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Abstract

The general expression of the power transfer from a high-energy ion beam to a background electrostatic plasma wave is obtained as a function of an arbitrary wave amplitude. The injected ions are assumed to slow down through classical transport processes and form a weakly destabilizing distribution function. It is verified that phase space gradients produced by a finite amplitude wave enhances the power transfer significantly, as was also indicated in earlier work.

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In an earlier work,[1] the expression for the power transfer from an ion beam (injected at high energy and undergoing particle annihilation) to an electrostatic plasma wave, was derived in the two limiting cases; the small (corresponding to Landau damping) and large amplitude limits. It was not possible to completely describe the power transfer rate in the transition region of the linear to nonlinear theory. In the present work, we construct a solution to this problem which holds for intermediate amplitudes as well and we explicitly demonstrate the existence of an enhanced maximum of the power transfer (relative to linear theory) at finite amplitude. This calculation demonstrates a case where the response of a nonlinear wave[2,3] can differ dramatically from simple Landau damping predictions as was also discussed in Ref. 1. If the background dissipation can be described by linear theory, the saturation level of the nonlinear wave may be much larger than the expectation of simple dimensional arguments. This arises in this problem because the source of particles establishing the steady state cannot feed particles in the trapping region. As a result a large phase space gradient arises at the separatrix of passing to trapped particles, which then causes an effective enhancement of the power transfer at a rate unrelated to the linear power transfer drive. Similar enhancements due to phase space gradients were observed in a drift wave calculation even without a particle annihilation mechanism present.[4]

We assume that a high-energy beam is injected into a background plasma where a single mode of an electrostatic plasma wave (or an ion acoustic wave) is present. The injected ions simultaneously slow down and annihilate due to classical drag and charge exchange processes and thereby form a weakly destabilizing distribution function to the background plasma wave. We also assume that the distribution of the initial velocity v_0 of the injected ions is given by $I_b(v_0) = Q\, S_I(v_0)$, with the normalization $\int_{-\infty}^{\infty} S_I(v_0) dv_0 = 1$. Considering the one-dimensional problem as in Ref. 1, we write the distribution function of the fast ions as $F(t,x,v) = \int_{-\infty}^{\infty} f\, S_I(v_0) dv_0$, where the Green's function $f = f(t,x,v;v_0)$ satisfies the

following kinetic equation

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + \frac{q\mathcal{E}}{m}\,\frac{\partial f}{\partial v} = -\nu_a f + a\,\frac{\partial f}{\partial v} + Q_0\,\delta(v - v_0)\ . \tag{1}$$

Here each fast ion with the mass m and the charge q is assumed to slow down due to collisions with the background plasma with a drag coefficient (deceleration) a and to annihilate at a rate ν_a. The electrostatic electric field $\mathcal{E}$ is taken as $\mathcal{E} = -\partial\varphi/\partial x$, with $\varphi = \varphi_0 \cos(kx - \omega t)$, where φ_0 is treated as constant under the assumption that the growth rate of the wave is small. Transforming the independent variables to $\psi = kx - \omega t (k > 0)$ and $E = \tilde{u}^2/2 - \Phi\cos\psi - \alpha\psi$, where $\Phi = q\varphi_0/m$, $\alpha = a/k$ and $\tilde{u} = v - \omega/k$, we rewrite Eq. (1) as

$$\pm u\,\frac{\partial f^{\pm}}{\partial \psi} = -\nu f^{\pm} + \frac{gQu}{k}\,\delta\left(E - \Phi\cos\psi - \alpha\psi - \frac{1}{2}u_0^2\right)\ . \tag{2}$$

where $u(\psi, E) = |\tilde{u}| = \sqrt{2(E - \Phi\cos\psi - \alpha\psi)}$, $\nu = \nu_a/k$, $u_0 = v_0 - \omega/k$,

$$g = \begin{cases} 1 & \text{if} \quad \tilde{u} \geq 0 \\ 0 & \text{if} \quad \tilde{u} < 0 \end{cases} \quad \text{and} \quad f = \begin{cases} f^{+}(\psi, E) & \text{if} \quad \tilde{u} \geq 0 \\ f^{-}(\psi, E) & \text{if} \quad \tilde{u} < 0\ . \end{cases}$$

