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Abstract

The possibility of using a population of energetic electrons produced in a small well-defined region as a tracer for magnetic field lines and a diagnostic for anomalous transport is examined in its simplest form. The population is produced by electron cyclotron resonance heating, and the needed power input estimated; it is detected by the emitted bremsstrahlung and the flux is calculated. If this is inadequate to permit the tracing of field lines, information can be gained by examining the spectrum, since electrons passing twice through the local heating region will gain double energy, hence near rational surfaces may be identified. The method appears promising, but open questions are indicated.

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I. Introduction

There is little question that a knowledge of the magnetic field geometry in a confined plasma is critical to understanding transport. Equilibrium requires a very special magnetic geometry—closed and nested magnetic surfaces, and any departure from this must yield excess transport; however, we actually know very little about the internal field, and any experiment which gives promise of exploring that internal field is worth examining.

One such experiment has been suggested by Guest and Dandl. It has long been known that electron cyclotron resonant heating, especially at the second harmonic, can produce, at least in mirror geometries, a local population of extremely energetic electrons, which are easily detected by the excess soft X-rays they produce as bremsstrahlung. These authors suggest that the same kind of heating in a tokamak could populate a small set of field lines with energetic electrons, and these can be followed, hence the field lines traced out, by observing the enhanced soft X-ray emission. Alternatively, if the heating is local but sustained, those electrons that pass repeatedly through the heating zone—which will happen only on rational surfaces with a small winding number—will gain large amounts of energy, and the spectrum of the emitted X-rays, rather than their spatial distribution, will give information about the flux surfaces.

Many questions are raised by this suggestion: (a) How many hot electrons are needed to provide a readily detectable signal? (b) What r.f. power is needed to produce adequate heating? (c) What is the effect of multiple passes through the heating region? (d) How many electrons are heated? (e) If the field is irregular, i.e., strays from the magnetic surfaces, how many are lost, and what effect does this have on the X-ray spectrum?

To get some preliminary answers to these questions, and to define the problems, we will explore an extremely oversimplified model of this process, which can be developed to give a good deal of information about the promise of the experiment.

II. The Model

Consider a very diffuse plasma confined to an infinite cylinder by a helical magnetic field—produced perhaps by external conductors, including a wire down the axis of the cylinder. Suppose the plasma is heated by a radiation field that fills a thin disk across the cylinder, and is repeated at fixed distances l along the cylindrical axis. Finally assume that collisions do no more than spoil any phase relations between the particles and the fields in successive heating episodes, and assume further that there is no D.C. electric field. In this, the simplest of contexts, we then will try to answer some of the questions raised above.

A. What fraction of the electrons must be heated to give an adequate X-ray signal?

To get at this we use the simple nonrelativistic form of the bremsstrahlung cross section, and write for the total flux of photons having frequency ν produced by collisions between ions, and electrons of energy $\mathcal{E}$

$$I(\nu) = n_+ n_- \tilde{\phi} \frac{m_0 c^2}{h\nu} g \log \left\{ \sqrt{\frac{\mathcal{E}}{h\nu}} + \sqrt{\frac{\mathcal{E}}{h\nu} - 1} \right\}^2 , \tag{1}$$

where g is the relative velocity on collision.

The useful measurable quantity is probably the total energy radiated above some cut-off frequency ν_0, which may be written

$$\frac{d\mathcal{E}}{dt}(\nu_0) = n_+ n_- \tilde{\phi} m c^2 \int_{\mathcal{E}=h\nu_0}^{\infty} d^3v\, v f(v) \int_{h\nu_0/\mathcal{E}}^{1} d\left(\frac{h\nu}{\mathcal{E}}\right) \log \left[\sqrt{\frac{\mathcal{E}}{h\nu}} + \sqrt{\frac{\mathcal{E}}{h\nu} - 1} \right]^2 , \tag{2}$$

which, for a Maxwellian distribution becomes

$$\frac{d\mathcal{E}}{dt}(\nu_0) = \frac{2}{\sqrt{\pi}} n_+ n_- \tilde{\phi} m c^2 v_\theta F\left(\frac{h\nu_0}{kT}\right),$$

where $\frac{1}{2} m v_\theta^2 = kT$ and

$$F(\alpha) = \alpha^2 \int_1^\infty dx\, x e^{-\alpha x} \int_{1/x}^{1} dy \log \left(\frac{1 + \sqrt{1-y}}{1 - \sqrt{1-y}} \right).$$

Integrating by parts eliminates the logarithm and reduces F to

$$F(\alpha) = 4\alpha \int_1^\infty dt\, e^{-\alpha t^2} \sqrt{t^2-1}$$
$$= -\frac{2\alpha}{\sqrt{\pi}} \left(1 + \frac{d}{d\alpha}\right) e^{-\alpha/2} K_0\left(\frac{\alpha}{2}\right).$$

This approaches $2/\sqrt{\pi}$ as $\alpha \to 0$ and for large α becomes

$$F(\alpha) \Rightarrow \frac{e^{-\alpha}}{\sqrt{\alpha}},$$

thus, for $h\nu_0 \gg kT$,

$$\frac{d\mathcal{E}}{dt} \Rightarrow n_+ n_- \frac{2}{\sqrt{\pi}} \tilde{\phi} m c^2 v_\theta \sqrt{\frac{kT}{h\nu_0}} e^{-h\nu_0/kT}.$$

If the heated electrons could also be represented as Maxwellian at a temperature T_1, the ratio of the added flux to the background becomes

$$R = \frac{\Delta\, d\mathcal{E}/dt}{d\mathcal{E}/dt} = \frac{\Delta n}{n} \frac{T_1}{T_0} \exp \frac{h\nu_0}{kT} \left(1 - \frac{T}{T_1}\right).$$

