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THE FEASIBILITY OF A RING STABILIZED PLASMA REACTOR IN THE LEE-VAN DAM LIMIT

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THE FEASIBILITY OF A RING STABILIZED PLASMA REACTOR
IN THE LEE VAN DAM LIMIT

By

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INTRODUCTION

Experiments during the past 20 years at Oak Ridge National Laboratory (O. R. N. L.) and other laboratories around the world have established that electron cyclotron resonant heating (e. c. r. h.) can be used to produce and sustain hot electron plasmas[1]. In 1967, it was shown by R. A. Dandl et. al.[2], that these plasmas took the form of rings or annuli made up of very hot, often relativistic, electrons (from about .05-1.0 Mev) with values of beta ranging upward to a sizable fraction of unity (around .4). It was further shown that these rings measured a few Larmour radii in thickness and that they were macroscopically stable when they existed in a sufficiently dense core plasma. This discovery came at the same time that

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1. N. A. Uckan, "OVERVIEW" (EBT Ring Physics Proceedings Of The Workshop, December 3-5, 1979, Oak Ridge , Tennessee)

2. R.A. Dandl,H.O. Eason, P.H. Edmonds, A.C. England, G.E. Guest, C.L. Hedrick, J.T. Hogan, and J.C. Sprott, Plasma Physics and Controlled Nuclear Fusion (International Atomic Energy Agency, Vienna, 1971) Vol. II p.607

3. W.L. Stirling, Phys. Fluids, <u>15</u>, 688, (1972)

refined energy balance calculations showed grave difficulties in producing net power output from simple magnetic mirror configurations.[3] Though in no way causally connected, these two discoveries can be said to have set the stage for the development of E.B.T. (see figure 1 and 2)

Again in 1967, R.A. Dandl[4] proposed that (high beta) electron rings could be formed into a bumpy torus and that the local magnetic wells created by the rings would stabilize the core plasma in which they were formed. Specifically, by creating a well of sufficient depth, these rings would stabilize a plasma against flute and interchange instabilities; and at high core betas, would enhance stability against ballooning modes[5]. This motivated the first E.B.T. experiment. The device was built and and lived up to expectations. Experiments established that microwave heating did indeed produce hot electron rings and that these rings did enhance the stability of the toroidal plasma in a bumpy torus[6].

4. R.A. Dandl, H.O. Eason, P.H. Edmonds, A.C. England, G.E. Guest, C.L. Hedrick, J.T. Hogan, and J.C. Sprott, Plasma Physics and Controlled Nuclear Fusion (International Atomic Energy Agency, Vienna, 1971) Vol. II p.607

5. Dandl, et. al. pp.607

On closer inspection, however, the problem is not a simple one. The stability requirements of the hot electron rings and the toroidal core plasma are closely coupled. From both kinetic theory models and experiments, it is a well-founded fact that the relative densities of cold (core) plasma and hot (electron ring) plasma are a critical factor in the overall stability of the configuration[7]. Further, the stability of the core plasma requires that the magnetic well produced by the rings be strong enough that $U' < 0$, where $U = \frac{\int \partial l}{B}$, near the edge of the plasma, to support a pressure gradient there[8]. As will be seen later, these stability criteria provide limits on the betas of both hot and cold plasmas which define the regions where they are stable.

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6. G.E. Guest EBT Ring Physics: Proceedings Of The Workshop Dec. 3-5, 1979, Oak Ridge, Tennessee, p.163