We note that E is a constant of motion of a slowing down particle and the particle is reflected by the effective potential $\Phi_{\text{eff}}(\psi) = \Phi\cos\psi + \alpha\psi$ at the turning point ψ_t, which is defined as the minimum value of ψ_t that satisfies $u(\psi_t, E) = 0$; the other roots ψ_t of $u(\psi_t, E) = 0$ (if they exist) are inaccessible to particles born to the left of the turning points (Fig. 1). The boundary conditions of Eq. (2) are, for a fixed E, $f^{+} \to 0$ as $\psi \to -\infty$ and $f^{+} = f^{-}$ at the turning point $\psi = \psi_t$. For a given E, there are a finite number ($N(E)$) of "birth" points $\left\{\psi_b^{(i)}\right\}$ $(1 \leq i \leq N(E))$ where the delta function of the right-hand side of Eq. (2) takes non-zero values. Here $\psi_b^{(i)}$ satisfies $E = u_0^2/2 + \Phi\cos\psi_b^{(i)} + \alpha\psi_b^{(i)}$. The solution to Eq. (2) is then found to be

$$f^{\pm}(\psi, E) = \sum_{i=1}^{N(E)} \frac{Q}{k\,u|\partial u/\partial\psi|_{\psi=\psi_b^{(i)}}} \exp\left(-\nu\int_{\psi_b^{(i)}}^{\psi_t} \frac{d\psi}{u} \mp \nu\int_{\psi_t}^{\psi} \frac{d\psi}{u}\right)\ . \tag{3}$$

We now calculate the power transfer P_{total} to the fast ions from the plasma wave

$$P_{\text{total}} = \frac{kq}{2\pi}\int_0^{2\pi/k} dx \int_{-\infty}^{\infty} dv\,\mathcal{E}v\,f = \int_{-\infty}^{\infty} du_0\,P(u_0)S_I(u_0 + \omega/k)\ , \tag{4}$$

where

$$P(u_0) = \frac{\omega q \varphi_0}{2\pi} \int_0^{2\pi} d\psi \int_{\Phi\cos\psi+\alpha\psi}^{\infty} dE \, \frac{\sin\psi}{u(\psi, E)} \left(f^+ + f^- + \mathcal{O}\left(\frac{ku}{\omega}\right)\right) . \tag{5}$$

In the following, we assume that $S_I(v_0)$ is a peaked function about $v_0 = V_0$ with a relatively narrow width ΔV, where ΔV satisfies $(\Phi + \alpha)/U_0 \ll \Delta V \ll U_0 \equiv V_0 - \omega/k$. In Eq. (5) it is assumed that the contribution to P is from a narrow region in velocity space where the speed of particles is near zero in the wave frame, so that quantities of $\mathcal{O}(ku/\omega)$ are ignorable. Since $f^{\pm}$ satisfies $f^{\pm}(\psi + 2\pi,\, E + 2\pi\alpha) = f^{\pm}(\psi, E)$, we use the transformation $E \to E - 2\pi n\alpha$, $\psi \to \psi - 2\pi n$ (where n takes on successive positive integer values) and then find that we can invert the order of integration of Eq. (5) as

$$\int_0^{2\pi} d\psi \int_{\Phi\cos\psi+d\psi}^{\infty} dE \cdots = \int_{E_1}^{E_1+2\pi\alpha} dE \int_{-\infty}^{\psi_t(E)} d\psi \cdots , \tag{6}$$

where E_1 denotes a local maximum of the function $E = \Phi\cos\psi + \alpha\psi$ at $\psi = \psi_{t0}$ (as shown in Fig. 1). Using the relation

$$\int_{y_1}^{y_2} \frac{d\psi}{u} = -\frac{1}{\alpha} u \Big|_{y_1}^{y_2} + \frac{\Phi}{\alpha} \int_{y_1}^{y_2} \frac{\sin\psi}{u} d\psi ,$$

we readily obtain from Eq. (3)

$$f^+ + f^- \simeq \sum_{(i)} \frac{2Q}{ku_0 \left|\partial u/\partial\psi\right|_{\psi=\psi_b^{(i)}}} \exp\left(-\frac{\nu u_0}{\alpha}\right) \left(1 - \frac{\nu\Phi}{\alpha} \int_{\psi_b^{(i)}}^{\psi_t} \frac{\sin\psi}{u} d\psi\right) \tag{7}$$