The power radiated at energies greater than 5 keV by a plasma with density $n = 10^{14}$ and a temperature of 1 keV is

$$\frac{d\mathcal{E}}{dt} = 1.6 \times 10^{15} \mathrm{eV/cm^3/sec}$$
$$= 250 \mu\mathrm{watt/cm^3},$$

and the ratio of the excess energy to background for a population at 2 keV is

$$R = \frac{\Delta\, d\mathcal{E}/dt}{d\mathcal{E}/dt} = 24.5 \frac{\Delta n}{n}.$$

B. What r.f. power is needed for a given $\Delta\mathcal{E}$?

Since the electric field induces a change in velocity of Δv where, in "circularly polarized" rotation $E_0 = \mathbf{E} \cdot \mathbf{h} = E_z$, $E_{\pm 1} = \frac{1}{\sqrt{2}} (E_x \pm i E_y)$

$$\Delta v_\sigma = \frac{e}{m} \int_0^t dt'\, E_\sigma(x', t')\, e^{i\sigma\Omega(t-t')}$$

and

$$\frac{d\mathcal{E}}{dt} = e\mathbf{E}_\sigma \cdot \Delta v_\sigma,$$

hence,

$$\Delta\mathcal{E} = \frac{e^2}{m}\int_0^t dt\, E^*_{-\sigma}\left(\bar{x},\bar{t}\right) e^{i\sigma\Omega\bar{t}} \int_0^{\bar{t}} dt'\, E_\sigma\left(x',t'\right) e^{-i\sigma\Omega t'}.$$

The trajectory integrals can be carried out in familiar fashion yielding

$$\Delta\mathcal{E} = \frac{e^2}{m}\cdot\int_0^t dt\, E^*_{-\sigma}E_\sigma\left(\bar{x},\bar{t}\right)\sum J_n J_m\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{i(m-n)\phi}\int_0^t e^{i\left[\omega+(n-\sigma)\Omega+k_{\|}v_{\|}\right]t}.$$

The phases are random variables here, hence the Bessel functions must be of the same order, while the time integral is small unless

$$\omega + (n-\sigma)\Omega + k_{\|}v_{\|} \to 0.$$

If k is small, and $\omega = 2\Omega$ this requires n and $\sigma = -1$ and if, moreover, $k_\perp v_\perp/\Omega = 2v_\perp/c$ and is small, then

$$\Delta\mathcal{E} \cong \frac{e^2}{m}\frac{k_\perp^2 v^2}{\Omega^2}E^2 t_0 t_1,$$

where t_1 is the correlation time for the field, measured along particle trajectories, and t_0 the total acceleration time. These times could be written as l_0, $l_c/v_{\|}$ where l_0 is the acceleration length, and l_c the correlation length for the r.f. field

$$l_c = \left\langle \frac{\sin\left[\left(\Delta\omega + k_{\|}v_{\|}\right) l_0/v_{\|}\right]}{\left(\Delta\omega + k_{\|}v_{\|}\right) l_0/v_{\|}} \right\rangle.$$

If detuning is small $l_c \cong l_0$ and the energy acquired becomes

$$\Delta\mathcal{E} \doteq \left(\frac{e^2}{mc^2}\right) 4\left(\frac{\mathcal{E}_\perp}{\mathcal{E}_{\|}}\right)\cdot E^2 l_0^2,$$

where, because the plasma density is low, we have assumed $\omega/k = c$.

From this the power needed to provide a significant energy gain may be determined. If $\Delta\mathcal{E} = 1\,\text{keV}$ and $\mathcal{E}_{\|} = \mathcal{E}_\perp$ then $Pl^2 \cong 175\,\text{kwatts}$.

The actual power needed depends on the system used to excite the plasma. For a resonator, P is replaced by QP where Q is the quality factor. P clearly decreases as the

acceleration and correlation lengths are increased. However, requirements seem fairly modest.

C. What is the effect of multiple passes?

The correlation times depend only on the $v_{||}$ which is unaltered by the transverse heating fields, and if collisions, while destroying phase relations between particles and fields, do not interchange $\mathcal{E}_{||}$ and $\mathcal{E}_{\perp}$ then the energy gain depends only on $\mathcal{E}_{\perp}$. Hence, if the first passage gives an energy gain of $s\mathcal{E}_0$ so that the energy after passing through the heating section becomes $(1+s)\mathcal{E}_0$ the energy after N passages becomes $(1+s)^N\mathcal{E}_0$.

If scattering reduces the ratio $\mathcal{E}_{\perp}/\mathcal{E}_{||}$ by a factor Δ between the heating sections the perpendicular energy gain becomes

$$\mathcal{E}_N = (1+s-\Delta)^N\mathcal{E}_0.$$

If the redistribution factor is enough to keep the ratio constant, the gain in total energy becomes

$$\mathcal{E}_N = (1+NS)\mathcal{E}_0.$$

It is the multiplication factor at low collisions that makes multiple heating attractive.

Note that in the absence of scattering, the fraction of the particles heated is determined by the first cycle, thereafter all the initially heated particles will accelerate in the same way, since the correlation length depends only on $v_{||}$ which is not changed by this purely transverse heating process.

D. How many electrons are heated in a pulse?

The first question to be answered here is what fraction of the electrons are in the neighborhood of the resonant field line. This depends on the bandwidth of the r.f. field, and on the

Doppler shift, $k_{||}v_{||}$. The bandwidth of the excitation is determined by requiring that

$$\left(\Delta\omega + k_{||}v_{||}\right) t_0 < \pi,$$

since the integral which determines the correlation time—or length—falls off for larger values of the argument.

