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Abstract

The scaling of the diffusion coefficient with the amplitude of the fluctuation is studied in conjunction with the $\mathbf{E} \times \mathbf{B}$ velocity correlation function. For high fluctuation amplitude the velocity correlation function is split into two parts. One part is the correlation for the integrable case which does not contribute to the net transport. The other part is the correlation which results from the stochasticity. The diffusion coefficient obtained from the integrals of the second part of correlation, scales as $\tilde{\phi}^0$ or $\tilde{\phi}^{-1}$ depending on the cases considered.

I. Introduction

For low-frequency ($\omega \ll \omega_{ce}, \omega_{ci}$) and long wavelength ($k_\perp \rho \ll 1$) fluctuations typically present in plasmas, the motion of the particles are governed by drift equations. As is well known, for broad frequency and amplitude fluctuation spectra, the system easily becomes stochastic. Less well known is that even for the simplest two-wave systems the motion also becomes stochastic under easily satisfied conditions.[1,2] In the two-wave regime, however, the scaling of the diffusion coefficient on the amplitude of the fluctuation has been in controversy.[2,3]

As noted by Kleva and Drake,[2] the usual quasilinear theory seems to be inappropriate to the simple two-wave systems for moderate to high amplitude of the fluctuations. Dupree's improvement on the quasilinear theory includes the orbit corrections due to the fluctuations. But the Dupree work does not have the physics necessary to describe the exponential divergence of nearby orbits, because his work is before the advent of modern stochastic theory.

In the present work, we investigate the correlation function and the diffusion coefficient in the simplest systems. We study the transport of the guiding centers in a homogeneous constant magnetic field, supporting two transversely propagating, electrostatic, fluctuations. It is well known that even this two-wave system shows stochastic behavior when certain conditions are satisfied.[2–3] We consider electrostatic waves propagating perpendicularly to the direction of density gradient and magnetic field, i.e., the diamagnetic direction. These drift-type waves are easily observed in the tokamak or other fusion devices which have a confining magnetic field and density or temperature gradient.

The previous studies of Horton[3] and Kleva and Drake[2] show some difference on the diffusion coefficient with respect to the amplitude of the fluctuations. At high amplitude Horton finds two results $D \propto \tilde{\phi}$ and $\tilde{\phi}^{-1}$ depending on the value of the wave phase velocities, whereas

Kleva and Drake obtain $D \propto \ell n\tilde{\phi}$. We study numerically the two-time velocity correlation function of this system. As is well known, the time integral of this correlation function is the diffusion coefficient. We show that when the amplitude of the fluctuation is small, the correlation time τ_c is independent of the fluctuation amplitude, a result which supports the quasilinear theory with $D \propto \tilde{\phi}^2$. In this case, the correlation function $C(\tau)$ shows decaying oscillators having time scale τ_c which is independent of the fluctuation amplitude $\tilde{\phi}$. When the fluctuation amplitude becomes large, the correlation function eventually begins to decrease as time elapses. In this regime, we can normalize the correlation function by $C(\tau = 0)$ and scaling the time proportional to the fluctuation amplitude, i.e., $\tau' = \tilde{\phi}\tau$ to obtain a self-similar correlation function. As a result of these scalings, the correlation function shows similar behavior for different fluctuation amplitudes given by

$$\frac{C(\tau)}{C(0)} \simeq F_0(\tau') + \frac{1}{\tilde{\phi}} F_1(\tau') + \frac{1}{\tilde{\phi}^2} F_2(\tau') \ . \qquad (1)$$

This result implies that the correlation time τ_c is inversely proportional to the fluctuation amplitude. If we consider only the $F_0(\tau')$ part of the correlation, the diffusion coefficient should increase proportionally with $\tilde{\phi}$. But, as we note, $F_0(\tau')$ is the correlation which results when the system is integrable. We therefore, suspect and attempt to prove that $F_0(\tau')$ does not contribute to the net transport. Thus, the transport may scale as $\tilde{\phi}^0$ or $\tilde{\phi}^{-1}$ depending on whether $F_1(\tau') = 0$ or not.

The paper is organized as follows. In Sec. II we introduce the fluctuating electrostatic potential, which takes the status of the Hamiltonian of the problem giving the perpendicular equation of the motion. In Sec. III we relate the velocity correlation function with the diffusion coefficient and analyze the scaling of the correlation time on the rudimentary manner. In Sec. IV we investigate the problem through numerical experiments and verify the conclusion of Sec. III. In Sec. V the results are summarized and discussed.

II. Equation of Motion

We assume that there exists a uniform magnetic field along the $\mathbf{z}$-direction i.e., $\mathbf{B} = B\mathbf{z}$, where $\mathbf{z}$ is the unit vector along the z-direction. Suppose that there are electrostatic waves propagating perpendicularly to the magnetic field. If such waves have long-wavelength ($k_\perp \rho \ll 1$) and low-frequency ($\omega/\omega_c \ll 1$), the motion of the particles can be described by drift equation. That is, the particle motion is given by $\mathbf{E} \times \mathbf{B}$ velocity

$$\mathbf{V}_E = c\frac{\mathbf{E} \times \mathbf{B}}{B^2} = c\frac{\mathbf{z} \times \nabla\Phi}{B} \tag{2}$$

where Φ is the electrostatic wave propagating perpendicular to the magnetic field. In general $\Phi = \overline{\Phi}(x) + \widetilde{\Phi}(x, y, t)$. In component form the equations of motion become

$$\frac{dx}{dt} = -\frac{\partial}{\partial y}\left(\frac{c}{B}\,\Phi(x, y, t)\right) \tag{3}$$

$$\frac{dy}{dt} = \frac{\partial}{\partial x}\left(\frac{c}{B}\,\Phi(x, y, t)\right) . \tag{4}$$

Thus our system is a Hamiltonian system with canonical momentum and coordinate $(p, q) = (x, y)$ and with the Hamiltonian $H = \frac{c}{B}\,\Phi(x, y, t)$.