if $\nu\sqrt{\Phi}/\alpha \ll 1$. With the use of Eqs. (5)–(7), $P(u_0)$ is calculated as

$$P(u_0) = \frac{\omega\varphi_0 q Q}{\pi k} \exp\left(-\frac{\nu u_0}{\alpha}\right) \left[\int_{I(u_0^2)} d\psi_b \int_{-\infty}^{\psi_t(E(u_0^2,\psi_b))} \frac{\sin\psi}{u(\psi, E(u_0^2,\psi_b))} d\psi \right.$$

$$\left. - \frac{\nu\Phi}{\alpha} \int_{I(u_0^2)} d\psi_b \left(\int_{-\infty}^{\psi_t(E(u_0^2,\psi_b))} \frac{\sin\psi}{u(\psi, E(u_0^2,\psi_b))} d\psi \right)^2 \right] , \tag{8}$$

where we used $E(u_0^2, \psi_b) = u_0^2/2 + \Phi\cos\psi_b + \alpha\psi_b$ and

$$\int_{E_1}^{E_1+2\pi\alpha} dE \sum_{i=1}^{N(E)} \frac{1}{u_0 \left|\partial u/\partial\psi\right|_{\psi=\psi_b^{(i)}}} = \int_{I(u_0^2)} d\psi_b$$

with the domain $I(u_0^2)$ of ψ on which the energy $E(u_0^2, \psi) = u_0^2/2 + \Phi \cos\psi + \alpha\psi$ lies between E_1 and $E_1 + 2\pi\alpha$ for a given u_0. As explained in the caption of Fig. 1, the total length of the domain $I(u_0^2)$ is 2π. In deriving Eq. (8), we used the relations $dE = (\alpha - \Phi \sin\psi_b^{(i)}) d\psi_b^{(i)}$ at each birth point $\psi_b^{(i)}$ for a fixed u_0 and $u\partial u(E, \psi)/\partial\psi = \Phi \sin\psi - \alpha$ for a fixed E. The lower limit of ψ-integral $\psi_b^{(i)}$ of Eq. (7) has been replaced by $-\infty$ in Eq. (8) since $u_0 \gg \Phi^{1/2}$.

It is easy to show from Eq. (8) that $\mathcal{P}(u_0^2) \equiv P(u_0) \exp(\nu u_0/\alpha)$ is a periodic function of u_0^2 with period $4\pi\alpha$. Therefore, the total power transfer of Eq. (4) is given by

$$P_{\text{total}} = \frac{1}{2} \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\nu\sqrt{z}/\alpha} \mathcal{P}(z) S_I(\sqrt{z} + \omega/k)$$

$$\simeq e^{-\nu U_0/\alpha} \frac{1}{2} \sum_{n=0}^{\infty} S_I(\sqrt{z_n} + \omega/k) \frac{\Delta z}{\sqrt{z_n}} \cdot \frac{1}{\Delta z} \int_{z_n}^{z_n+\Delta z} dz\, \mathcal{P}(z) \ , \tag{9}$$

where $z = u_0^2$, $z_n = n\Delta z\, (n \geq 0)$ and $\Delta z = 4\pi\alpha$. In deriving Eq. (9) we used the fact that $\exp(-\nu\sqrt{z}/\alpha)$ and $S_I(\sqrt{z} + \omega/k)/\sqrt{z}$ are slowly varying functions of z compared to the spread $U_0 \Delta V$ of the initial kinetic energy of the particles and the period $\Delta z = 4\pi\alpha$ of $\mathcal{P}(z)$, respectively. Since the average $\overline{P}$ of $\mathcal{P}(z)$ over a period Δz is independent of z_n and

$$\frac{1}{2} \sum_{n=0}^{\infty} S_I(\sqrt{z_n} + \omega/z) \frac{\Delta z}{\sqrt{z_n}} \simeq \int_0^\infty S_I\left(u_0 + \frac{\omega}{k}\right) du_0 = 1 \ , \tag{10}$$

we obtain

$$P_{\text{total}} = \frac{\exp\left(\dfrac{-\nu\, U_0}{\alpha}\right)}{2\pi\alpha} \int_{z_n}^{z_n+\Delta z} dz\, \mathcal{P}(z)$$