The width of the resonant region then is determined by requiring that the gyrofrequency lies within this bandwidth

$$\Delta\Omega < \left(\Delta\omega + k_{||}v_{||}\right) < \frac{\pi}{t_0}.$$

From this

$$\delta r = \frac{\pi}{\Omega t_0} L_s = \frac{\pi v_{||}}{\Omega}\frac{L_s}{l_0} \cong \frac{\pi r_L}{l_0} L_s,$$

where $L_s = (B.dB/dx)^{-1}$ is the magnetic field shear length. We may use these arguments to write an expression for the number of particles that are accelerated. This is per unit axial length

$$\begin{aligned}\Delta n(r,\mathcal{E}) &= 2\pi r \delta r f\left(\mathcal{E}_{||}, \mathcal{E}_{\perp}\right)\\ &= 2\pi^2 r \frac{r_L L_s}{l_0} f\left(\mathcal{E}_{||}, \mathcal{E}_{\perp}\right).\end{aligned}$$

To find the number of particles which are accelerated to an energy $> \mathcal{E}_0$ on the first pass, we evaluate the number at $\mathcal{E} - \Delta\mathcal{E}$, then integrate from $\mathcal{E}_0 \to \infty$. If $\Delta\mathcal{E}$ is not too large, this yields

$$\Delta n\left(> \mathcal{E}_0\right) = -2\pi^2 r \frac{\bar{r}_L L_s}{l_0} \int_{\mathcal{E}_0}^{\infty} d^3v \frac{v_{||}}{v_\theta} \Delta\mathcal{E} \frac{\partial f}{\partial \mathcal{E}},$$

where

$$\Delta\mathcal{E} = \frac{4e^2}{mc^2} E^2 l_0^2 \frac{v_\perp^2}{v_{||}^2}$$

in the simplest approximation. The integral becomes

$$-\frac{1}{\pi^{3/2}}\frac{1}{kT}\int_{1/2mv^2=\mathcal{E}_0}^{\infty} \frac{dv}{v_\theta^4} v^2 \int d\Omega\, v \frac{\sin^2\theta}{|\cos\theta|} e^{-(v/v_\theta)^2} = \delta \frac{1}{\sqrt{\pi}}\frac{1}{kT}\left(|\log\mu_0| - 1\right)\cdot\left(1 + \frac{\mathcal{E}}{kT}\right) e^{-\mathcal{E}/kT},$$

where the $\log \mu_0$ is the result of a cutoff used to avoid the singularity at $v_{\|} = 0$. This singularity arose from approximating the field correlation as $l/v_{\|}$. To determine a suitable value for this we must examine the correlation time more carefully.

The correlation time was defined as

$$\left\langle \int_0^{t_0} dt\, e^{i\Delta\omega t} e^{ik_{\|} v_{\|} t} \right\rangle, \qquad \text{where} \qquad t_0 = \frac{l_0}{v_{\|}}$$

which is indeed proportional to $1/v_{\|}$ provided $v_{\|}$ is not too small. However if $k_{\|} v_{\|}$ is small enough, so that it is $\ll \Delta\omega$ then it may be neglected, the integral becomes

$$\int dt\, e^{i\Delta\omega t} d\Delta\,\omega = \int F(t) dt,$$

where F is the envelope of E, and the correlation time is determined by the bandwidth.

As a specific and relevant example consider the r.f. source as a beam propagating across the magnetic field in a wedge, with a small opening angle ϕ_0 so that $k_{\|} = k \sin\phi$ and $k_{\perp} = k\cos\phi_0$ for $\phi < \phi_0$. Then the

$$\begin{aligned} t_c &= \int_0^{t_0} dt \int \frac{d\phi}{2\phi_0} \exp i \left(\Delta\omega + k \sin\phi v_{\|}\right) t \\ &\simeq \int_0^{t_0} dt\, e^{i\Delta\omega t} \frac{\sin k\phi_0 v_{\|} t}{k\phi_0 v_{\|} t}. \end{aligned}$$

The second factor depends only on the ratio between the acceleration length $v_{\|} t$ and the field correlation length $1/k\phi_0$, and if this is small, reduces to unity, and the correlation time is determined by

$$\frac{1}{t_0} \int_0^{t_0} dt\, F(t) \int_0^t dt'\, F(t').$$

If $t_0 = l_0/v_{\|}$ is small, this does yield $l_0/v_{\|}$, since the F may be treated as constants; but if t_0 is large, this is the autocorrelation time for the driving field and is given by the reciprocal of the bandwidth $1/\langle\Delta\omega\rangle$. The minimum value of $v_{\|}$ is then $v_{\|} > \langle\Delta\omega\rangle\, l_0$, and the minimum value of μ is $\mu_0 = \frac{\langle\Delta\omega\rangle l_0}{v}$, and we approximate v by the thermal speed, and write

$$\mu_0 = \frac{\langle\Delta\omega\rangle}{2\Omega} \frac{2l_0\Omega}{v_\theta} = \left(\frac{\langle\Delta\omega\rangle}{\omega} \frac{2l_0}{r_L}\right) \ll 1.$$

Then the accelerated fraction becomes

$$\frac{n(>\mathcal{E})}{n_0} = 2\pi^{3/2} r \frac{r_L L_s}{l_0} \left[\frac{\langle \Delta \mathcal{E} \rangle}{kT} \left(2 \log \left(\frac{\omega r_L}{\Delta \omega} l_0 \right) - 1 \right) \left(1 + \frac{\mathcal{E}_c}{kT} \right) e^{-\mathcal{E}/kT} \right] \Big/ \text{unit length.}$$

This is only valid if the bracketed quantity is less than 1, since no particles are created. If $\langle \Delta \mathcal{E} \rangle \cong kT$ the fraction can be quite large. The logarithmic term, which represents the steady heating of very slow particles will be removed if the model included a steady electric field or significant scattering or if nonlinear acceleration processes are important; but it appears that a large fraction of the particles on the resonant surface could be accelerated (although only a small fraction of the total number of particles).