For the one wave case, the system is integrable with recourse to the elliptic functions and changing of the coordinate system to that of moving with the phase velocity of the wave. For multiple waves having the same phase velocities, the situation is the same. The integrable cases are thoroughly studied by Horton.[1,2]

For the two-wave cases with different phase velocities, the criterion for the onset of stochasticity is given in Ref. 3. Since we wish to study stochastic transport in the Hamiltonian system, we choose a simple two-wave fluctuating potential (equivalently Hamiltonian) which results in stochasticity. If we normalize the length and time appropriately, we can take the stochastic two-wave Hamiltonian[3] as follows

$$H = H_0(x) + \widetilde{\phi}(\sin(x)\cos(y) + \cos(x)\cos(2(y - t))) \tag{5}$$

where

$$\frac{dH_0(x)}{dx} = -\frac{\omega_1 - k_{y_1}\, v_E(x)}{\Omega_E}$$

and the two waves have equal amplitude $\tilde{\phi}$. The maximum transport occurs for $u(x) = \frac{dH_0}{dx} = 0$. Perpendicular equations of motion then become

$$\frac{dx}{dt} = \tilde{\phi}\,[\sin(x)\sin(y) + 2\cos(x)\sin(2(y-t))] \tag{6}$$

$$\frac{dy}{dt} = \tilde{\phi}\,[\cos(x)\cos(y) - \sin(x)\cos(2(y-t))]\ . \tag{7}$$

The transformation to these dimensionless equations, as given in Ref. 3, uses x, y, t in units $k_{x_1}^{-1}$, and Ω_E^{-1} where $\Omega_E = c\, k_{x_1}\, k_{y_1}\, A_1/B$ for the first wave. Here $\tilde{\phi} = \Phi_1/A_1$ is the dimensionless amplitude with respect to the reference potential A_1. The dimensionless relative velocity of the second wave relative to the first wave is $v = (\omega_2/k_{y_2} - \omega_1/k_{y_1})/(ck_{x_1} A_1/B)$ and is taken as unity. Here A_1 is a reference amplitude for the first electrostatic wave which may be taken as either $\Delta\Phi$, T_e/e or $(\rho_s/L_N)T_e/e$ depending on the physical circumstances.

Equations (6)–(7) describe the $\mathbf{E}\times\mathbf{B}$ system with the maximum diffusion rate since there are no equilibrium flow terms. This situation occurs approximately for the η_i-mode in the absence of shear flow.[1] A more general transport system is described in Ref. 3 including a radial electric field $E_r(x)$. In the wave frame the effect of the radial electric field is to introduce an equilibrium flow $u(x)$ modifying Eq. (7) to

$$\frac{dy}{dt} = u'x + \tilde{\phi}\,[\cos(x)\cos(y) - \sin(x)\cos(2(y-t)]$$

where the dimensionless shear flow parameter u' is given by $u = k_{y_1}\, v_E'/k_{x_1}\, \Omega_E$. The effect of v_E' is to produce an $\mathbf{E}\times\mathbf{B}$ convective cell with the separatrix width $\Delta x = (\tilde{\phi}/u')^{1/2} = k_{x_1}(cA_1/Bv_E')^{1/2}$. Thus, the effect of the shear cannot be neglected when $\Delta x \lesssim \pi$. For a shear flow layer of width Δr_E with potential drop $\Delta\Phi$ the convective cell island width is $\delta r = \Delta r_E(\tilde{\phi}/\Delta\Phi)^{1/2}$ and the condition for strong shear flow modification of $\mathbf{E}\times\mathbf{B}$ transport is $\Delta x = k_x\, r_E(\tilde{\phi}/\Delta\Phi)^{1/2} < \pi$.

III. Correlation Function

When the global stochasticity is set up, the behavior of the system of test particles can be described by the diffusion process. The basic assumption for the diffusion approximation is that the particles experience a short correlation time along its trajectory. The short correlation time is given by the global and the intrinsic stochasticity of the system. The definition of the diffusion coefficient follows from the formal integration of the equation of motion

$$x(t) - x(0) = \int_0^t v_E\left(x(t_1), y(t_1), t_1\right) dt_1 \tag{8}$$

where v_E is shorthand for $-\frac{\partial H}{\partial y} = \dot{x}$. Introducing the average $\langle \quad \rangle$ over the initial conditions $(x(0), y(0))$ in the phase space, the relation

$$\left\langle (x(t) - x(0))^2 \right\rangle = \lim_{t\to\infty} \int_0^t dt_1 \int_0^t dt_2 \left\langle v_E(t_1) v_E(t_2) \right\rangle \tag{9}$$

$$\simeq 2Dt \qquad \text{as} \qquad t \to \infty \tag{10}$$

defines the diffusion coefficient.

Thus, we are led to study the very nature of the two-time velocity correlation functions. For simplicity of the notation, let us write the unnormalized correlation function as

$$C(\tau) = \langle v_E(t) v_E(t+\tau) \rangle \tag{11}$$

where we assume that the average over the initial conditions is time translationally invariant, thus eliminating the t dependence of the velocity correlation function. The definition for D can be developed further using the time translational invariance of the average to give

$$D = \lim_{t\to\infty} \frac{1}{2t} \int_0^t dt_1 \int_0^t dt_2\, C(t_1 - t_2) \tag{12}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} d\tau\, C(\tau) \tag{13}$$

where we changed integration variable from (t_1, t_2) to (τ, ξ), where $\tau = t_1 - t_2$ and $\xi = t_2$. That is, the diffusion coefficient is the time integral of the velocity correlation function which is a well established result.

It should be noted that because our approach is not a self-consistent field theory, that is, we start from the given waves, the velocity correlation function is known in principle. Since the system is not integrable, however, we cannot evaluate the correlation function exactly. But we can know the derivatives of any order at $t = 0$ although the algebra becomes extremely complex as the order of the derivatives becomes large. For illustration, we compute

$$C(0) = \langle v_E(0)\, v_E(0) \rangle \tag{14}$$

and

$$\frac{dC(\tau)}{d\tau} = \left\langle v_E(0) \left([v_E(\tau), H] + \frac{\partial v_E(\tau)}{\partial \tau} \right) \right\rangle \tag{15}$$

where $[f, g]$ is the Poisson bracket defined by

$$[f, g] \equiv \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial x} . \tag{16}$$