7. R.R. Dominguez EBT Ring Physics: Proceedings Of The Workshop,Dec. 3-5, 1979, Oak Ridge, Tennessee, p.383

8. N.A. Uckan, EBT Ring Physics: Proceedings Of The Workshop, Dec. 3-5, 1979, Oak Ridge, Tennessee, p.31

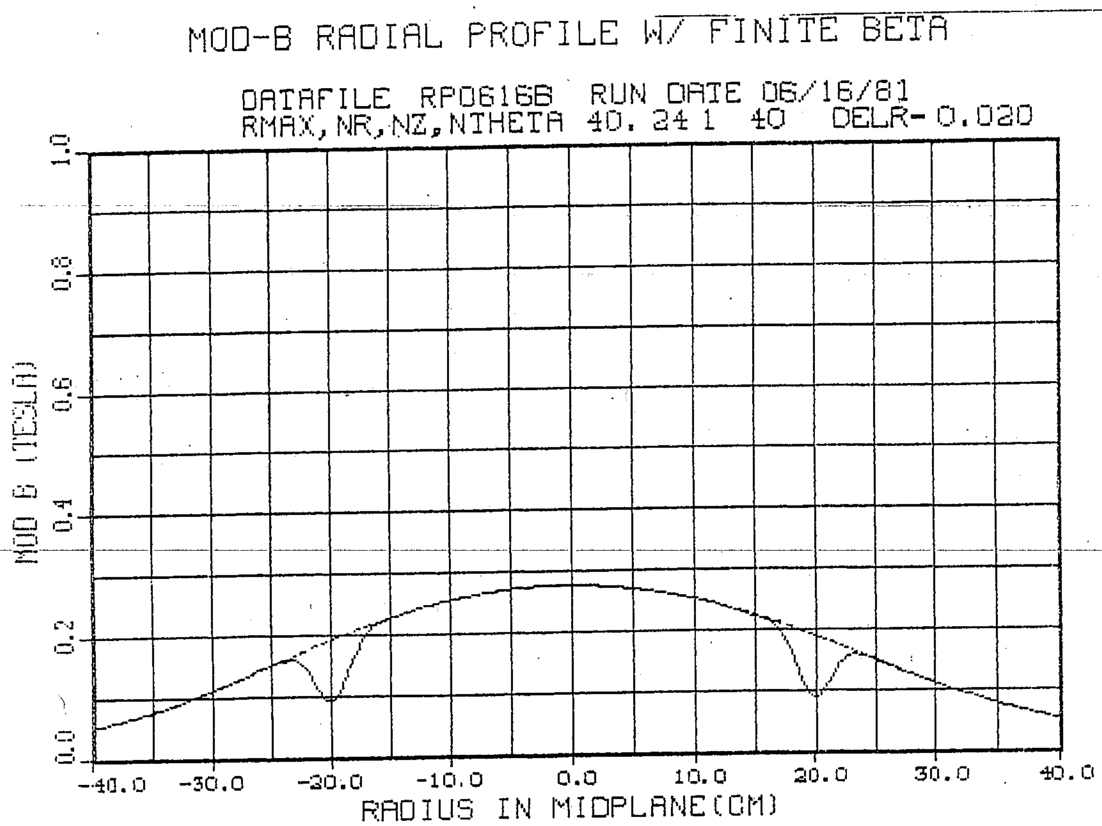
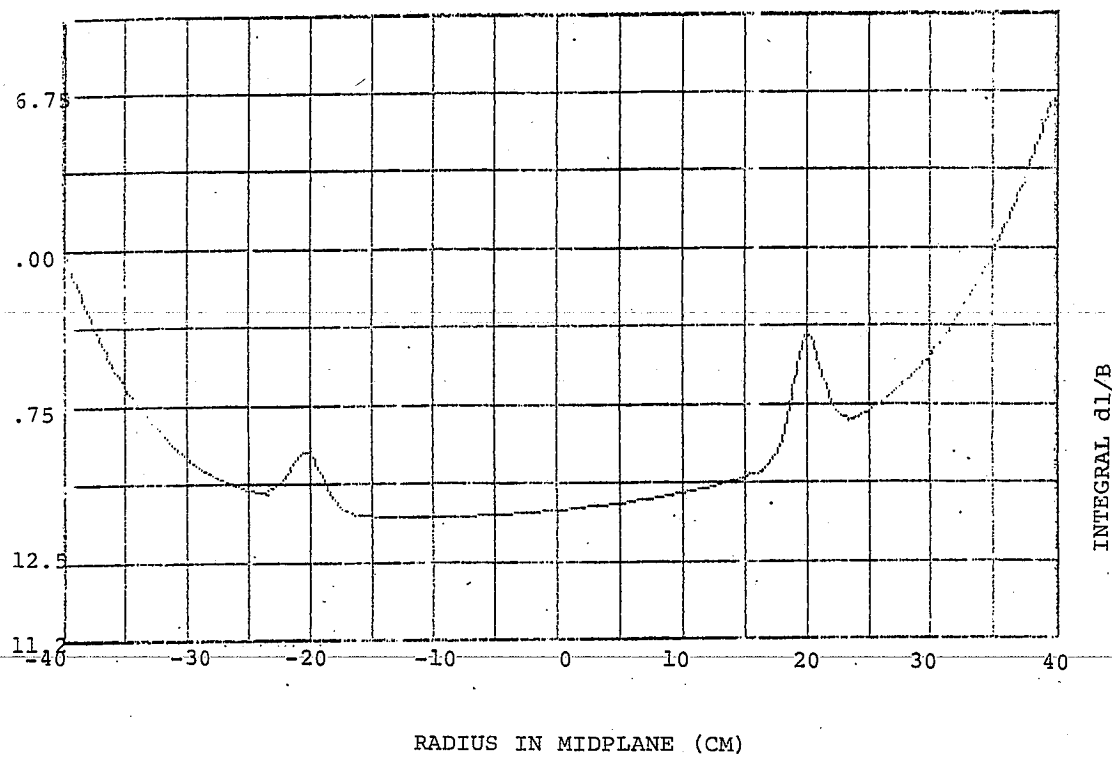