E. What is the effect of fluctuations in the magnetic field?

Suppose that the magnetic field is steady, but the field lines straggle slightly about the magnetic surfaces, in a random walk, characterized by the mean square angle of departure from the magnetic surface $\langle \delta\theta^2 \rangle$, and a correlation length l_B. The diffusion coefficient, yielding the rate at which particles diffuse from the surface as they move along the field line is then $D = l_B \langle \delta\theta^2 \rangle$ and the mean square displacement after traveling a distance L is $\langle \delta x^2 \rangle = L_s l_B \langle \delta\theta^2 \rangle$. When $\delta x = \delta r$ the width of the acceleration zone, $1/e$ of the particles, on average, will have left the resonant region, hence the distance travelled before acceleration stops is

$$L = \frac{\langle \delta r^2 \rangle}{l_B \langle \delta\theta^2 \rangle} = \frac{\langle \delta r^2 \rangle}{D}.$$

If the acceleration stages are separated by a distance l_0 then the number of possible stages of acceleration is $N = L/l_0$ and the maximum energy reached by a particle of initial energy $\mathcal{E}_0$ will be $\mathcal{E} = (1+S)^N \mathcal{E}_0$ and the X-ray spectrum should fall off at energies greater than this maximum evaluated at T.

The model that we started with had an azimuthally symmetric heating region, hence, all electrons on the resonant surface are heated on each pass. If, however, we were to restrict

that heating zone to a small sector, then to be heated, electrons would need not only to pass by the heating zone, but to have been on a field line whose winding number was such that it returned in the proper sector. If, for example, the azimuthal field was produced by a current flowing in a wire going down the center of the cylinder, so that $B \sim 1/r$, then with a constant axial field, the pitch of the field line would vary as $1/r^2$ and only those electrons on surfaces for which $r^2 = C/2N\pi$ with $C = \frac{B_\theta^0}{B_z^0} RL$ would be heated on every transit, thus at specific values of the frequency, where the resonant condition

$$\omega = \frac{eB_z}{mc}\left(1 + \frac{C^2}{r^2L^2}\right)$$

and the periodic condition are simultaneously satisfied, the heating rate would be maximized. If instead $r^2 = \left(\frac{M}{2\pi N}\right) C$, then heating would occur on every Mth transit, and field line diffusion would be more effective in limiting the spectrum. Observe, moreover, that the relation between the periodicity and the resonance condition depends on the field geometry, and in itself may give information about field geometry.

F. The signal

As we have seen doubling the electron energy enhances the energy radiated at 5 kT by a factor of 24/electron. If we consider the threshold at 10 keV the factor is increased to 148, although the total energy is reduced by $\sqrt{2}x$ the same factor, while the signal being reduced by $1/\sqrt{2}$. Of course, a Maxwellian implies that the distribution relaxes, but since the mean free path is very long, this requires either a long observation time or some anomalous relaxation process. In the absence of such, the collisionless treatment may be a better representation.

Indeed, in the simplest case, in which $E_{||}$ remains constant, the perpendicular energy becomes after N passes

$$E_\perp^+ = E_\perp^0 \left(1 + \frac{S}{E_{||}}\right)^N$$

and the distribution function may be obtained by writing it as the original function, Maxwellian—in our case—with the original variables given in terms of their modified values,

so that

$$f\left(E_{\|},E_{\perp}\right)d^3v = f_0\left(E_{\|},\frac{E_{\perp}}{\left(1+S/E_{\|}\right)^N}\right)\frac{d^3v}{\left(1+S/E_{\|}\right)^N}.$$

To evaluate the radiation produced we must form $f(\mathcal{E})$, i.e., integrate over the angles. Normalizing all energies to T yields

$$f(\mathcal{E})d\mathcal{E} = \frac{2}{\pi^{3/2}}\oint d\varphi\int_{-1}^{1}d\mu\, d\mathcal{E}\frac{\mathcal{E}\exp\left(-\mathcal{E}\left[\mu^2+(1-\mu^2)/(1+S/\mathcal{E}\mu^2)^N\right]\right)}{(1+S/\mathcal{E}\mu^2)^N}.$$

As we have observed, the E dependence in the energy gain saturates at a minimum value of

$$\begin{aligned}\mu=\mu_0(v) &= \frac{\langle\Delta\omega\rangle\, l_0}{v}\\ \mu_0^2(v) &= \frac{\langle\Delta\omega^2\rangle\, l_0^2}{v^2}\\ &= \frac{\langle\Delta\omega^2\rangle\, l_0^2}{v_\theta^2}\frac{kT}{E}=\frac{\mu_0^2}{\mathcal{E}}.\end{aligned}$$

We may then approximate the integral over μ by steepest descents, noting that the term in the exponent is a decreasing function of μ and that its second derivative is approximately $2\mathcal{E}$. Thus we replace μ by its minimum value, and write the integral as

$$f(\mathcal{E})d\mathcal{E} = 2\sqrt{2}d\mathcal{E}\sqrt{\mathcal{E}}\left(\frac{\mu_0^2}{\mu_0^2+S}\right)^N\exp\left[-\mu_0^2-\mathcal{E}\left(\frac{\mu_0^2}{\mu_0^2+S}\right)^N\right].$$

The radiated power then becomes

$$\frac{d\mathcal{E}}{dt} = n_+\Delta n_-\tilde{\phi}mc^2v_\theta 2\sqrt{2}e^{-\mu_0^2}\left(\frac{\mu_0^2}{\mu_0^2+S}\right)^{-N}G\left(\frac{h\nu_0}{kT}\left(\frac{\mu_0^2}{\mu_0^2+S}\right)^N\right),$$

where

$$\begin{aligned}G(\alpha) &= \alpha^2\int_1^\infty dx\sqrt{x}e^{-\alpha x}\int_{1/x}^1 dy\log\left(\frac{1+\sqrt{1-\mu}}{1-\sqrt{1-\mu}}\right)\\ &= \sqrt{\pi\alpha}e^{-\alpha}\left[1-\alpha e^\alpha\int_\alpha^\infty e^{-x}\frac{dx}{x}\right].\end{aligned}$$

For large α, $G\to\sqrt{\pi/\alpha}e^{-\alpha}$ while for α small,

$$G\to\sqrt{\pi\alpha}(1+\alpha\ln\gamma\alpha):\gamma=1.78\,.$$

Thus for large α, $G = \sqrt{2\pi}F$ and if the temperature ratio is taken as $\left(\frac{\mu_0^2+S}{\mu_0^2}\right)^N$,

$$R = \frac{\Delta\, d\mathcal{E}/dt}{d\mathcal{E}/dt} = 2\sqrt{2}\pi e^{-\mu_0^2}\frac{T_1}{T_0}e^{\frac{h\nu_0}{kT}\left(1-\frac{T_0}{T_1}\right)}.$$

For $h\nu_0/kT = 5$ and $T_1/T = 2$ this ratio is $R = 170\Delta n/n$ the total power being 42 milliwatt/cm^3.