Thus we can evaluate $\frac{dC(\tau)}{d\tau}|_{\tau=0}$ explicitly. Recursively, we can compute the higher derivatives

$$\frac{d^{n+1}C(\tau)}{d\tau^{n+1}} = \left\langle v_E(0) \mathcal{D}^{n+1} v_E(\tau) \right\rangle \tag{17}$$

where

$$\mathcal{D}^{n+1} v_E(\tau) = [\mathcal{D}^n v_E(\tau), H] + \frac{\partial}{\partial \tau} \mathcal{D}^n v_E(\tau) \quad \text{and} \quad \mathcal{D}^0 v_E(\tau) = v_E(\tau) .$$

From these derivatives of the velocity correlation function at $\tau = 0$, we can form a Taylor series expansion about $\tau = 0$:

$$C(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n C(\tau)}{d\tau^n}\bigg|_{\tau=0} \tau^n . \tag{18}$$

Although this expansion is useless for the evaluation of diffusion coefficient, since the series is not in a closed form, we may hope to infer the time scale of the correlation function as $\widetilde{\phi}$ varies. We explicitly computed the first few low order derivatives with help of the symbolic manipulator MACSYMA. The result is

$$C(\tau) = \widetilde{\phi}^2 \left(\frac{5}{4} - \frac{95\widetilde{\phi}^2 + 128}{64} \tau^2 + \frac{9391\widetilde{\phi}^4 + 27072\widetilde{\phi}^2 + 8192}{24 \cdot 512} \tau^4 + \cdots \right) . \tag{19}$$

We note that for the $\widetilde{\phi} \ll 1$ limit, the time scale is independent of $\widetilde{\phi}$, but that for the $\widetilde{\phi} \gg 1$ limit, the dependence is on $(\widehat{\phi}\tau)^n$ and thus $\tau_c \simeq 1/\widetilde{\phi}$.

We consider this approach more systematically. First, it is noted that the derivatives in Eq. (15) consist of two terms, the contribution from $\mathbf{E} \times \mathbf{B}$ convection and from the explicit time dependence of the drift velocity. It is noted that the convective contribution is $\mathcal{O}(\widetilde{\phi}^3)$ and the contribution from the explicit time dependence is $\mathcal{O}(\widetilde{\phi}^2)$ in Eq. (15). For low fluctuation amplitude ($\widetilde{\phi} \ll 1$) we may keep only the contribution from the explicit time dependence. In this case we can calculate the derivatives exactly to the infinite order. The Taylor series reduces to

$$C(\tau) = \widetilde{\phi}^2 \left(\frac{1}{4} + \cos(2\tau) \right) . \tag{20}$$

In this limit the correlation function is simply sinusoidal oscillations with period π. This is just the period of the driving Hamiltonian. Also we can see that as $\widetilde{\phi} \to 0$, the Taylor expansion of Eq. (19) agrees with the expansion of Eq. (20) confirming our calculations. For the other extreme, where $\widetilde{\phi} \gg 1$ case, we keep the contribution from the highest order terms. In this case it is immediately seen that $C_n \left(\equiv \frac{d^n C(\tau)}{d\tau^n}\big|_{\tau=0} \right)$ has the leading term proportional to $\widetilde{\phi}^{n+2}$. Thus we may write

$$C_n \simeq C(0) \left(a_n^n \widetilde{\phi}^n + a_{n-1}^n \widetilde{\phi}^{n-1} + a_{n-2}^n \widetilde{\phi}^{n-2} \right) \tag{21}$$

and

$$C(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} C_n \tau^n \simeq C(0) \left[\sum_{n=0}^{\infty} \frac{1}{n!} a_n^n (\widetilde{\phi}\tau)^n + \frac{1}{\widetilde{\phi}} \sum_n \frac{1}{n!} a_{n-1}^n (\widetilde{\phi}\tau)^n \right.$$

$$+\frac{1}{\tilde{\phi}^2}\sum_n \frac{1}{n!}a_{n-1}^n(\tilde{\phi}\tau)^n\Bigg] . \tag{22}$$

If we scale the time variable through $\tau' = \tilde{\phi}\tau$ and if we define

$$F_i \equiv \sum_n \frac{a_{n-i}^n}{n!}(\tilde{\phi}\tau)^n \tag{23}$$

then we may write

$$\frac{C(\tau)}{C(0)} \simeq F_0(\tau') + \frac{1}{\tilde{\phi}}F_1(\tau') + \frac{1}{\tilde{\phi}^2}F_2(\tau') . \tag{24}$$

From this we infer

$$D = \frac{1}{2}\int_{-\infty}^{+\infty} C(\tau)d\tau = \frac{5}{4}\tilde{\phi}\left[\int_0^{+\infty} F_0(\tau')d\tau' + \frac{1}{\tilde{\phi}}\int_0^{+\infty} F_1(\tau')d\tau' + \frac{1}{\tilde{\phi}^2}\int_0^{+\infty} F_2(\tau')d\tau'\right] . \tag{25}$$

In the above expression (25) we note the following:

1. $F_0(\tau)$ is the highest order contribution in $\tilde{\phi}$, which results from the summation for the convective contribution only. In other words, if we ignore the time dependence of the Hamiltonian, we should get F_0 as the exact velocity correlation function. Therefore, we may say that the correlation $F_0(\tau')$ is the integrable contribution of the Hamiltonian system. Since $\partial_t = 0$ implies integrability in a 1D ($d = 2$) system. From the above argument we suppose that $F_0(\tau')$ represents at most an aperiodic oscillation and will not contribute to the integration which is the transport coefficient. In other words, we expect

$$\int_0^{+\infty} F_0(\tau')d\tau' = 0 \tag{26}$$

due to the integrability of the 1D Hamiltonian system.

2. From the above arguments we infer that the diffusion coefficient should scale as either $\tilde{\phi}^0$ or $\tilde{\phi}^{-1}$ depending on the circumstances.

IV. Numerical Experiments and Discussions

A. Numerical Experiments

We measure the velocity correlation function through numerical techniques. Since the problem is periodic in phase space with period 2π, we replaced the phase space average by the average over the reduced phase space of $[-\pi,\pi]\times[-\pi,\pi]$. For measurement of the velocity correlation function, we locate N particles randomly in $[-\pi,\pi]\times[-\pi,\pi]$. We integrate the equations of motion of these particles with given initial conditions. During the integration we compute

$$C(\tau)\simeq\frac{1}{N}\sum_{i=1}^{N}\left(v_E(0)v_E(\tau)\right) \tag{27}$$

and take this as an approximation for the velocity correlation function. In this work we take $N=128\sim 1024$ particles and the integration of the equation of motion was done by a 5-6 order adaptive Runge-Kutta method (DVERK from IMSL). The truncation error e_T per step was set between $10^{-3}<e_T<10^{-5}$ and the effect of integration error is controlled by varying e_T so as not to result in erroneous values of $C(\tau)$.