FIGURE #1

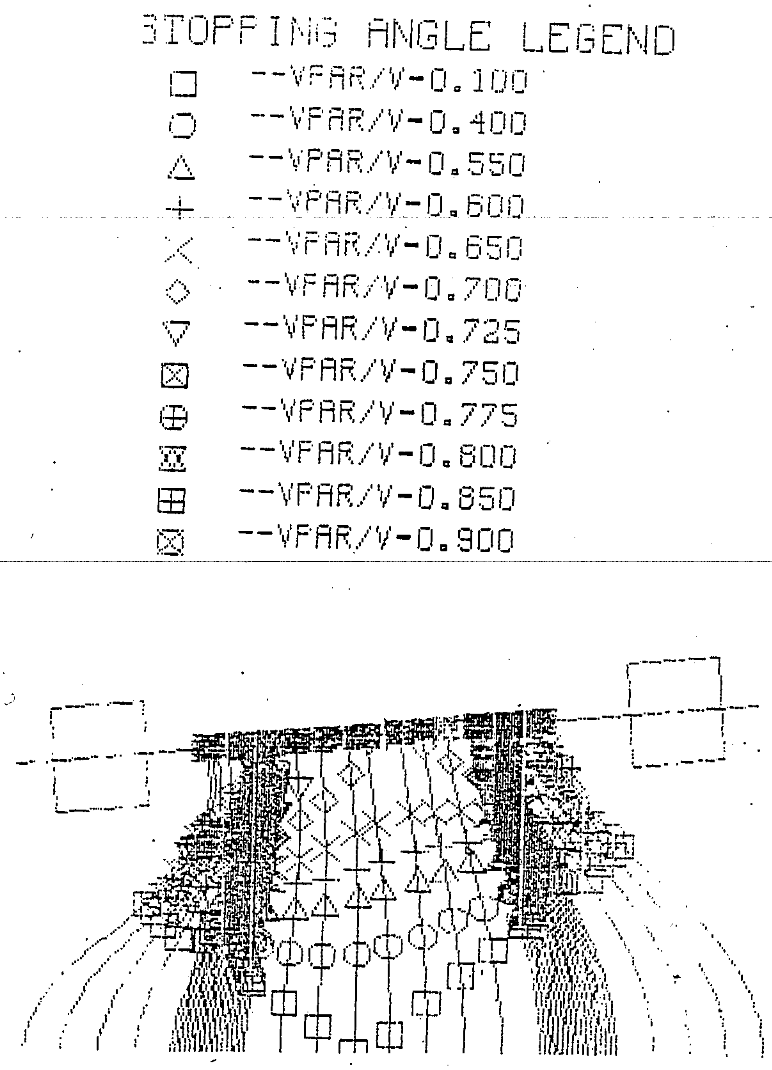
Mod-B vs. Radius for EBT configuration with a major radius of 3.601 meters and 36 humps..

FIGURE # 2



dl/B vs. radius in the midplane for anEBT configuration with a major radius of 3.601 meters and 36 bumps.

FIGURE #3



- Field line and stopping angle diagram for EBT configuration. Major radius=3.601 meters number of bumps=36

Today an evolution of knowledge is occuring along this very line of thought. Two models have been advanced to describe these regions of stable and unstable plasma configurations. The rigid ring model assumes that the only effect of the rings is to produce a local minimum in the magnetic field and there is no further interaction between ring and core plasmas on MHD time scales[9]. This model predicts that stable confinement of the core plasma (in the ideal MHD limit) will occur if β_{core} is less than β_{hot}; provided that the ring beta is of sufficiently high value to produce a local magnetic well[10]. The interacting ring theory alters these results in the region, β_{hot} = 10-30%[11]. Unfortunately, this is the region where a fusion reactor of the E.B.T. design must operate to be economically feasible as an energy source[12], so the point of departure of the two stability models is extremely significant.