$$= \frac{\omega\varphi_0\, qQ}{\pi k} \frac{\exp\left(\dfrac{-\nu\, U_0}{\alpha}\right)}{4\pi\alpha} \int_{z_n}^{z_n+\Delta z} du_0^2 \int_{I(u_0^2)} d\psi_b\, g(E(y_0^2, \psi_b)) \ , \tag{11}$$

where

$$g(E(u_0^2, \psi_b)) = \int_{-\infty}^{\psi_t} \frac{\sin\psi}{u} d\psi - \frac{\nu\Phi}{\alpha} \left(\int_{-\infty}^{\psi_t} \frac{\sin\psi}{u} d\psi \right)^2 . \tag{12}$$

Transforming the variable u_0^2 of Eq. (11) to $E = E(u_0^2, \psi_b)$, we have

$$\frac{1}{2} \int_{z_n}^{z_n+\Delta z} du_0^2 \int_{I(u_0^2)} d\psi_b = \iint_\Omega dE\, d\psi_b = 2\pi \int_{E_1}^{E_1+2\pi\alpha} dE \ , \tag{13}$$

where the domain Ω of the integration is transformed to a rectangular region as illustrated in Fig. 2(a).

We consider the contribution to P_{total} from the first term of the right-hand side of Eq. (12). Averaging Eq. (11) over u_0^2 with the use of Eq. (13) leads us to consider

$$G_2' = -\frac{\Phi}{\alpha} \int_{E_1}^{E_1+2\pi\alpha} dE \int_{-\infty}^{\psi_t(E)} \frac{\sin\psi}{u(\psi, E)} d\psi \ . \tag{14}$$

We define ψ_{t1} as the coordiate satisfying $E_1 \equiv \Phi \cos\psi_{t0} + \alpha\,\psi_{t0} = \Phi\cos\psi_{t1} + \alpha\psi_{t1}$ as in Fig. 1. Note that we take $\psi_{t0} \le \psi_{t1} \le \psi_{t2} + 2\pi$ (see Fig. 1) and $\psi_{t1} = \psi_{t0}$ only if $\alpha/\Phi \ge 1$. We split the integration over ψ in Eq. (14) into the two domains $(-\infty, \psi_{t1})$ and $(\psi_{t1}, \psi_t(E))$ and exchange the order of the integrations over E and ψ. Changing the variable from E to $u = u(\psi, E)$ yields

$$G_2' = -\frac{\Phi}{\alpha} \left(\int_{-\infty}^{\psi_{t1}} d\psi \int_{u_1(\psi)}^{u_2(\psi)} du \sin\psi + \int_{\psi_{t1}}^{\psi_{t2}} d\psi \int_0^{u_2(\psi)} du \sin\psi \right) \ , \tag{15}$$

where $u_1(\psi) \equiv u(\psi, E_1)$ and $u_2(\psi) \equiv u(\psi, E_1 + 2\pi\alpha)$. Performing the integration over u in Eq. (15) and using the relations $u_2(\psi - 2\pi) = u_1(\psi)$ and $\sin\psi = \Phi^{-1}(u_1\, du_1/d\psi + \alpha)$, we obtain

$$G_2' = \int_{\psi_{t0}}^{\psi_{t1}} d\psi\, u_1(\psi) \ .$$

Clearly $G_2' = 0$ if $\Phi \le \alpha$ (when $\psi_{t0} = \psi_{t1}$). Therefore the total power transfer is given by

$$P_{\text{total}} = \left[G_1\left(\frac{\Phi}{\alpha}\right) + \beta \frac{\omega}{k\alpha^{1/2}} G_2\left(\frac{\Phi}{\alpha}\right) \right] P_{\text{L}} \ , \tag{16}$$

where $\beta = \dfrac{ak}{\nu_a \omega} \approx \mathcal{O}(1)$, and

$$G_1\left(\frac{\Phi}{\alpha}\right) = \frac{1}{\pi^2} \int_0^{2\pi} d\xi \left(\int_{-\infty}^{\psi_t(\xi)} \frac{d\psi \sin\psi}{[\xi - (\Phi/\alpha)\cos\psi - \psi]^{1/2}} \right)^2 \ , \tag{17}$$

