{P}$ and the minimum at $E = 0$. The homoclinic (solitary) wave is realized for the Case II; i.e. $W = \Phi_M$. Case I gives a nonlinear periodic wave train, Cases III-V unbounded (unacceptable) solutions.

3. The pseudopotential Φ for Case B with real E. Case I can give two period waves as well as one period wave around $E = \sqrt{P}$ or $-\sqrt{P}$. The solitary wave is realized when $W = 0$. Case II: Periodic waves for $W < 0$, but no solitary wave possible. Case III: Solitary wave is possible for $W = 0$.

4. The pseudopotential $\tilde{\Phi}$ [Eq. (94)] as a function of amplitude R. Case A potential is in (a), Case B potential in (c), while the dark soliton for Case A is shown in (b). In both Case A and Case B nonlinear peridic waves are also possible.

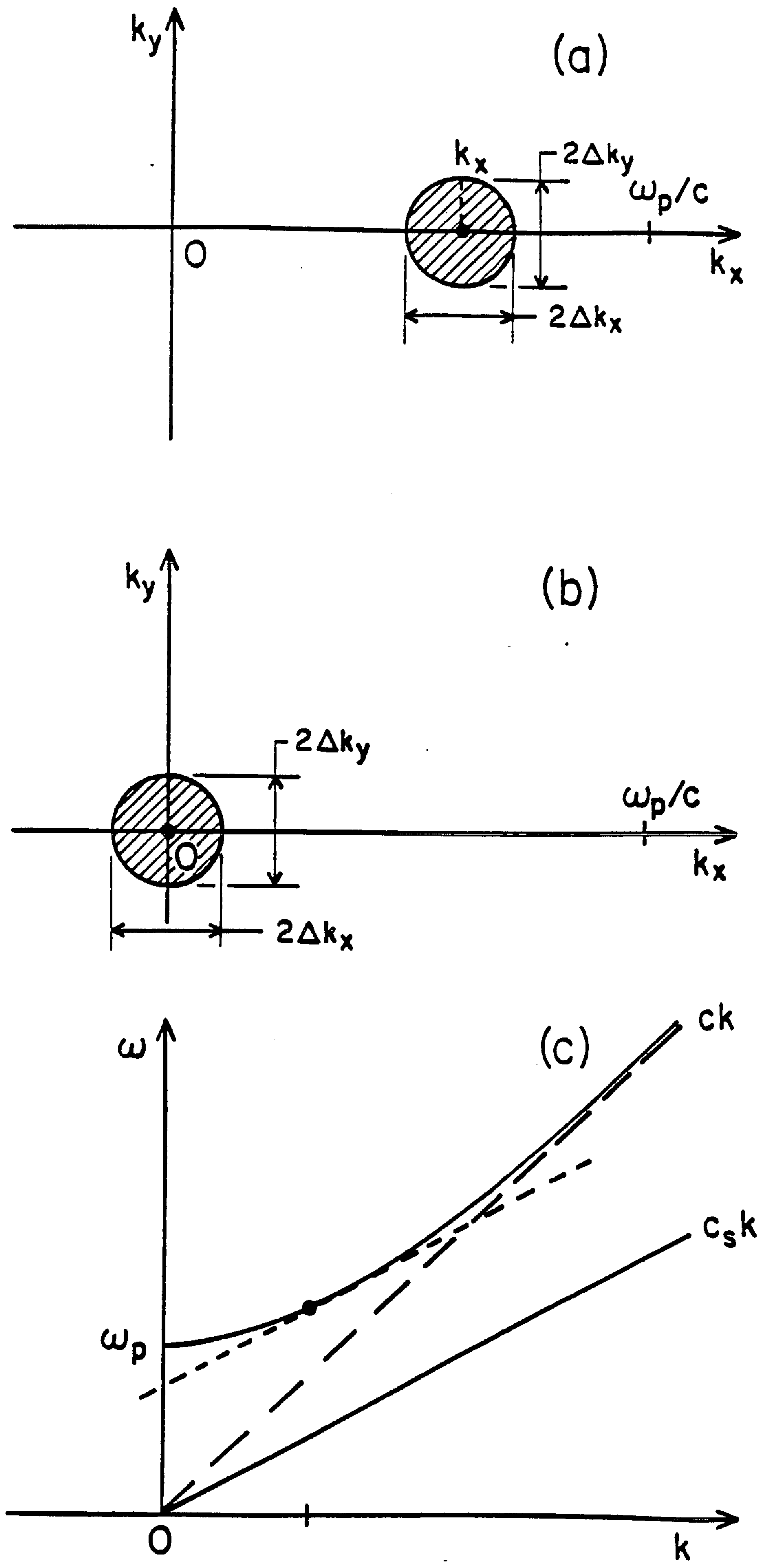
5. Numerical time integration of Eqs. (74) and (75) for v and E for Case A(I) with real E. Periodic wave case. The phase velocity of the structure was measured to be V.