In Fig. 1 we show the effect of the finite number of test particles, which can be regarded as the effect of the Monte-Carlo (MC) simulation. For this case fluctuation amplitude $\tilde{\phi}=3$, $e_T=10^{-4}$, $N=1024$ and we superimpose 5 different runs. We note that although there is a finite difference for the runs, the error seems to be small and the structure of the velocity correlation functions is obviously not affected. We also see that the correlation function decays to zero while oscillating from the maximum at $\tau=0$. This is a general structure of the computed correlation function, and it is physically appealing. Also note that the velocity correlation function is normalized to its $\tau=0$ value. The $C_{\mathrm{MC}}(0)/C(0)$ is 1.025 showing an error of few percent compared with the exact 1.

In Fig. 2 we show the velocity correlation function for the $\tilde{\phi}\ll 1$ limit. In Fig. 2(a) $\tilde{\phi}=0.01, 0.02, 0.03$. The plot shows a steady oscillation. As $\tilde{\phi}$ becomes smaller, the correlation is more nearly periodic in agreement with Eq. (20) in Sec. III. Also we note that the oscillating time scale is the same for $\tilde{\phi} = 0.01, 0.02, 0.03$ and the period of oscillation is estimated to be 3.1 agreeing with the periodicity of the driving Hamiltonian π. To obtain diffusion in this small amplitude regime there must be a small background diffusion D_0 due either to collisions or small scale turbulence. We now add the effect of D_0 giving small random kicks to δx, δy each time step Δt. The resulting decay of the two-time correlation function is shown in Fig. 2(b).

In Fig. 3, we plot $C(\tau)/C(0)$ as a function of non-scaled time τ and scaled time $\tau' \equiv \tilde{\phi}\tau$ for different $\tilde{\phi}$. From Fig. 3(a) we can guess that the correlations may resemble each other if we scale the abscissa. That is, if we compress or stretch one of the curves, the correlation may show similar behavior. The result of scaling is shown in Fig. 3(b), where we may say the velocity correlation function contains a dominant contribution which does not depend on $\tilde{\phi}$ when regarded as a function of τ'. Therefore we suggest the correlation function may be written as

$$C(\tau)/C(0) \simeq F_0(\tau') + \mathcal{O}(\tilde{\phi}^{-1}) \tag{28}$$

where $F_0(\tau')$ is independent of $\tilde{\phi}$ when $\tilde{\phi} \gg 1$. The possibility of this self-similar scaling is anticipated in Sec. III from the power series expression of the correlation. In Fig. 3 we use $\tilde{\phi} = 20, 30$, $N = 1024$, and $e_T = 10^{-4}$.

Finally, we also measure the diffusion coefficient using the usual rule

$$D(t) \simeq \frac{1}{2t}\frac{1}{N}\sum_{i=1}^{N}(x(t) - x(0))^2 \ . \tag{29}$$

When $D(t)$ appears convergent, we regard that $D(t)$ is ergodic and replace the ensemble average by its time average. We obtain from the time series of $D(t)$

$$\overline{D} = \frac{1}{T - T_0}\int_{T_0}^{T} D(t)dt \tag{30}$$

and estimate the standard deviation by

$$\delta D = \left[\frac{1}{T - T_0}\int_{T_0}^{T}\left(D(t) - \overline{D}\right)^2 dt\right]^{1/2} \tag{31}$$

where T_0 is the time when convergence is observed.

From a series of simulations we plot the diffusion coefficient D as a function of the fluctuation amplitude $\tilde{\phi}$ in Fig. 4. For $\tilde{\phi} \gg 1, D(\tilde{\phi})$ decreases linearly with as $\tilde{\phi}$ increases. Using a least square fit, we find that the $D(\tilde{\phi})$ for $\tilde{\phi} \gg 1$ is given by

$$D \simeq A\tilde{\phi}^{-\alpha} \tag{32}$$

where $A = 9.28$ and $\alpha = 1.03$. This power dependence of $\tilde{\phi}^{-1.03}$ should be compared with $\tilde{\phi}^{-1}$ which is the conclusion of the analysis in Sec. III. It seems that this inverse power dependence on amplitude is in reasonable agreement with the theory in Sec. III taking into account the limitations of the simulations and the theory.

The units of the dimensional diffusion coefficient are Ω_E/k_x^2 so that Fig. 4 shows that D_x is bounded by $\Omega_E/k_{x_1}^2 \cong 0.93(k_{y_1}/k_{x_1})(cA_1/B)$ with the upper bound reached at $\tilde{\phi} \cong \Phi_1/A_1 \cong 10$. In terms of the reference potential A_1 of the first wave the maximum diffusion rate is $D_{\max} \cong 0.93(k_{y_1}/k_{x_1})(cA_1/B)$, thus when the reference potential is $A_1 = T_e/e$ we obtain that the Bohm diffusion coefficient is the upper bound to the diffusion rate.

Now we consider the reduction of the transport by the presence of a small velocity shear flow giving $dy/dt = u'x + \partial\tilde{\phi}/\partial x$. The Hamiltonian now has the usual kinetic energy term $H = \frac{1}{2m}p^2 + V(p,q)$ with $p = u'x = mx$. The reduction of the diffusion at fixed $\tilde{\phi} = 10$ for increasing shear parameter u' is shown in Fig. 5. The reduction follows

$$D(u', \tilde{\phi}) = D(\tilde{\phi})\exp(-\alpha|u'|)$$

with $\alpha \simeq 4.8 - 5.0$. In the regime where $m = |u'| \gg \tilde{\phi}$ the system becomes integrable and the diffusion vanishes by the Chirikov resonance overlap condition.[5]

B. Discussion

In this subsection we would like to note various shortcomings and the physical plausibility of the results presented here.