The maximum energy is determined essentially by the bandwidth, for as the energy increases, the gyrofrequency shifts by a factor

$$\frac{\Delta\Omega}{\Omega} = \frac{\Delta E}{E_0}.$$

For TEXT with $r = 3.810$ cm, and with a correlation length of 1 cm, a bandwidth of .002 permits an energy gain of 1 keV, and yields $= 0.4$. With a gain factor of $S = 0.16$ a single pass doubles the energy, and for a multiple pass heating, the gain/pass would need to be significantly less than this.

For the model we have picked, with the azimuthal field provided by a current I carried along the axis by a wire, so that $B_\theta = 2I/r$, $B_z =$ const. then the pitch of the field line is $d\theta/dz = B_\theta/B_z = 2I/B_z r$ and if L_D is the axial distance between accelerating regions, the rotational transform in going between these is

$$= 2IL_D/B_z r^2.$$

If the accelerating region is not symmetric in azimuth, but confined to a narrow sector, then the rotational transform must be $2n\pi$ if a second acceleration is to occur. At a radius $r = r_0$, $r_0^2 = 2IL_D/2\pi B_z$, $\delta\theta = 2\pi$ so that a particle passes through the next accelerator having gone a distance

$$\begin{aligned} d_0 &= L_D\left[1 + \frac{4\pi I}{B_z L_D}\right]^2 \\ &= \left[L_D^2 + (2\pi r_0)^2\right]. \end{aligned}$$

There are other resonant radii, $r_{m,n} = r_0\sqrt{m/n}$ for which a particle will pass through the nth accelerating sector after having made m revolutions about the axis, and travelling a distance

$$d_{n,m} = nL_D\left[1 + \frac{4m\pi I}{B_z L_D}\right]^{1/2}.$$

It is however clear that if the pulse length is more than a few microseconds, most of the low order resonant surfaces will be saturated—indeed if the pulse length is 100 microseconds the particle can travel for 1000 m and with $L_D = 5$ m, then all particles on rational surfaces with $n < 10$ will be accelerated to saturation, and only the frequency can discriminate between surfaces.

If the field lines cannot be traced, the modification to the overall spectrum is obtained by multiplying the flux from the accelerated particles by the fraction accelerated. If acceleration occurs at $r = r_0$ and the total radius is R then this fraction becomes

$$f = \frac{2\pi^2 r_0}{\pi R^2} r_L \frac{L_s}{l_0}\frac{\Delta\theta}{2\pi},$$

where $\Delta\theta$ is the width of the heated sector. If $R = 30$ cm, $r_0 = R/2$, $L_s = 100 l_0$, $r_L = .02$ cm, and $\Delta\theta = 0.1(2\pi)$, then since the flux at $h = 5$ keV is 170 times background, the signal should be easily recognized for higher energies.

This is an indication of how accelerated electrons might be used for investigating magnetic field structures. If the X-ray intensity is great enough, it may be possible to visualize the magnetic field lines directly, but if this is not possible, the effects of multiple transits through the heating zone produce field geometry dependent modifications to the X-ray spectrum.

To improve the analysis, we should consider those effects that might limit the energy gain, the simplest of which is the effect of interparticle collisions.

Very roughly, collisions have three consequences—they exchange energy between parallel and transverse motion, thus reducing the heating—they induce diffusion across the field lines, just as do magnetic fluctuations. Finally, if the magnetic configuration is to be maintained, and requires a plasma current as is the case in a tokamak, then a steady electric field is needed

and this can induce an independent population of fast electrons—the runaway population—which itself can be modified by the ECRH.

III. Collisions

We content ourselves here with the simplest linear treatment of collisions. In this approximation, the distribution function is written as $f = f_0 + f_1$ where f_0 is taken as Maxwellian, hence is not affected by collisions. The Fokker-Planck equation then becomes

$$\frac{\partial f_1}{\partial t} + v_{\|}\frac{\partial}{\partial z}f_1 - \frac{\partial}{\partial \mathbf{v}}\left(\mathbf{F} + \mathbf{D}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f_1 = 0$$

with the drag and diffusion coefficients given by $F, D = \Sigma F_i, D_i$

$$\begin{aligned}
\mathbf{F}_i &= \frac{2}{\sqrt{\pi}}\nu_i \mathbf{v} G\left(\frac{v}{v_\theta}\right) \qquad \nu_i = \frac{4\pi \log \Lambda}{M_r^2 v_\theta^3} \\
D_i &= \frac{2}{\sqrt{\pi}}\nu_i \frac{v_\theta^2}{2}\left[\frac{1}{2}(\mathbf{1} - \hat{\mathbf{v}}\hat{\mathbf{v}}) H\left(\frac{v}{v_\theta}\right) - \frac{1}{2}(1 - 3\hat{\mathbf{v}}\hat{\mathbf{v}}) G\left(\frac{v}{v_\theta}\right)\right]
\end{aligned}$$

the sum being over the target species, to which the thermal speeds and reduced masses also refer. If the electron energy significantly exceeds the ion thermal speed, the ion contributions simplify to

$$\begin{aligned}
F &= \frac{4(\pi)\log\Lambda}{M^2 v^3}\mathbf{v}, \qquad \text{since} \qquad G \to \frac{\sqrt{\pi}}{2}\frac{1}{x^3} \\
\mathbf{D} &= \frac{2\pi \log \Lambda}{M^2 v}\frac{1}{2}(\mathbf{1} - \hat{\mathbf{v}}\hat{\mathbf{v}}).
\end{aligned}$$