$$G_2\left(\frac{\Phi}{\alpha}\right) = \frac{2}{\pi^2} \frac{\alpha^{3/2}}{\Phi^2} G_2' = \frac{2\sqrt{2}}{\pi^2} \left(\frac{\alpha}{\Phi}\right)^2 \int_{\psi_{t0}}^{\psi_{t1}} d\psi \left[\frac{\Phi}{\alpha}(\cos\psi_{t1} - \cos\psi) + \psi_{t1} - \psi \right]^{1/2} \ , \tag{18}$$

$$P_{\text{L}} = P_{\text{total}}(\Phi \to 0) = -\frac{\pi m \omega Q}{2} \Phi^2 \frac{\nu_a}{a^2} \exp\left[-\frac{\nu_a}{a}(V_0 - \omega/k) \right] \ . \tag{19}$$

Note that $G_1(y \to 0) = 1$, $G_1(y \to \infty) \to 64/\pi^3 y$, $G_2(y) = 0$ if $y \leq 1$ and $G_2(y \to \infty) = 16/\pi^2 y^{3/2}$. The contribution to P_{total} from G_2 agrees with Ref. 1 for all y while the contribution from G_1 agrees with Ref. 1 in the asymptotic limit $\Phi/\alpha \gg 1$. The result for $\Phi/\alpha \approx 1$ is a new result. The numerical structure of $G_1(y)$ and $G_2(y)$ is given in Fig. 3.

We note that linear theory applies if $\Phi/\alpha \ll 1$ and the power transfer changes scale when $\Phi/\alpha > 1$. For $\Phi/\alpha < 1$, $G_2 = 0$, and the power transfer rate $G_1 P_{\text{L}}$ is comparable to the predicted linear Landau damping rate. If the G_2 term were not present (which can be shown to be the case if annihilation of particles trapped in the separatrix region did not exist), the power transfer rate, due to the nonlinearity of Φ in the G_1 term, would gradually change for finite Φ/α and for large Φ/α it would be reduced by a factor $64\alpha/\pi^3\Phi$. If in addition to the destabilizing drive, a linear dissipative power transfer of the wave to the background plasma was present at a rate $G_d P_{\text{L}}$ (note $G_d < 1$ if linear instability is to occur) the level of saturation is determined by the zero power transfer condition $(G_1 - G_d)P_{\text{L}} = 0$ (if $G_d \ll 1$ the saturation level is $\Phi/\alpha = 64/\pi^3 G_d$). However, particle annihilation forces the presence of the G_2 term which completely changes the scale of the saturation level. We note that the power transfer is amplified by a large factor $\omega/k\alpha^{1/2}$. The critical point, $\Phi/\alpha = 1$, just occurs when a separatrix arises and the particles slowing down from the source are unable to penetrate the trapping region. Saturation then occurs when $\Phi/\alpha \gg 1$. Then neglecting G_1, the zero power transfer condition, $(\beta\omega\, G_2/k\alpha^{1/2} - G_d)\ P_{\text{L}} = 0$, predicts that saturation occurs when $\Phi = (16\omega\,\beta\alpha/\pi^2 k\, G_d)^{2/3}$, roughly a factor $(\omega^2/k^2\alpha)^{1/3}$, larger than would be inferred from examining parameters arising in linear theory.

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Fig. 1. The birth-point curve $E = \frac{1}{2}u_0^2 + \Phi\cos\psi + \alpha\psi$ (A) and the effective potential curve $E = \Phi_{\text{eff}} = \Phi\cos\psi + \alpha\psi$. (B). For a given E, the birth points $\psi_b^{(i)}(i = 1, 2, 3)$ and the turning point ψ_t are also shown. Here $\psi_{t2} = \psi_{t0} + 2\pi$. We note that o indicates inaccessible points ψ that satisfy $u(\psi, E) = 0$. The total length of the integral region $I(u_0^2)$ is in general 2π, which is easily seen in the special case of this figure by moving the section a to a'.

Fig. 2. The integral regions of Eq. (13). The hatched region of (A) indicates Ω, which may be transformed to the hatched region of (B) by transforming $a \to a'$, $b \to b'$ and $c \to c'$ in (A).