1. We arbitrarily regrouped the series representation of the velocity correlation function in powers of $\tilde{\phi}$ to split the correlation function into several parts. Since the velocity correlation function is intrinsically a conditionally convergent function, the procedure can not be supported on a mathematical basis. This shortcoming may be supplemented on a physical basis.

2. The correlation function for $\tilde{\phi} \ll 1$ was computed exactly in the text. However, it does not have the necessary requirement for evaluation of the transport coefficient. That is, it does not converge when integrated over long times. To have convergence, another mechanism, such as a finite residual collision frequency or a background bath of small amplitude fluctuations is required. By adding small random kicks to $\delta x, \delta y$ we may restore the convergence of $\int^{\infty} C(\tau) d\tau$ as shown in Fig. 2(b).

3. The general behavior of the correlation function $C(\tau)$ is decaying while oscillating as expected on the basis of physical intuition. For $\tilde{\phi} \ll 1$. The correlation function from the simulations agree well with the $\tilde{\phi} \ll 1$ theory. We regard this as the integrable regime. For $\tilde{\phi} \gg 1$, the time scale of the decaying oscillations is proportional to $\tilde{\phi}^{-1}$. This consequence is expected since, in this strongly nonlinear regime, the convective contribution is dominant although we show that the pure convection limit does not contribute to the net transport.

4. We calculate the diffusion coefficient numerically for the Hamiltonian given by Eq. (5). We estimate the dependence of the diffusion coefficient on the fluctuation amplitude. Rather surprisingly, we find a different dependence of D on $\tilde{\phi}$ from the previous

authors.[2,3] We find that D steadily increases until about $\tilde{\phi} = 10$, but for $\tilde{\phi} > 10$, D starts to decrease as $1/\tilde{\phi}$. The only difference between the earlier works and us was that we use random initial conditions over the unit cell instead of using fixed x_0 as in Refs. 2 and 3. We think this can not cause essential difference in the behavior of D. Furthermore, we fitted D on $\tilde{\phi}$ as $D \simeq A\tilde{\phi}^{-\alpha}$. With a good correlation coefficient in the fitting we obtain $\alpha \simeq 1.03$ which agrees with our theoretical argument of $\alpha = 1$.

5. To decide theoretically whether $D \simeq \tilde{\phi}^0, \tilde{\phi}^{-1}$ for $\tilde{\phi} \gg 1$, we need to calculate all coefficients a_i^n given in Eq. (21). It seems that $a_i^n = 0$ for odd i for the Hamiltonian (5). Thus, for our case D should have the dependence of $D \simeq \tilde{\phi}^{-1}$.

V. Conclusions

We evaluate through the numerical technique the velocity correlation function for the Hamiltonian used by Horton.[3] The correlation function is decaying while oscillating, agreeing with physical intuition based on the $\mathbf{E} \times \mathbf{B}$ convection of the particles. Furthermore, it is shown that for low fluctuation amplitude $\tilde{\phi} \ll 1$, the correlation is a steady oscillation with a characteristic period almost independent of the fluctuation amplitude $\tilde{\phi}$. The period of this oscillation agrees with the periodicity of the driving Hamiltonian. This result is calculated analytically and verified through the numerical computation. For high fluctuation amplitude $\tilde{\phi} \gg 1$, the correlation function decays but shows similar behavior for different $\tilde{\phi}$ when written as $C(\tau)/C(0) \simeq F_0(\tau') + \mathcal{O}(\tilde{\phi}^{-1})$ where τ' is the amplitude scaled time $\tau' = \tilde{\phi}\tau$ and $F_0(\tau')$ is that part of the correlation which results from the Hamiltonian when only coherent convection is present. Therefore we suppose that $F_0(\tau')$ is at most an aperiodic oscillation and does not contribute to the transport. Since the stochasticity and stochastic transport comes from the time dependent convection and separatrix crossing, the argument that the integrable part of the correlation results in no transport is reasonable. Omitting the integrable part of the correlation we get scaling of D on $\tilde{\phi}$ given in Eq. (25) which has a power dependence of $D \sim \tilde{\phi}^0$ or $\tilde{\phi}^{-1}$. For a paradigm Hamiltonian we used the Horton model and compute the transport finding that $D \simeq A\tilde{\phi}^{-\alpha}$, with $\alpha \simeq 1.03$, agreeing with our theoretical argument.

We show that the diffusion rate is bounded above by unity in the dimensionless variables. The maximum occurs when the rotation rate Ω_E in the frozen potential is about five times the frequency of the two-wave system. Below this critical amplitude the diffusion rate increases linearly with wave amplitude. The correlation time τ_c of the particles in the monochromatic waves decreases inversely with amplitude for amplitudes below the critical amplitude. Above the critical amplitude the adiabatic invariance of the action associated with the rapid rotation in the convective cells reduces the transport to the boundary layer around the $\Phi(x,y,t) \sim 0$ contours.

For convection potentials of order T_e/e the upper bound on the diffusion rate obtained is approximately the Bohm diffusion rate.

The reduction of transport for higher fluctuation amplitudes comes from the "approach to the integrable system" as $\tilde{\phi}$ becomes large. For large $\tilde{\phi}$ the rotation rate Ω_E becomes rapid compared with rate of change of the separatrix contours and the corresponding action becomes a good invariant.

We acknowledge that the two-wave system is rather special and does not typically represent the situation occurring in actual fusion devices. In experimental situations there may be many small amplitude waves.[6] For such cases the correlation time is determined by either the dispersion of the waves $\Delta\omega^{-1}$ or the intrinsic orbital stochasticity from the nonlinearity as in our case depending on the competition between $\Delta\omega$ and $k_\perp^2 D$. As in Horton and Choi,[3] for such broad spectral cases, other more involved renormalized perturbation expansions may be necessary, and they may result in another scaling of D with $\tilde{\phi}$ and probably having a more direct relevance to the actual experimental situations in toroidal confinement systems.

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Figure Captions

1. Superposition of 5 different runs with same parameters to show the effect of numerical simulation. Parameters are $\tilde{\phi} = 3$, $e_T = 10^{-4}$, $N = 1024$.