6. Time integration for v and E for Case A(II) with real E. Solitary wave case (kink soliton) for E. Note that $|v|$ shows a bright soliton shape.

7. Time integration for v and E for Case B(I) with real E. The double periodicity pattern can be seen particularly in E. Note that v changes its sign.

8. Time integration for v and E for Case B(II) with real E. The bright soliton is observed. Note again that v changes its sign.

9. Time integration for v and amplitude R for Case A with complex E. The dark soliton is observed in the amplitude R of the electric field. Although $V > 0$, the solitary velocity v is locally negative with $V + v$ still being positive overall.



k_y
(a)
k_x
$2\Delta k_y$
ω_p/c
O
k_x
$2\Delta k_x$
k_y
(b)
$2\Delta k_y$
ω_p/c
O
k_x
$2\Delta k_x$
ω
(c)
ck
$c_s k$
ω_p
O
k

Fig. 1

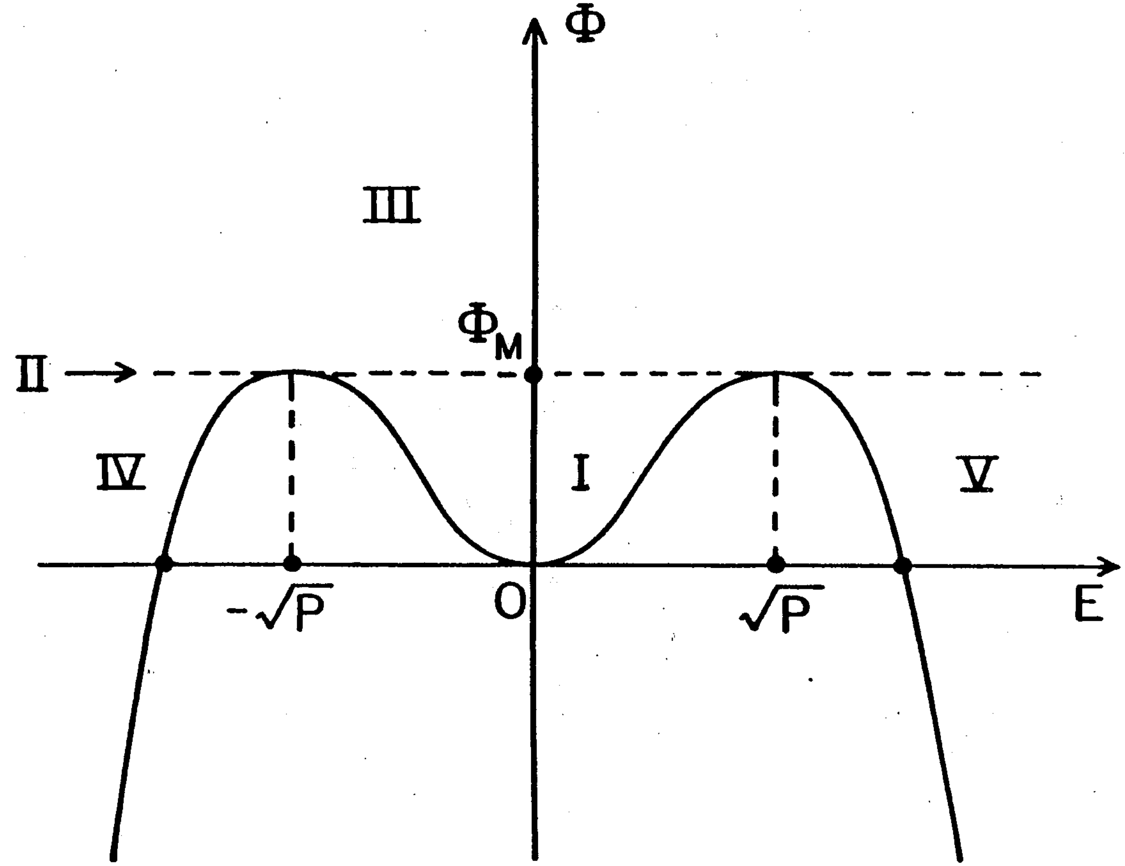


Fig. 2

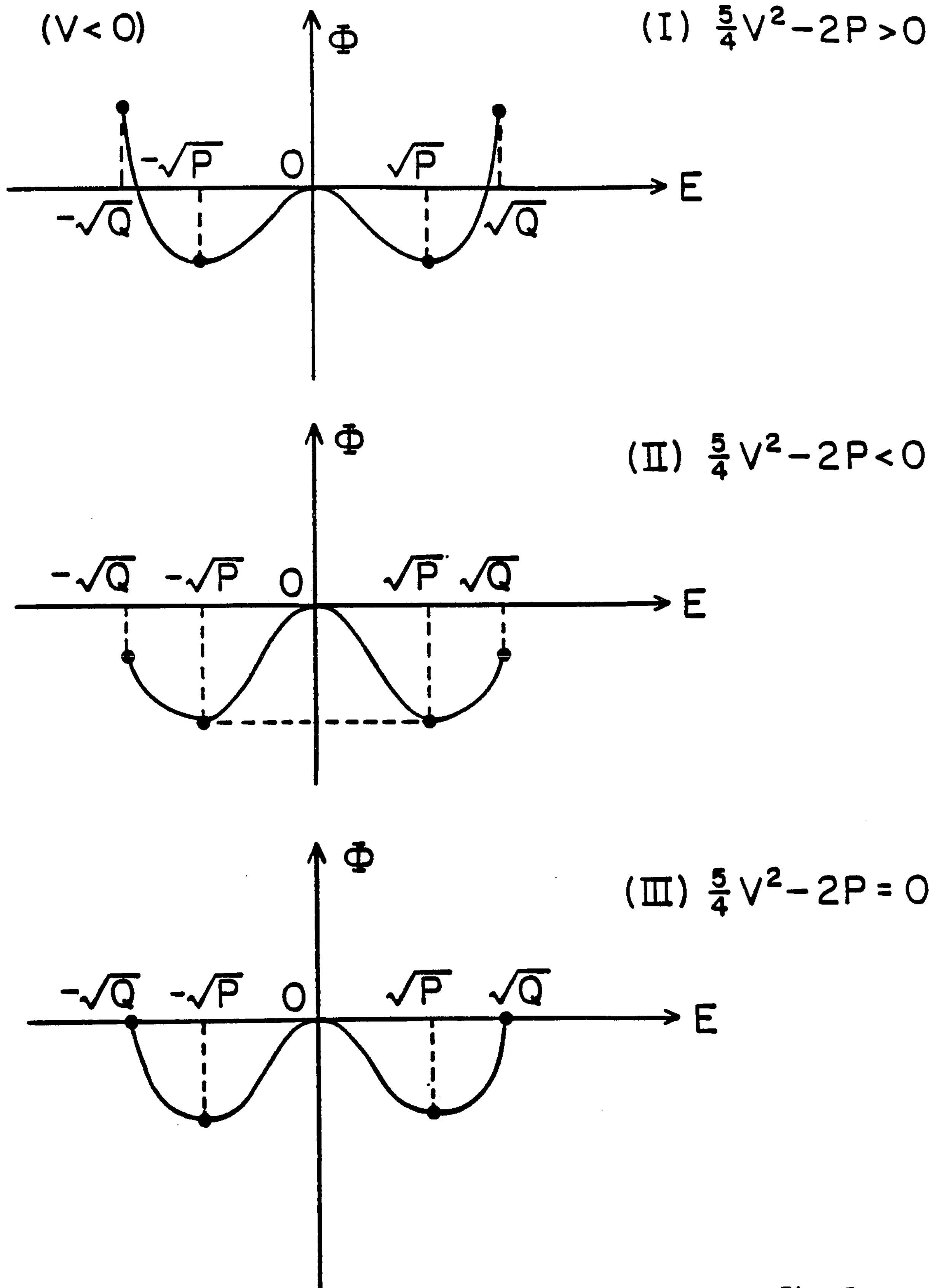
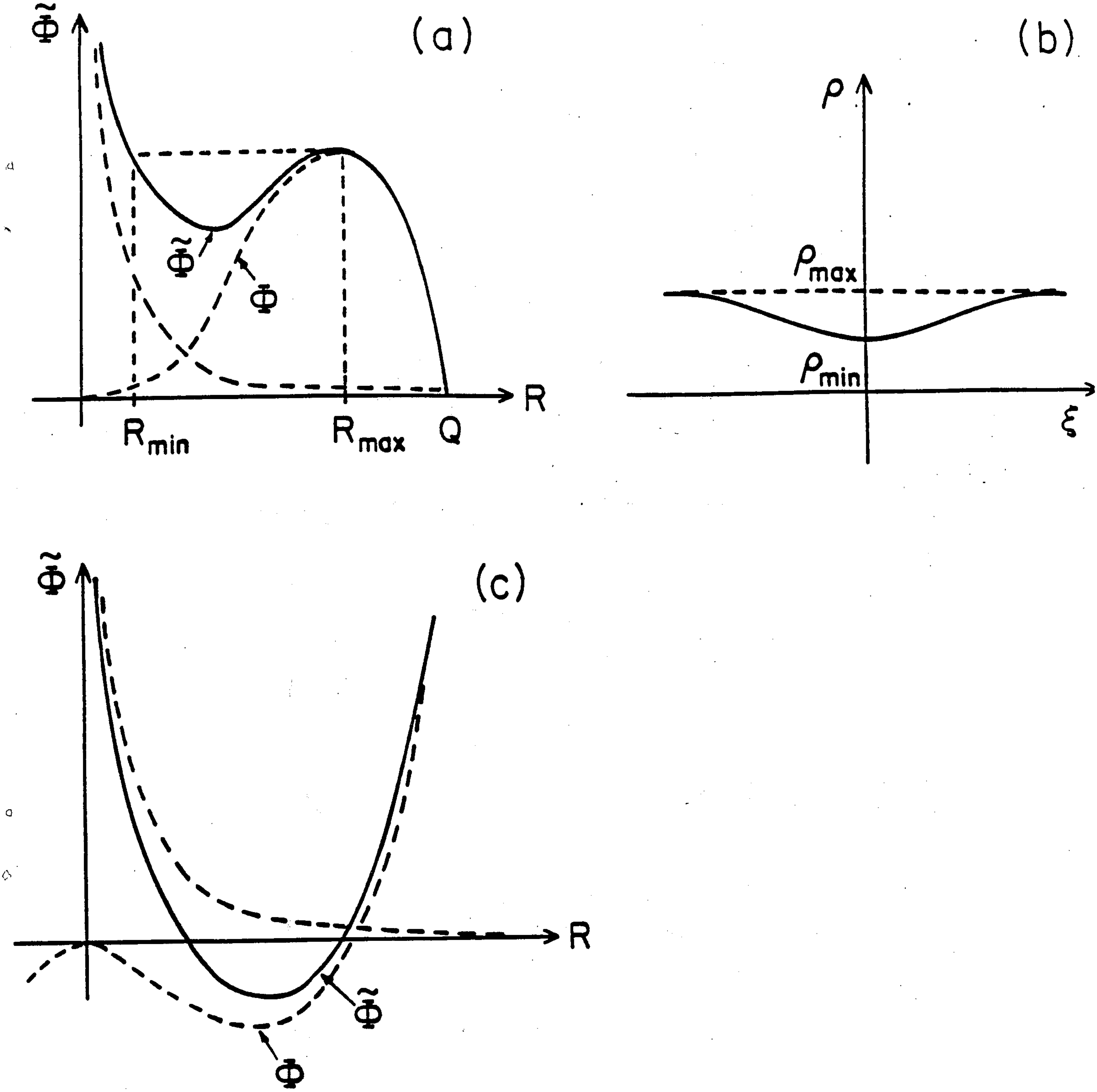