Fig. 3. Plots of the functions $G_1(y)$ and $G_2(y)$. For larger y, $G_1(5.0) = 0.20$, $G_2(5.0) = 0.096$; $G_1(10.0) = 0.14$, $G_2(10.0) = 0.041$; $G_1(25.0) = 0.068$, $G_2(25.0) = 0.012$. For $y = 25$ the asymptotic form is about 10% larger than the numerical value for G_2 and 20% for G_1.

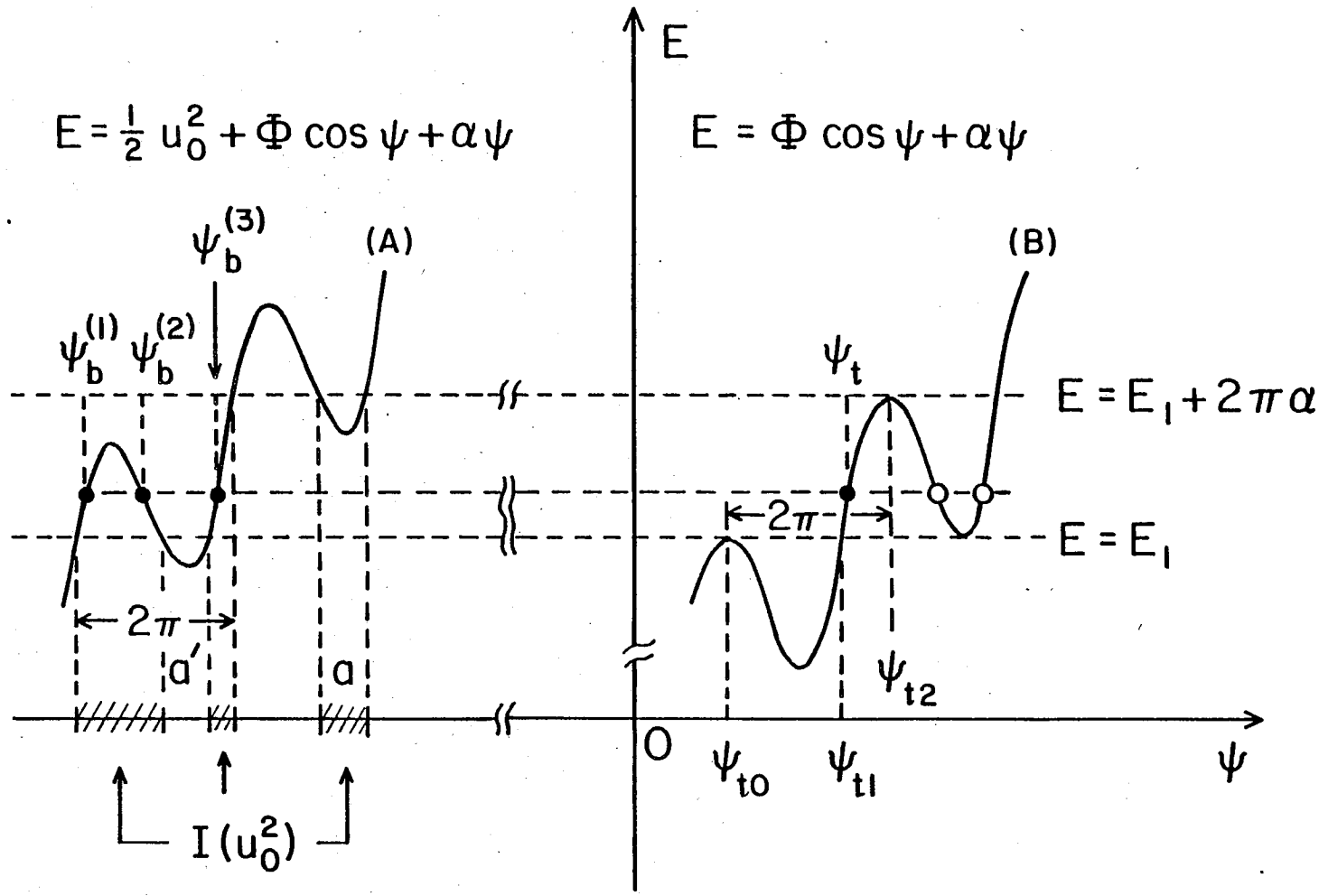
E
$E = \frac{1}{2} u_0^2 + \Phi \cos\psi + \alpha\psi$
$E = \Phi \cos\psi + \alpha\psi$
$\psi_b^{(3)}$
(A)
(B)
$\psi_b^{(1)}$
$\psi_b^{(2)}$
ψ_t
$E = E_1 + 2\pi\alpha$
2π
$E = E_1$
2π
a'
a
ψ_{t2}
O
ψ_{t0}
ψ_{t1}
ψ
$I(u_0^2)$