2. Measured correlation for $\tilde{\phi} = 0.01, 0.02, 0.03$. Here $N = 1024$ and $e_T = 10^{-4}$.

   (a) pure Hamiltonian flow (b) Hamiltonian flow with small background diffusion from $D_0 = \langle \delta x^2 \rangle / \Delta t$.

3. The correlation function $C(\tau)/C(0)$ (a) as a function of τ (b) as a function of $\tau' \equiv \tilde{\phi}\tau$, showing that $C(\tau)/C(0) \simeq F_0(\tau') + \mathcal{O}(\tilde{\phi}^{-1})$ for $\tilde{\phi} = 20$ and 30.

4. Dependence of D on the fluctuation amplitude $\tilde{\phi}$ for $N = 1024$ and $e_T = 10^{-4}$.

5. Dependence of D on the shear flow parameter u' at the fixed amplitude $\tilde{\phi} = 10$.

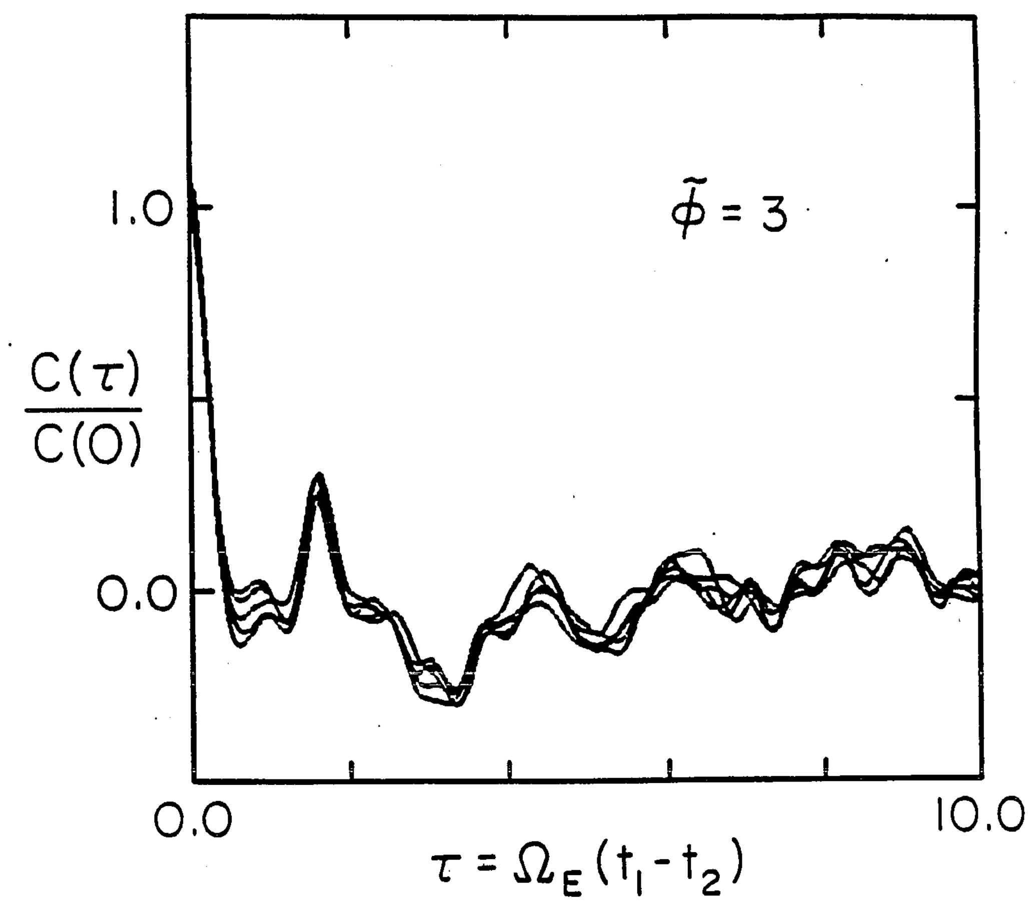
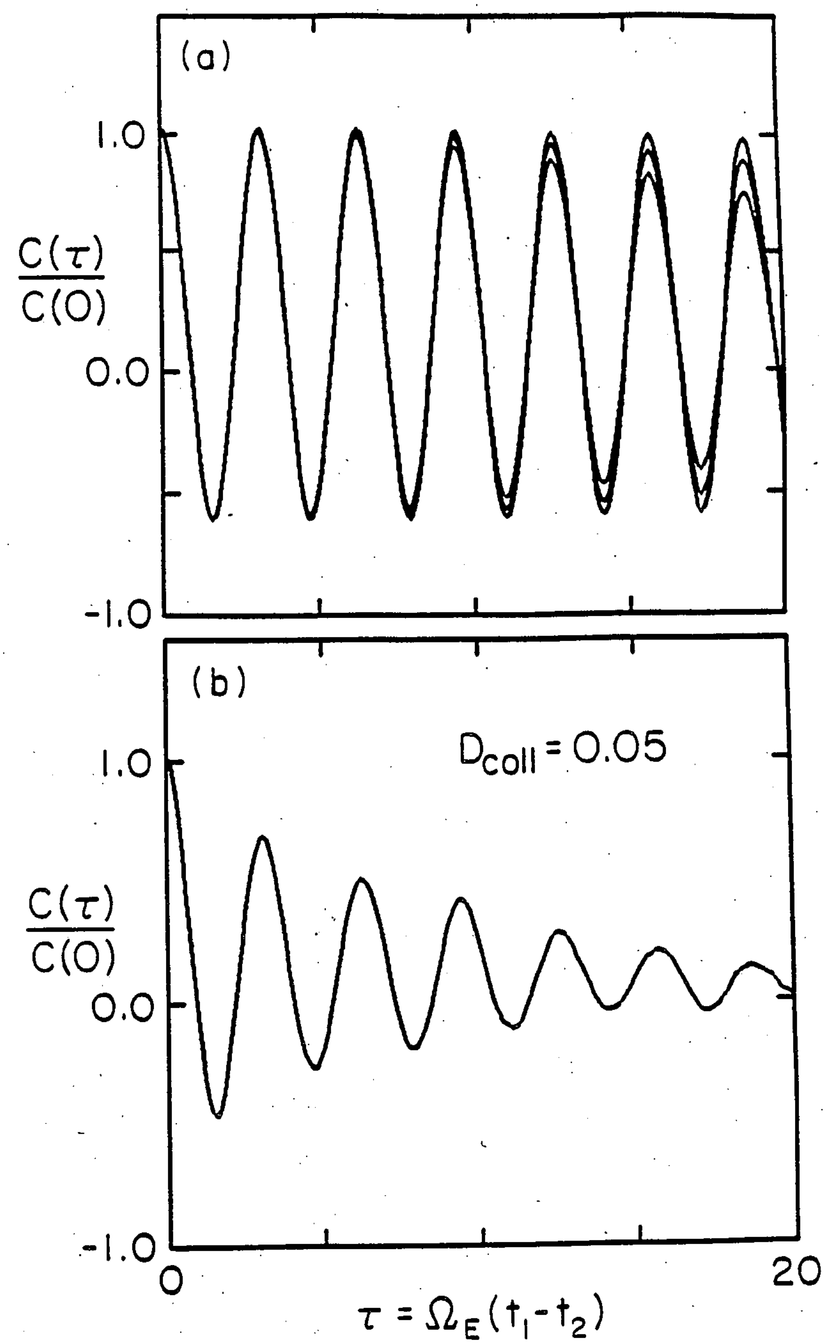