We will be mostly interested in the scattering of energetic electrons, and if the electron velocity is greater than about twice the thermal speed, the electron coefficients also simplify. Since the ion coefficients are now independent of temperature, while the reduced masses are m (the electron mass) in the ion terms and $m/2$ in the electron terms, the two terms may be combined, written in terms of the electron collision frequency

$$\nu = \frac{4\pi e^4 \log \Lambda}{m^2 v_\theta^3}$$

and the electron thermal speed to which v_θ now refers. The Fokker-Planck equation becomes

$$\frac{\partial f}{\partial t} + v_{\|}\frac{\partial f}{\partial z} - 5\nu\frac{\partial}{\partial \mathbf{v}}\cdot\left\{\left(\frac{v_\theta}{v}\right)^3 \mathbf{v} + \frac{1}{4}\frac{v_\theta^3}{v}(1-\hat{v}\hat{v}) - \frac{4}{5}\left(\frac{v_\theta}{v}\right)^2 [1-3\hat{\mathbf{v}}\hat{\mathbf{v}}]\frac{\partial}{\partial \mathbf{v}}\right\} f = 0.$$

We use this first to determine the relaxation rate—i.e., how rapidly the ratio $\mu^2 = \mathcal{E}_{\|}/\mathcal{E}$ relaxes to its equilibrium value $= 1/3$.

Writing $f = f_0\mu$ reduces the Fokker-Planck equation to

$$f_0\left[\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial z}\right)\mu^2 - \frac{5\nu_-}{4}\frac{v_\theta^2}{v}(1-\hat{\mathbf{v}}\hat{\mathbf{v}})[H-G] : \frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}}\mu^2\right] = 0$$

for

$$\mathbf{D}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = -\mathbf{F} f_0 \qquad \text{and} \qquad \frac{\partial}{\partial \mathbf{v}}\mu^2 \perp \hat{\mathbf{v}}.$$

Moreover,

$$(1-\hat{\mathbf{v}}\hat{\mathbf{v}}) : \frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}}\mu^2 = \frac{1}{v^2}\left(1-3\mu^2\right)$$

hence

$$v_{\|}\frac{\partial}{\partial z}\left(1-3\mu^2\right) + \frac{15}{4}\nu_-\left(\frac{v_\theta}{v}\right)^3\left(1-3\mu^2\right) = 0.$$

Hence the relaxation length becomes

$$z \simeq \left(\frac{v}{v_\theta}\right)^4 \cdot \bar{\lambda}_f = \left(\frac{v}{v_\theta}\right)^4 \frac{4}{15}\frac{v_\theta^-}{\nu} = \frac{4}{15}\left(\frac{v}{v_\theta}\right)^4 \lambda,$$

where λ is the mean free path.

For a density of 1×10^{13} and a temperature of 1 keV this becomes about 4000 meters, roughly 670 transits of the TEXT tokamak and except for very large winding numbers is unimportant. Cross-field diffusion is determined by the diffusion coefficient

$$D_c = \nu r_L^2 \rightarrow \frac{r_L^2}{\lambda}\left(\frac{v_{\|}}{v_\theta}\right)^4 \text{ for spatial diffusion}$$

and this means that a minimum magnetic field fluctuation is needed in order to dominate the loss of heated electrons. Comparing collisional diffusion with the field diffusion in Sec. II.E requires

$$\left\langle \delta\theta^2 \right\rangle > \frac{r_L^2 v_\theta^4}{l_B \lambda v_{\|}^4}.$$

IV. Runaways

Finally we turn to the effect of a D.C. electric field. For fast electrons, the motion is determined by the electric acceleration and the drag, so that the equation of motion becomes

$$\frac{dv_{||}}{dt} + \nu_- \left(\frac{v_\theta}{v}\right)^3 v_{||} = \frac{e}{m}E.$$

If we consider the case for which $v_\perp = 0$ and $v_{||}$ is large enough initially, we obtain

$$v_{||}\frac{\partial v_{||}}{\partial z} + \nu_- \frac{v_\theta^3}{v_{||}^2} = \frac{e}{m}E; \qquad x\frac{dx}{dz} + \frac{1}{x^2} = C$$

which can be integrated to yield

$$\frac{1}{2}mv_{||}^2 - eEz = K - \frac{kT}{C}\log\left(\frac{Cx^2-1}{Cx_0^2-1}\right),$$

where

$$C = \frac{1}{2}\frac{eE\lambda_f}{kT}; \qquad x = \frac{v_{||}}{v_\theta}.$$

If on the other hand $v_\perp$ is large initially the motion in $v_{||}$ is given by

$$\frac{d}{dt}v_{||} + \nu_- v_\theta^3 \frac{v_{||}}{\left(v_\perp^2 + v_{||}^2\right)^{3/2}} = \frac{e}{m}E$$

which can be written as

$$x\frac{dx}{dz} + \frac{x}{(1+x^2)^{3/2}} = \bar{C} = C\frac{v_\perp^2}{v_\theta^2}.$$

The function $x/\left(1+x^2\right)^{3/2}$ has its maximum at $x^2 = 1/2$ where its value is $2/\sqrt{27} = 0.385$, and if $\bar{C}$ exceeds this, then any electron will run away in the D.C. field. If $\bar{C}$ is less than this there will be a minimum parallel velocity needed for runaway, given for small $\bar{C}$ by $x = 1\big/\sqrt{\bar{C}}$.

Again the equation of motion may be integrated, although that operation is a little complex, with the result

$$\frac{1}{2}mv_{||}^2 - eEz = K - F\left(x, \bar{C}\right),$$

where the energy loss F is again logarithmic for large values of its argument, which is now $\left(v + v_{\|}\right)/v_{\theta}$.