(V< O)
Φ
(I) $\frac{5}{4}V^2-2P>0$
$-\sqrt{P}$
O
$\sqrt{P}$
E
$-\sqrt{Q}$
$\sqrt{Q}$
Φ
(II) $\frac{5}{4}V^2-2P<0$
$-\sqrt{Q}$
$-\sqrt{P}$
O
$\sqrt{P}$
$\sqrt{Q}$
E
Φ
(III) $\frac{5}{4}V^2-2P=0$
$-\sqrt{Q}$
$-\sqrt{P}$
O
$\sqrt{P}$
$\sqrt{Q}$
E

Fig. 3



(a)
$\tilde{\Phi}$
$\tilde{\Phi}$
Φ
R
R_{min}
R_{max}
Q
(b)
ρ
ρ_{max}
ρ_{min}
ξ
(c)
$\tilde{\Phi}$
R
$\tilde{\Phi}$
Φ

Fig. 4

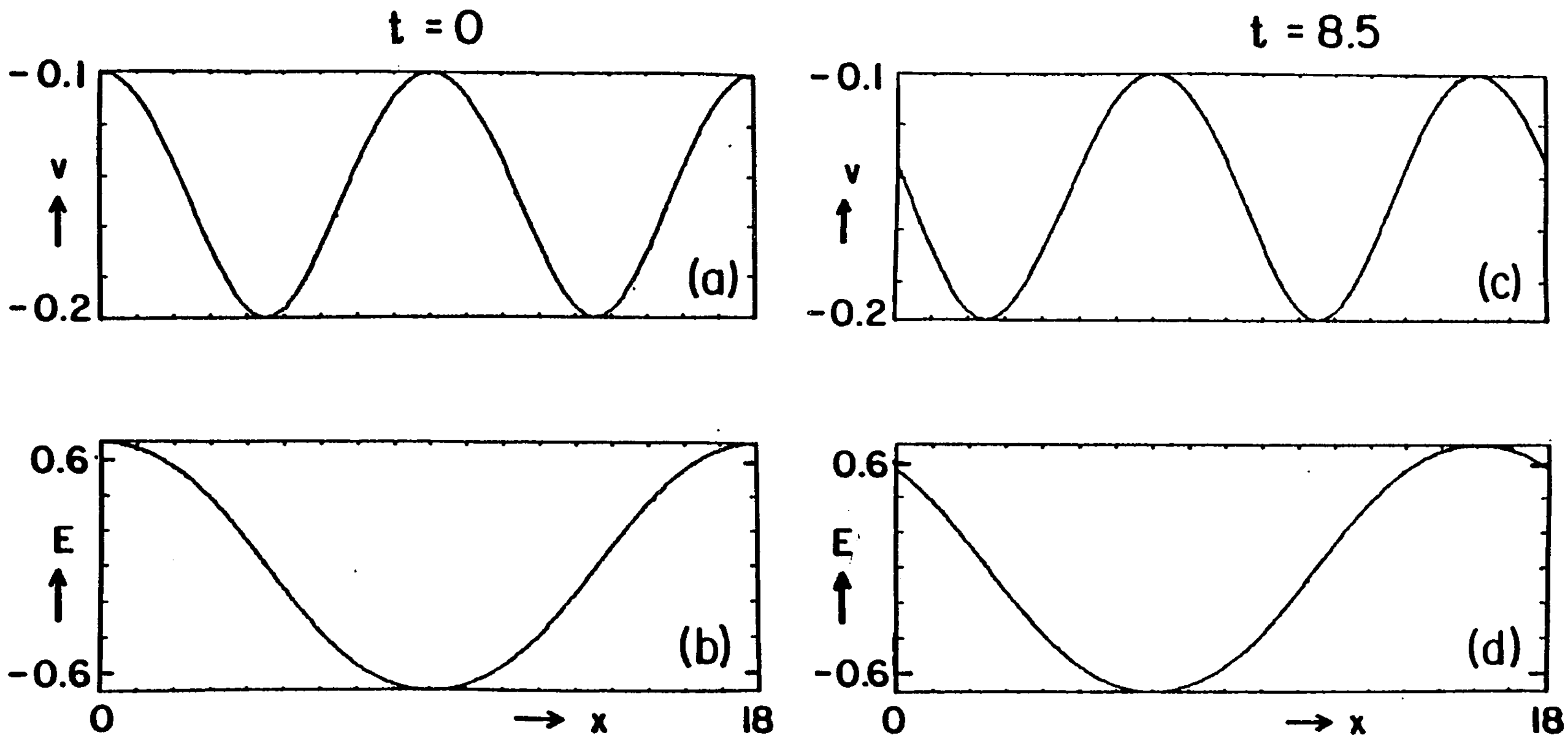


Fig. 5

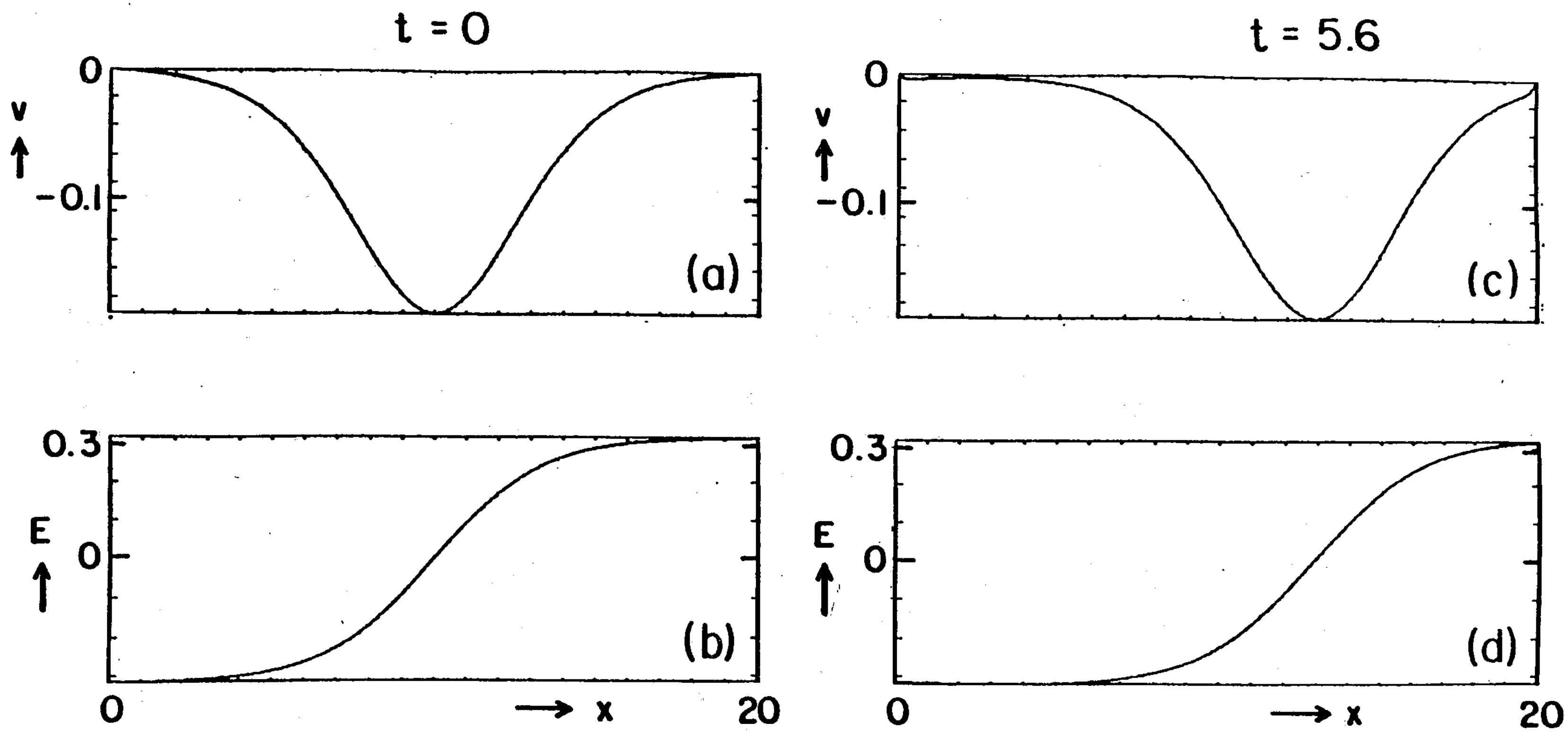