9. D.B. Nelson and C.L. Hedrick Nuclear Fusion, 19, 283, (1979)

10. G.E. Guest, C.L. Hedrick, and D.B. Nelson, Phys. Fluids, 18, 871, (1975)

11. J.W. Van Dam and Y.C. Lee, EBT Ring Physics: Proceedings Of The Workshop, Dec. 3-5, 1979, Oak Ridge, Tennessee, p.471

12. N.A. Uckan, EBT Ring Physics: Proceedings Of The Workshop, Dec. 3-5, 1979, Oak Ridge, Tennessee, p.33

The theory which evolved from the rigid ring approach is the interacting ring model of stability. It predicts much more pessimistic regions of stability with respect to the betas of the core plasma and the rings, especially in the region of reactor operation. To be specific, the rigid ring model assumes that the rings are unaffected by low frequency perturbations in the core plasma[13]. On the other hand, Vam Dam and Lee, using the interacting ring assumption that the rings respond significantly to low frequency perturbations of the core plasma, especially in the parallel component of its magnetic field, contend that ring-core interactions will produce compressional effects which lead to destabilization, if the core beta exceeds a very low value[14]. More details will appear later, as well as an expansion of this distinction at a more basic level, but suffice it to say here, that this evolution of the knowledge of the interactions between ring and core plasmas still continues.

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13. G.E. Guest, C.L. Hedrick, and D.B. Nelson, Phys. Fluids, 18, 871, (1975)

14. J.W. Van Dam and Y.C. Lee, EBT Ring Physics: Proceedings Of The Workshop, Dec. 3-5, 1979, Oak Ridge, Tennessee, p.471

Someday it is hoped that with interacting ring theory using a more realistic toroidal (r-dependent) geometry rather than a slab geometry and with the inclusion of so far neglected effects such as ring kinetic effects and effects of the backround plasma, that a comprehensive, clear picture of stability in ring stabilized plasmas will emerge. However, what stands out, enhanced by this evolution of knowledge, is crystal clear. For the bumpy torus concept of plasma confinement to succeed in building a fusion reactor, the physics of the hot electron plasma as well as the physics of the core plasma must be understood. And they must be understood not just separately, but as they interact as species in a self-contained unit. Further, this basic physics must be engineered into a magnetic configuration. The magnetic well created by the rings in fact exists as do the stubborn instabilities of all presently known mirror confined plasmas, both with and without annuli. To interface these two areas of knowledge is a major task as well as a major hope in successfully building a fusion reactor and will be the subject of this paper.

USE OF VLASOV-MAXWELL EQUATIONS AND MHD THEORY IN RING STABILIZED PLASMAS

The most striking feature of what has been classifed in this paper as ring stabilized plasmas is that they consist of a very hot species, the electron rings, whose energies can run into the Mev range, and a relatively cold species, the core plasma, whose energies are of the order of Kev. Thus in order to understand the physics of these plasmas; and as a consequence, their regions of stable operation, one must deal with a plasma whose component species can and do exhibit totally different physical characteristics.

On a purely intuitive level, one would expect the two species to obey two different sets of equations. The Vlasov approximation to the B. B. G. K. Y. hierarchy deals with a collisionless regime and would seem best suited for describing the behavior of the electron rings. On the other hand, a fluid-type MHD description seems more applicable to the colder core plasma. Yet to describe the behavior of ring stabilized plasmas, these two essentially different plasma regimes must be coupled into a consistent, closed set of equations. Indeed, the difficulty shows up immediately in elementary plasma physics. For the MHD equations to be valid, all drift

1. G.E. Guest, C.L. Hedrick, and D.B. Nelson, Phys. Fluids 18,871,(1975)

velocities must be small compared to the ion thermal velocity[1]. Yet in the hot electron rings, the drift velocity of the electrons normally exceeds the ion thermal velocity.