In TEXT, however, the field amounts to only a few volts/transit, and again, except for very large winding numbers, the parallel energy gain is not important, although the increase in runaway population may be dramatic.

In TEXT with a density of 6×10^{13}, a temperature of 0.92 keV, and a current density of 400 amp/cm, the constant $C = .05$, and an initial energy of 7 kT is needed to yield runaways, and the population is 3×10^{-3}. However, if the electrons are locally heated by a factor of 2, then there is a local increase in the runaway fraction to 4.8%, an increase of a factor of 10, and this added population may be useful in studying internal turbulence.

This oversimplified analysis suggests a number of important questions, among them:

a) What is the effective field correlation length for a given radiation pattern, and how is this altered by the presence of the plasma, and by uncontrolled variations in its properties?

b) What is the radiation pattern produced in an open resonator of a given type? Can any such structure significantly increase the plasma heating, by enhancing the field strength while maintaining the correlation? This requires determining the details of the diffraction pattern, and in determining its sensitivity to the effect of the plasma.

c) In a given radiation pattern which is intended to heat the electrons significantly, what are the important nonlinear effects? Note that a plane wave is not an adequate model.

d) Is the enhanced bremsstrahlung the best diagnostic for the fast electrons? Can the field lines be visualized in some way? If so, this would be preferable to looking at the spectrum.

e) What effect do the irradiated electrons have on the magnetic structure they are supposed to investigate? It is conceivable that the local heating might alter tearing modes

in such a way that the hot electrons find themselves in a closed rational flux tube, threading the heated region. In other places the field might be stochastic, but this intrusive diagnostic would never know.

f) Can this method distinguish between the various kinds of disturbances that can produce cross-field diffusion? Perhaps the electrons are going rapidly, and may not be much influenced by electric fields, for example. What scale can be determined: clearly the minimum must exceed the gyroradius of the energetic electrons, but by how much?

g) Are there instabilities, and fluctuations, induced by the fast electron population itself, which might redistribute electron energy or induce cross-field transport that has nothing to do with the unmodified plasma?

There are many other questions, including such obvious ones as the actual motion of the energetic electrons in a tokamak that need to be answered, but any diagnostic that provides even a promise of getting at the internal field geometry, is worth a little effort!

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Appendix 1: The Runaway Problem

If $v > 2v_\theta$ and $v_\perp = 0$, the equation of motion becomes

$$y\frac{dy}{dz} + \frac{1}{y^2} = C,$$

where y is the parallel velocity normalized to the thermal speed, $y = v_\parallel / v_\theta$ C is the normalized electric field $= \frac{1}{2}\frac{eE\lambda}{kT}$ and the displacement Z is normalized to the mean free path $z = Z/\lambda$

$$\begin{aligned}\lambda &= \frac{1}{5}\frac{v_\theta}{\nu_-} \\ &= \frac{1}{5}\frac{(kT)^2}{\pi e^4 \log\Lambda}.\end{aligned}$$

Then

$$\frac{1}{2}y^2 - Cz = K + \frac{1}{C}\log\frac{Cy^2-1}{Cy_0^2-1}; \qquad Cy_0^2 = \frac{\frac{1}{2}mv_\parallel^2}{kT}\frac{\frac{1}{2}eE\lambda}{kT} > 1.$$

If $v_\perp = 0$ we normalize to $v_\perp$, $x = v_\parallel / v_\perp$ and λ acquires a factor $(v_\perp/v_\theta)^2$. The equation of motion becomes

$$x\frac{dx}{dz} + \frac{x}{(1+x^2)^{3/2}} = C$$

which may be integrated to

$$\frac{1}{2}x^2 + Cz = K - \int \frac{x^2 dx}{C(1+x^2)^{3/2} - x} = K - \mathcal{J}.$$

We need to show that for large argument the last term is not greater than

$$\log\frac{v_\parallel + v}{v_\theta} + \text{constant} = \log\left(x + \sqrt{1+x^2}\right).$$

We handle the integral by substituting $x = \sinh u$, with $u = \log\left(x + \sqrt{1+x^2}\right)$:

$$\begin{aligned}\mathcal{J} &= \int \frac{\sinh^2 u \cosh u}{C\cosh^3 u - \sinh u} du \\ &= \frac{1}{C}\int \frac{C\left(\cosh^3 u - \cosh u\right)}{C\cosh^3 u - \sinh u} \\ &= \frac{u}{C} - \frac{1}{C}\int \frac{C\cosh u - \sinh u}{C\cosh^3 u - \sinh u} \\ &= \frac{1}{C}\log\left(x + \sqrt{1+x^2}\right) - \frac{1}{C}\mathcal{J}_2.\end{aligned}$$

Now again change variable to $Z = \exp u$. Then

$$
\begin{aligned}
\mathcal{J}_2 &= \int \frac{\frac{C}{2}\left(Z+\frac{1}{Z}\right) - \frac{1}{2}\left(Z-\frac{1}{Z}\right)}{\frac{C}{8}\left(Z+\frac{1}{Z}\right)^2 - \frac{1}{2}\left(Z-\frac{1}{Z}\right)} \frac{dZ}{Z} \\
&= \int \frac{C\left(Z^2+1\right) - \left(Z^2-1\right)}{\frac{C}{4}\left(Z^2+1\right)^3 - Z^2\left(Z^2-1\right)} Z\, dZ \\
&= \frac{1}{2}\int \frac{[(C-1)\omega + C + 1]\, d\omega}{\frac{C}{4}(\omega+1)^3 - \omega(\omega-1)} \qquad \text{for} \qquad \omega = Z^2.
\end{aligned}
$$