Fig. 6

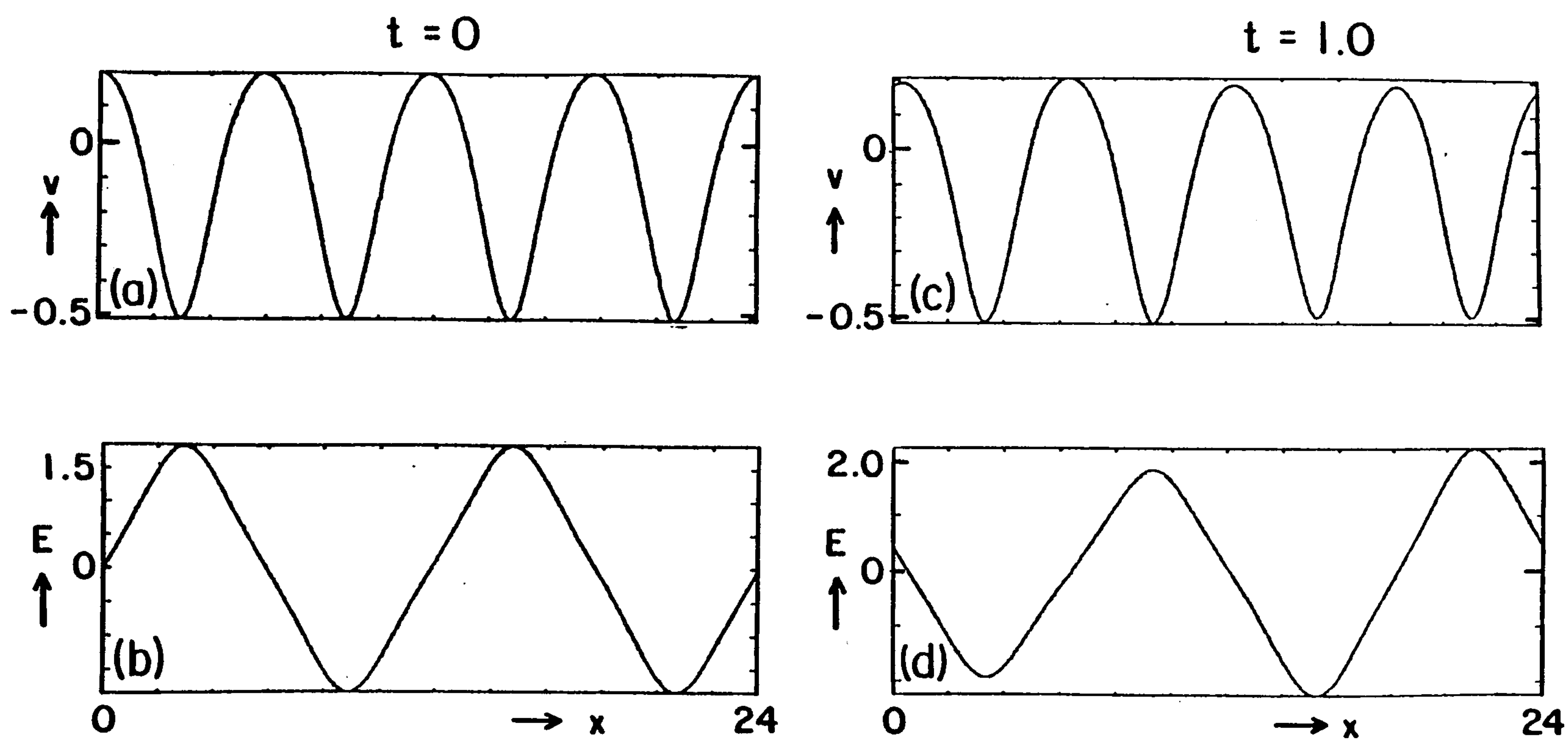


Fig. 7

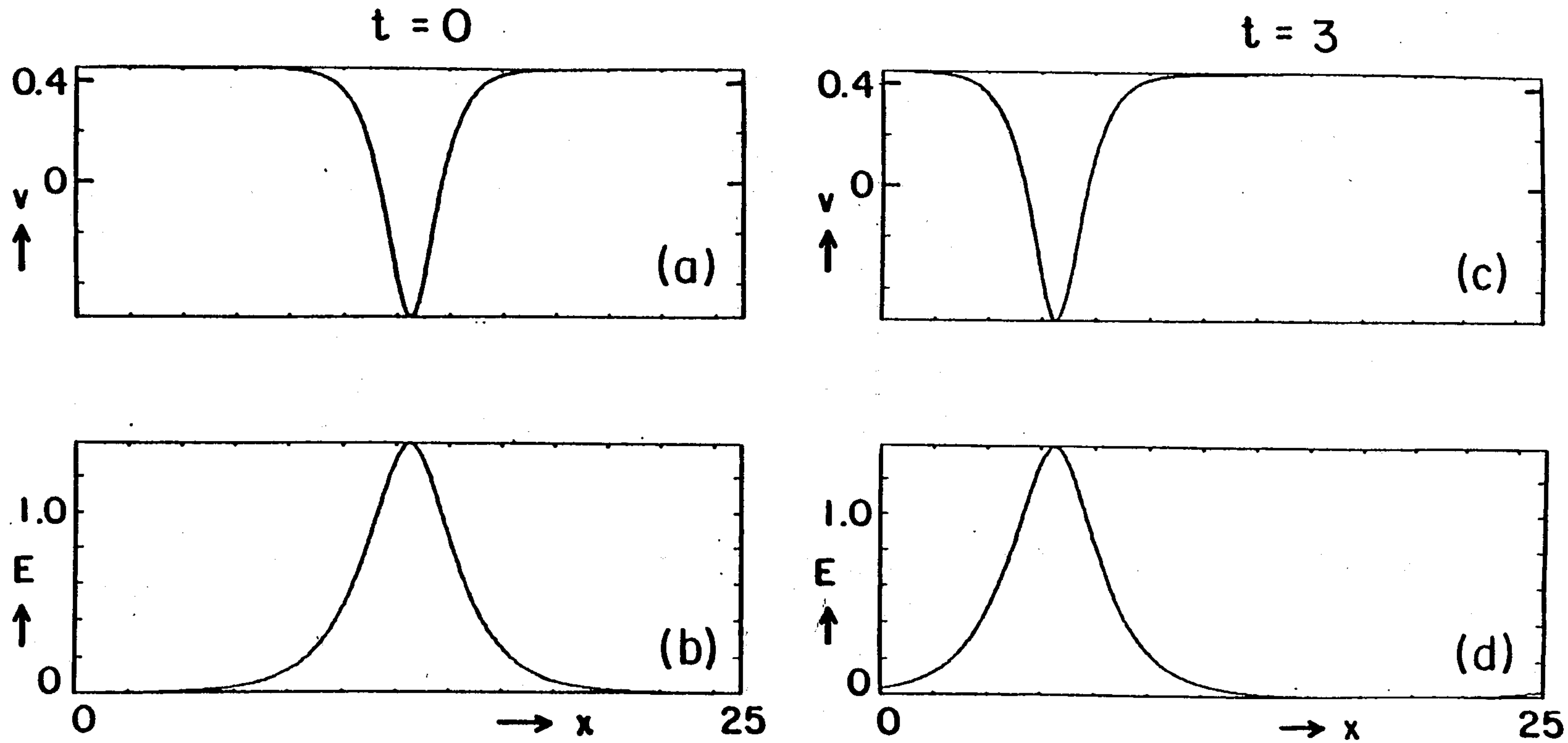


Fig. 8

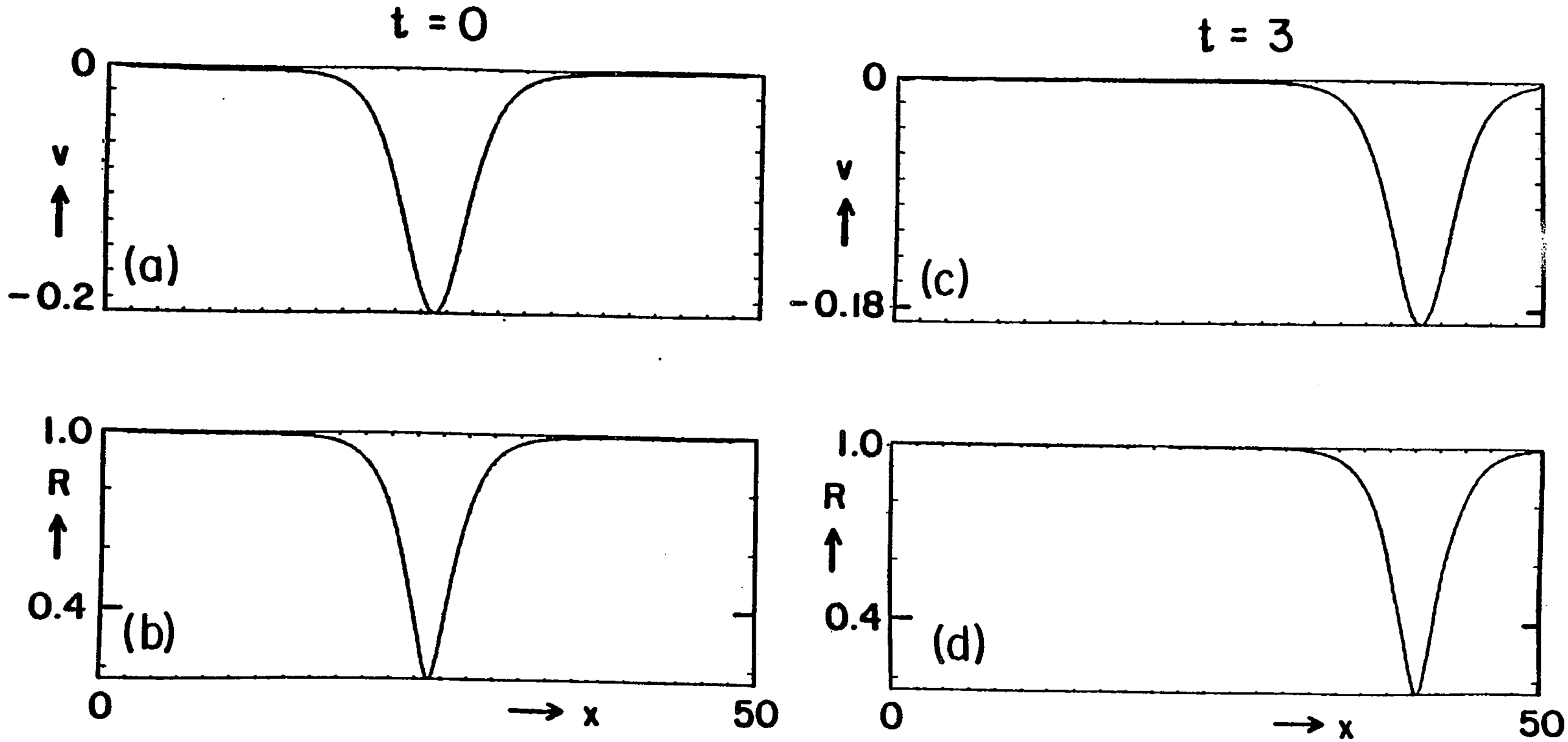


Fig. 9