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RELATION OF LINEAR RESPONSE TO NONLINEAR MOTION†

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Ponderomotive force is the time-averaged force on a particle in an oscillating field. Ponderomotive force was encountered many decades ago in the form of radiation reaction. Use of the ponderomotive force concept is widespread in plasma theory. Ponderomotive force has been proposed as a mechanism of plasma confinement[1-3]. It has been invoked as the dominant nonlinearity in wave coupling[4] and filamentation of radiation[5,6]. However, ponderomotive force calculations are typically specific to the problem at hand.

In contrast, the linear response of many physical systems has been thoroughly generalized. In plasma physics, the linear response is often studied by means of the Vlasov-Maxwell set of equations. In this system of equations, the evolution of the particle phase-space density is given by Liouville's equation,

$$\frac{\partial f}{\partial t} + \{f,h\} = 0 , \tag{1}$$

with the electromagnetic Hamiltonian

$$h(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{[\underset{\sim}{p} - e\underset{\sim}{A}(\underset{\sim}{q},t)/c]^2}{2m} + e\Phi(\underset{\sim}{q},t) .$$

We divide the electromagnetic field and phase-space density into equilibrium and perturbed parts: $\underset{\sim}{A} = \underset{\sim}{A}_0(\underset{\sim}{q}) + \underset{\sim}{A}_1(\underset{\sim}{q},t)$, etc. The equilibrium part is assumed to satisfy Eq. (1). Upon linearizing Eq. (1) we find that the perturbation satisfies

$$\frac{\partial f}{\partial t} + \{f_1,h_0\} + \{f_0,h_1\} = 0 . \tag{2}$$

The solution to Eq. (2), which may be obtained by integration along characteristics, can be used to calculate the linear response in the charge and current distributions:

$$\rho_1(\underset{\sim}{x},t) = \int d^3p\,d^3q\,f_1(\underset{\sim}{q},\underset{\sim}{p},t)\; e\delta(\underset{\sim}{x} - \underset{\sim}{q}) \tag{3a}$$

$$j_1(\underset{\sim}{x},t) = \int d^3p\,d^3q\,\Big[f_1(\underset{\sim}{q},\underset{\sim}{p},t)\; e\underset{\sim}{p}\delta(\underset{\sim}{x} - \underset{\sim}{q})/m$$

$$- f_0(\underset{\sim}{q},\underset{\sim}{p},t)\; e^2\underset{\sim}{A}_1(\underset{\sim}{q},\underset{\sim}{p},t)\delta(\underset{\sim}{x} - \underset{\sim}{q})/mc\Big] . \tag{3b}$$

Historically, these two quantities, the linear response and the ponderomotive force, have not been considered to be related, but recent work[7-9] shows that in fact they are. This work applies the methods of canonical perturbation theory to the problem of particle motion in an oscillating electromagnetic field. The result is that the time-averaged motion of the particle is given by a new quantity, the ponderomotive Hamiltonian, $K_{2\bar{\nu}}(\underset{\sim}{q},\underset{\sim}{p},t)$, in addition to the unperturbed Hamiltonian. The ponderomotive Hamiltonian is the generalization of the ponderomotive force to kinetic plasma theory. Furthermore, this analysis has uncovered a remarkable relation between the linear response and the ponderomotive Hamiltonian, viz.,

$$\int d^3q\,d^3p\,f_0(q,p)K_{2\nu}(q,p,t)$$
$$= \frac{1}{2}\left\langle\int d^3x\left[\rho_1(\underset{\sim}{x},t)\Phi_1(\underset{\sim}{x},t) - \underset{\sim}{j}_1(\underset{\sim}{x},t)\cdot\underset{\sim}{A}_1(\underset{\sim}{x},t)/c\right]\right\rangle_t , \tag{4}$$

where the angular brackets denote the time average.

For cold, homogeneous plasma this result yields the following expression for the ponderomotive potential,

$$\Phi_p(\underset{\sim}{x},t) = -\frac{1}{4\pi n}\,\underset{\sim}{\mathcal{E}}^*(\underset{\sim}{x},t)\cdot\underset{\approx}{\chi}(\omega)\cdot\underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t) \tag{5}$$

in which the electric field is assumed to have the form

$$\underline{E}(\underset{\sim}{x},t) = \underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t) + c.c. , \tag{6}$$

and the plasma species of density n has susceptibility $\underset{\approx}{\chi}$. Upon inserting the appropriate well-known susceptibilities for unmagnetized and magnetized plasma, we immediately obtain the ponderomotive potentials for unmagnetized plasma,

$$\Phi_p = \frac{e^2 |\underset{\sim}{E}|^2}{m\omega^2} ,$$

and magnetized plasma,

$$\Phi_p = \frac{e^2 |\hat{u}_z \cdot \underset{\sim}{E}|^2}{m\omega^2} + \frac{e^2 |\hat{u}_+ \cdot \underset{\sim}{E}|^2}{m\omega(\omega - \Omega)} + \frac{e^2 |\hat{u} \cdot \underset{\sim}{E}|^2}{m\omega(\omega + \Omega)} .$$

In this last equation $\Omega = eB_0/mc$ is the gyrofrequency, the magnetic field is assumed to be in the z-direction, and $\hat{u}_\pm \equiv (\hat{x} \pm i\hat{y})/\sqrt{2}$.

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