We can express $\mathcal{J}_2$ in terms of the roots of the cubic

$$
(\omega+1)^3 - \frac{4}{C}\omega(\omega-1) = 0.
$$

It is clear that for any value of C there is one negative real root r while for $C = C_0 = 2/3\sqrt{3}$, there is a real double root. For $C < C_0$ there are two positive real roots, while for $C > C_0$ the roots are complex conjugates. Only in the last case is the motion unbounded for arbitrary initial conditions; in the other two cases, the initial value of w must exceed the larger of the two positive roots. Then

$$
\begin{aligned}
\mathcal{J}_2 &= \frac{2(C-1)}{C}\int \frac{w + \frac{C+1}{C-1} dw}{(w+r)(w-r_1)(w-r_2)} \\
&= \frac{2(C-1)}{C}\int \frac{dw}{(w-r_1)(w-r_2)} + \left(\frac{C+1}{C-1} - r\right)\int \frac{dw}{(w-r)(w-r_1)(w-r_2)}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{1}{(w+r)(w-r_1)(w-r_2)} &= \frac{1}{(r+r_1)(r+r_2)}\left[\frac{1}{w+r} - \frac{1}{2}\frac{2w-(r_1+r_2)}{(w-r_1)(w-r_2)}\right. \\
&\qquad \left. + \frac{r + \frac{r_1+r_2}{2}}{(w-r_1)(w-r_2)}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{J}_2 &= A\log\frac{w+r}{\sqrt{(w-r_1)(w-r_2)}} + B\int \frac{dw}{(w-r_1)(w-r_2)} \\
&= A\log\frac{w+r}{\sqrt{(w-r_1)(w-r_2)}} + \frac{B}{(r_2-r_1)}\log\left(\frac{w-r_1}{w-r_2}\right),
\end{aligned}
$$

where

$$A = \frac{2(C-1)}{C}\left[\left(\frac{C+1}{C-1}-r\right)\frac{1}{(r+r_1)}\frac{1}{(r+r_2)}\right],$$

$$B = \frac{2C-1}{C}+\left(r+\frac{r_1+r_2}{2}\right)A.$$

Note that as $r_2 \to r_1$,

$$\frac{1}{r_2-r_1}\log\frac{w-r_1}{w-r_2} \to -\frac{1}{w-r_2},$$

while

$$\log\frac{w+r}{\sqrt{(w-r_1)(w-r_2)}} = \frac{1}{2}\log\left[\frac{(w+r)^3}{(w+r)^3-\frac{4}{C}w(w-1)}\right].$$

For large w

$$\frac{1}{r_1-r_2}\log\left(\frac{w-r_1}{w-r_2}\right) \to -\frac{1}{w},$$

while

$$\log\frac{(w+r)^3}{(w+r)^3-\frac{4}{C}w(w-1)} \to \left[3(r-1)+\frac{4}{C}\right]\frac{1}{w}.$$

Thus for large w

$$\mathcal{J} \to \frac{1}{C}\log\left(x+\sqrt{1+x^2}\right)+O\left(\frac{1}{w}\right)$$

as was required.

Appendix 2: Evaluation of $G(\alpha)$

Since this is not standard, we give the details

$$
\begin{aligned}
G(\alpha) &= \alpha^2 \int_1^\infty dx \sqrt{x} e^{-\alpha x} \frac{2}{\sqrt{x}} \sqrt{x-1} - \frac{1}{x} \log \frac{\sqrt{x} + \sqrt{x^2-1}}{\sqrt{x} - \sqrt{x^2-1}} \\
&= 2\alpha^2 \int_1^\infty dt\, e^{-\alpha t^2} \left[2t\sqrt{t^2-1} - \log \frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}} \right] \\
&= 2\alpha^2 \left[2e^{-\alpha} \int d\sqrt{t^2-1} e^{-\alpha(t^2-1)} t^2 - 1 - \mathcal{J} \right] \\
&= 2\alpha^2 \left[\frac{\sqrt{\pi}}{2(\alpha)^{3/2}} e^{-\alpha} - \mathcal{J} \right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{d\mathcal{J}}{d\alpha} &= \int_1^\infty e^{-\alpha t^2} t^2 \log \left(\frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}} \right) \\
&= -\frac{1}{2\alpha} e^{-\alpha t^2} t \log \Big|_1^\infty + \frac{1}{2\alpha} \int e^{-\alpha t^2} \log \left(\frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}} \right) dt \\
&\quad + \left[\frac{2}{2\alpha} \int e^{-\alpha t^2} \frac{t\, dt}{\sqrt{t^2-1}} = \frac{\sqrt{\pi}}{2\alpha^{3/2}} e^{-\alpha} \right]
\end{aligned}
$$

$$
\frac{d\mathcal{J}}{d\alpha} + \frac{1}{2\alpha}\mathcal{J} = \frac{1}{\sqrt{\alpha}} \frac{d}{d\alpha} \sqrt{\alpha}\mathcal{J} = -\frac{\sqrt{\pi}}{2\alpha^{3/2}} e^{-\alpha}
$$

$$
\mathcal{J} = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \int_\alpha^\infty \frac{e^{-x}}{x} dx
$$

$$
G = \sqrt{\pi\alpha} e^{-\alpha} \left[1 - \alpha e^{+\alpha} \int_\alpha^\infty e^{-x} \frac{dx}{x} \right].
$$

References

1. B. H. Quon and R. A. Dandl, "Preferential Electron Cyclotron Heating of Hot Electrons," to appear in Phys. Fluids (1989).

2. W. Heitler, *Quantum Theory of Radiation*, Second Edition, Oxford: Oxford University Press (1948).

3. R. A. Dandl, G. Guest, and B. H. Quon, Final Report to Air Force Geophysics Laboratory, AFGL-TR-88-0302.

4. J. E. Howard, Plasma Phys. **23**, 97 (1981).

5. W. M. Nivens, T. D. Rognlien, and B. I. Cohen, Phys. Rev. Lett. **59**, 60 (1987).

6. O. J. Kwon and P. H. Diamond, Nucl. Fusion **28**, 1931 (1988).

7. For example, W. B. Thompson, *Introduction to Plasma Physics*, Pergamon Press, Oxford, 1962, pp. 171–173, 230–232; also, J. Plasma Phys. **5**, 325 (1971).