Fig. 1

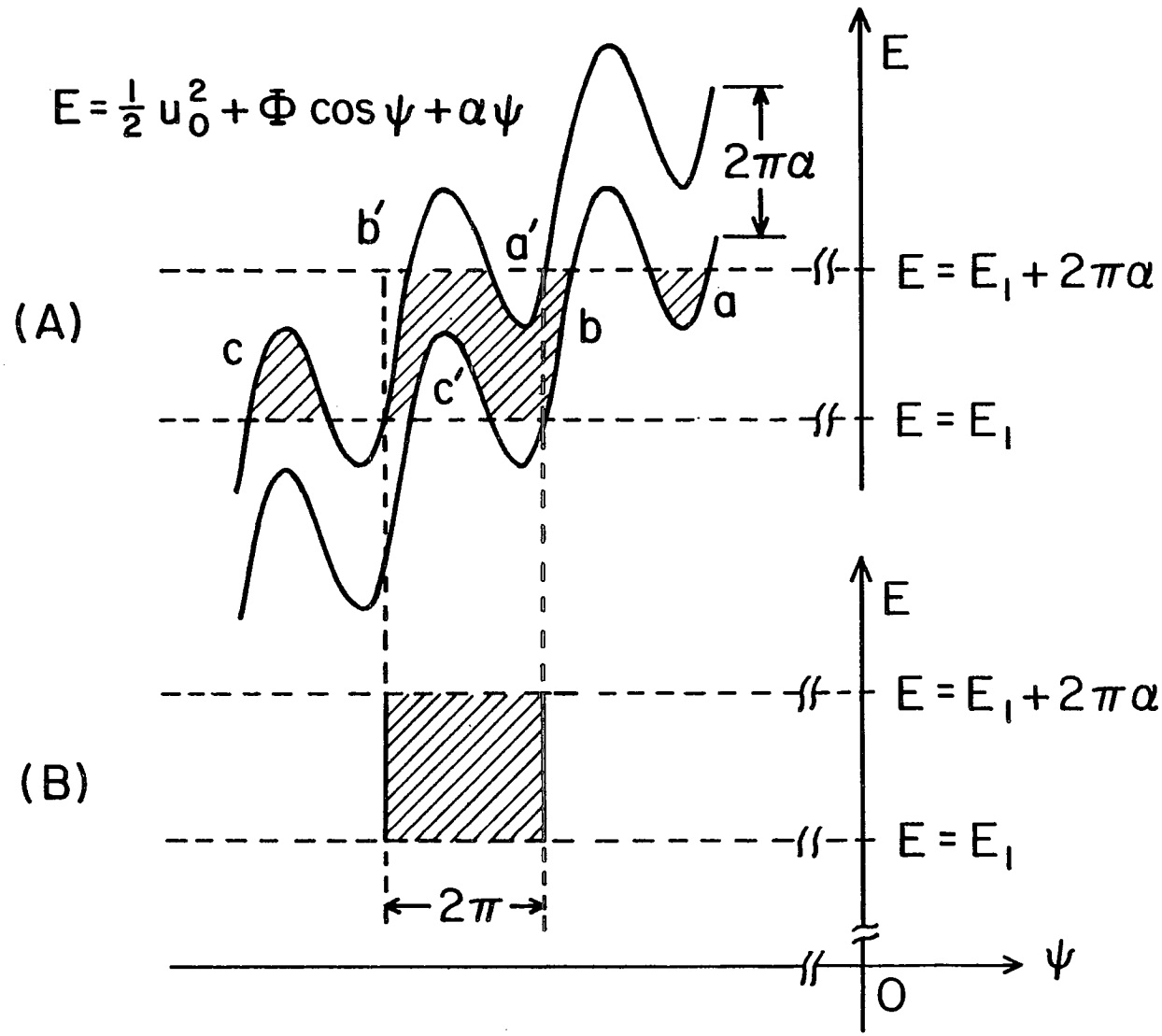
$E = \frac{1}{2} u_0^2 + \Phi \cos\psi + \alpha\psi$
(A)
(B)
E
$2\pi\alpha$
$E = E_1 + 2\pi\alpha$
$E = E_1$
a
b
c
a′
b′
c′
2π
O
ψ