Fig. 1



(a)
1.0
$\frac{C(\tau)}{C(0)}$
0.0
-1.0
(b)
$D_{coll} = 0.05$
1.0
$\frac{C(\tau)}{C(0)}$
0.0
-1.0
0
20
$\tau = \Omega_E(t_1 - t_2)$

Fig. 2

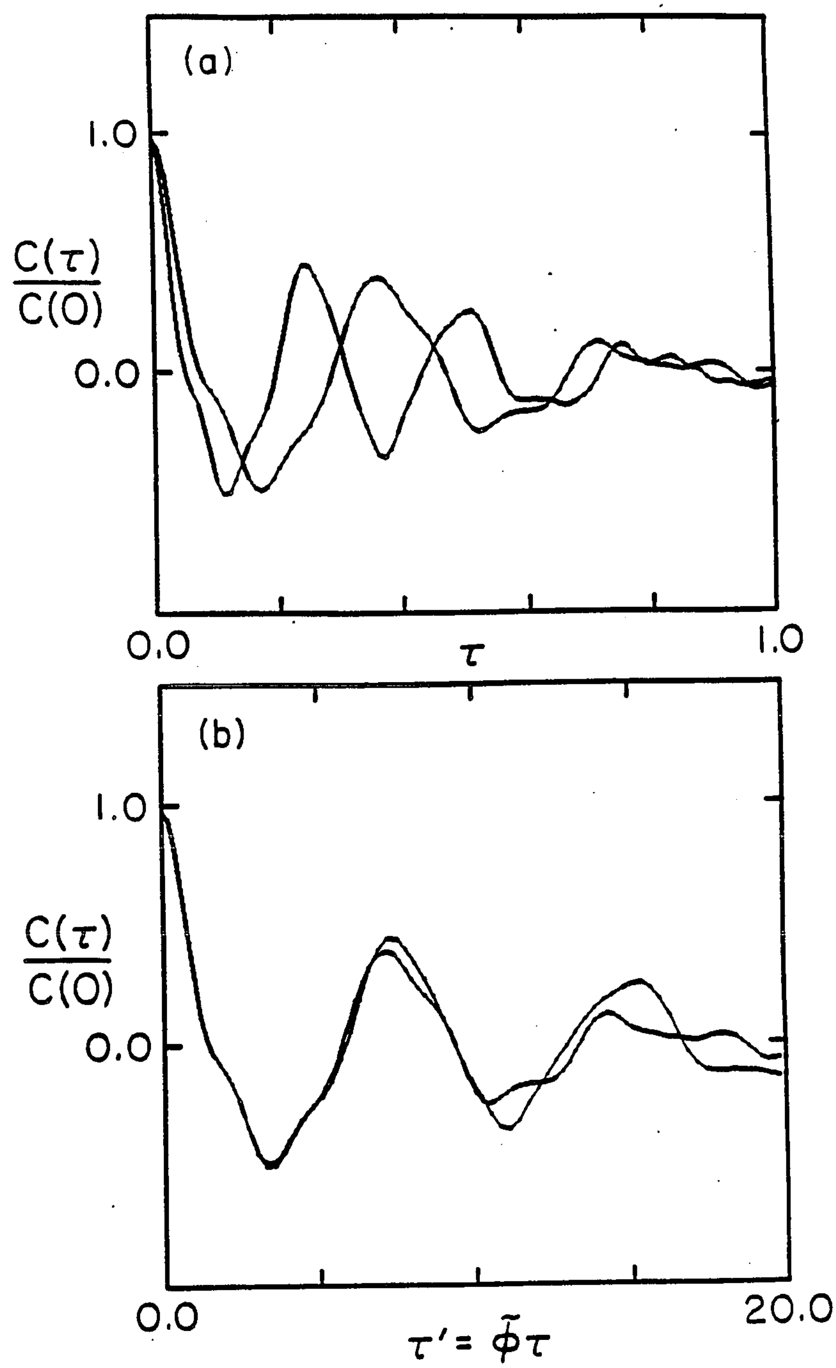

(a)
1.0
$\frac{C(\tau)}{C(0)}$
0.0
0.0
τ
1.0
(b)
1.0
$\frac{C(\tau)}{C(0)}$
0.0
0.0
$\tau' = \tilde{\phi}\tau$
20.0