Fig. 2

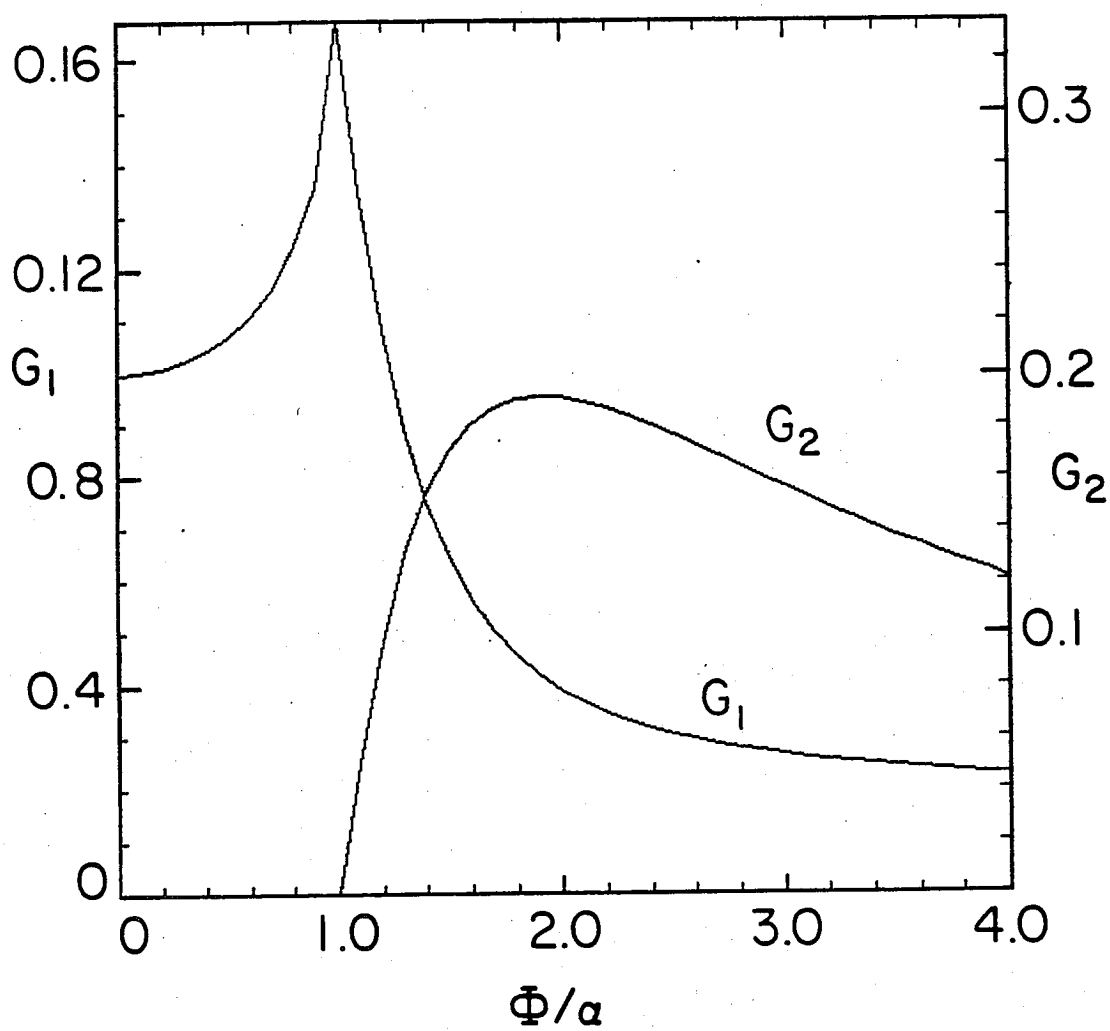

Fig. 3