Fig. 3

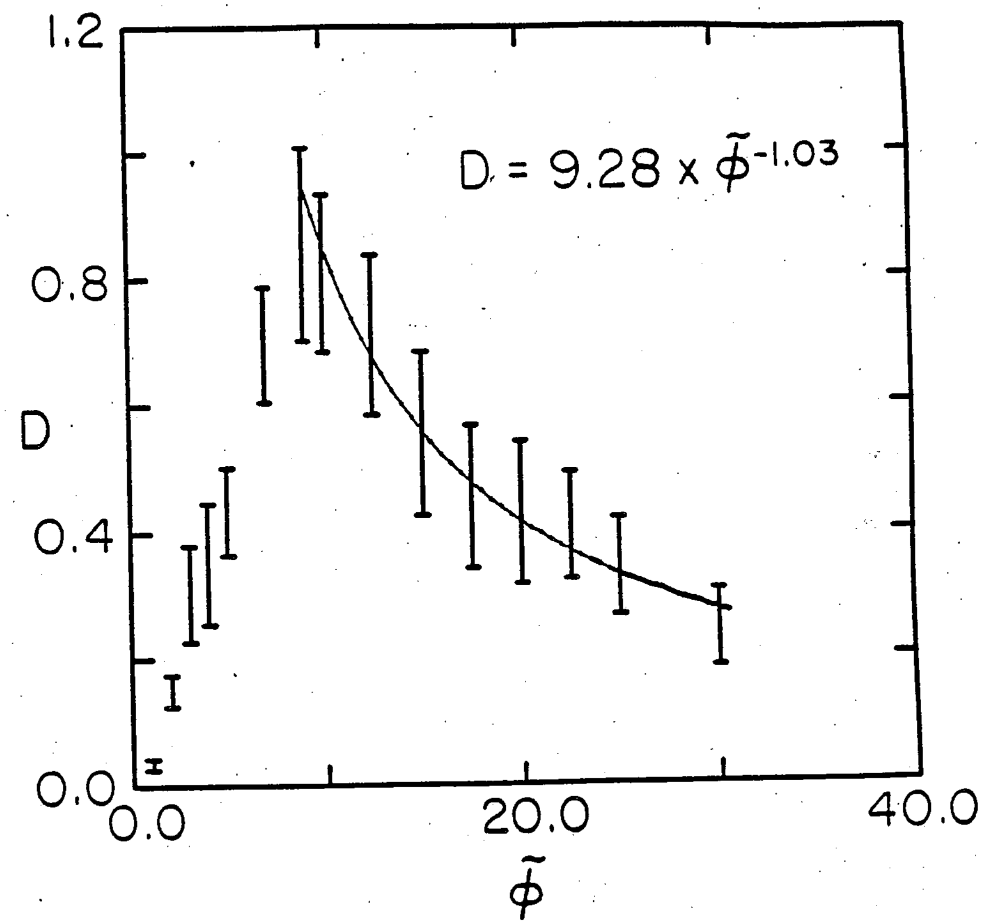
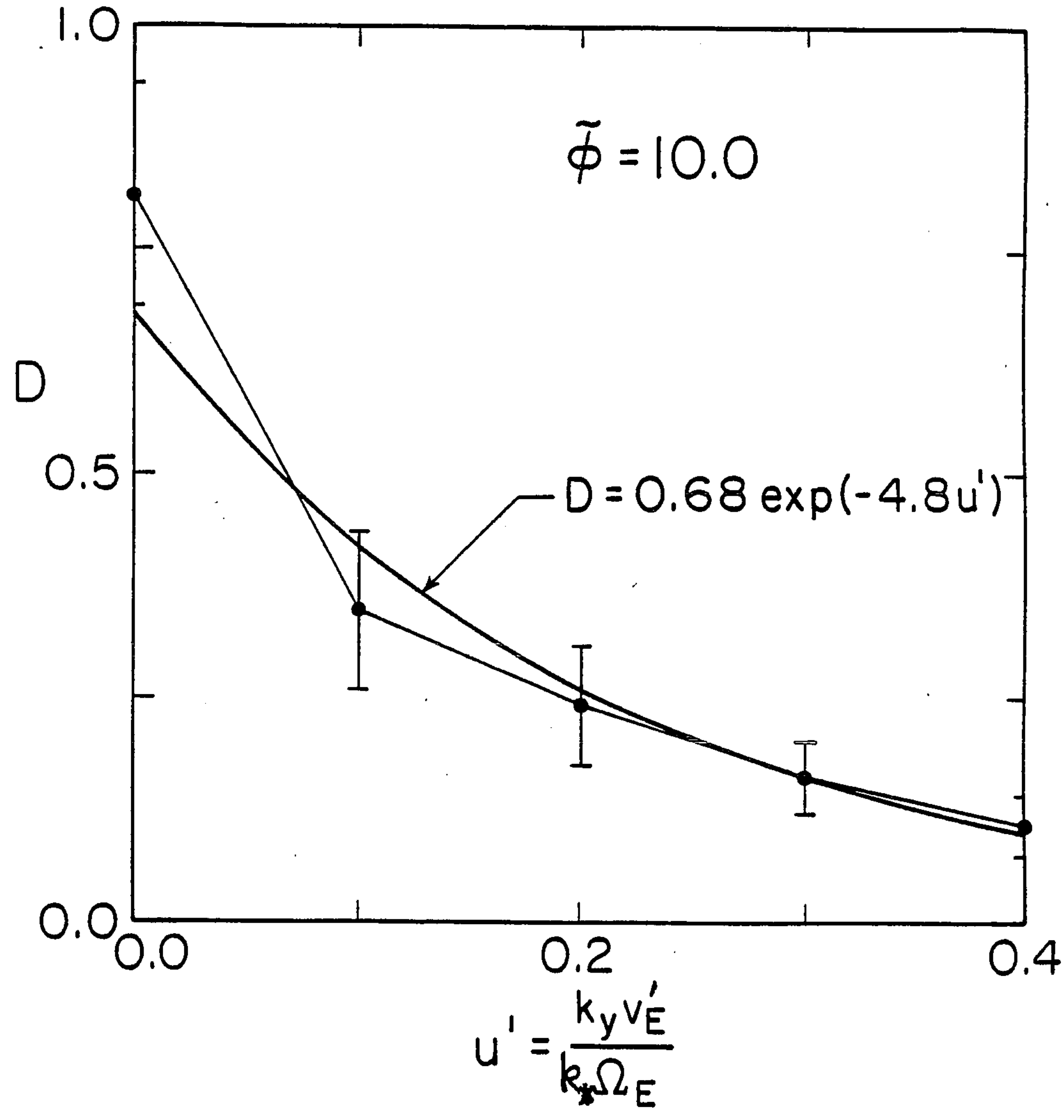


Fig. 4



1.0
0.5
0.0
D
$\tilde{\Phi} = 10.0$
$D = 0.68 \exp(-4.8u')$
0.0
0.2
0.4
$u' = \frac{k_y v_E'}{k_x \Omega_E}$

Fig. 5