This dilemna was postulated to be resolved by considering the parameters of interest in ring stabilized reactors (especially E.B.T.)[2]. In all known stability regions of E.B.T., the density of the hot electrons is much less than the density of the core plasma, while the temperature of the hot electrons is much greater. Under these circumstances, the interaction of the rings with the core plasma can be ignored and indeed, it will be shown that the dispersion relation derived from Vlasov theory agrees with that derived using MHD assumptions in the single fluid limit. This, however, is only the first step in this evolutionary process of understanding the relation between the rings and the core plasma. Soon after these MHD calculations were published, it was shown that having the hot electron drift frequency exceed the ion cyclotron frequency can produce effects not seen when using MHD analysis. This implies that differences

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2. Guest, Hedrick, Nelson, Phys. Fluids 18,871

3. Y.C. Lee, M.N. Rosenbluth, and J.W. Van Dam, IFS Report 12, March, 1981

between MHD theory predictions and the actual behavior of ring stabilized plasmas are significant[3]. Additionally, the single fluid plasma comparison seems to exclude the rings from consideration or at least to treat them as physically unrelated to the core plasma, thus assuming the validity of rigid ring theory rather than proving it.

The question as to whether MHD theory remains consistent in a hot-cold species plasma is the very essence of the difference between the two theories to be discussed. The advocates of rigid ring theory perform their stability analysis using the assumption that MHD theory works well enough to be able to modify and use when studying ring stabilized plasmas. On the other hand, interacting ring theory maintains that there is additional physics left out when using MHD theory (the frozen-in field line constraint must be replaced by flux conservation. See chapter 4-6). It is precisely this line of thinking that leads from the first attempts to explain stability regions in E.B.T. to the contemporary view.

In what follows, two methods will be developed to find the stability regions of ring stabilized plasmas. The first will incorporate rigid ring theory, the second interacting ring theory. It will be found that they agree in areas where a single fluid approximation is valid, but they will disagree when it is necessary to consider regions of reactor operation where the hot electron rings must be considered as a separate component of the plasma either with respect to their density or their pressure effects.

This will not be obvious at first, but by breaking the analysis into two parts, one in which $\delta = 0$ (single fluid approximation; δ = density of hot electrons/density of core plasma), and one in which $\delta \neq 0$ (multiple species plasma), it will be obvious that what is breaking down is the MHD approximation.

The complexity of mathematical treatment and the crudeness of results often obscures basic issues that hit at the very heart of the subject of discussion. The Vlasov-Maxwell equations in the MHD-approximation seem like a perfectly reasonable starting point in analyzing the stability of ring stabilized plasmas. Indeed for awhile nobody thought to question this approach. Yet the frequency domain of ring stabilized plasmas is inherently outside the domain of MHD theory. To replace the MHD approach with an approach that self-consistently includes the frequency range of the hot electrons seems a logical next step in understanding the behavior of ring stabilized plasmas. The results of this step and its comparison with modified MHD analysis will be explored in depth in what follows.

RIGID RING THEORY

The beginning of any discussion of rigid ring theory begins with the assumption that a modified MHD analysis is valid and that the author is free to use MHD equations. The idea is often justified by the fact that in the approximation that the plasma can be treated as a single fluid ($n_a \ll n_c; T_a \gg T_c$), Vlasov theory and MHD theory agree. Later, by allowing the hot electron component of the plasma to be perturbed, this justification was extended to include regions of operation in which the hot electron pressure was not too large[1].

The rigid ring theory was proposed by Nelson, Hedrick, et. al. and has been around for nearly a decade[2]. In it the hot electrons of the rings do not interact with the rest of the plasma on an MHD-time scale although they are in force balance with the plasma and the external magnetic field on a diffusion time scale. With these assumptions, the equation of motion of the plasma is:

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1. D.B. Nelson and C.L. Hedrick Nuclear Fusion 19, 283, (1979)

2. G.E.Guest, C.L. Hedrick, and D.B. Nelson Phys. Fluids, 18, 871 (1975)

$$\rho\frac{\partial \underset{\sim}{v}}{\partial t} = (\nabla \times \underset{\sim}{B}-J_e) \times \underset{\sim}{B}-\nabla p$$

where J_e is the current produced by the diamagnetic drift of the hot electrons. Further, in equilibrium, all species in the plasma satisfy the equations:

$$\underset{\sim}{J}_e \times \underset{\sim}{B} = \nabla \cdot \underset{\sim}{p}_e$$

$$\underset{\sim}{J}_p \times \underset{\sim}{B} = \nabla p$$

$$\underset{\sim}{J}_e + \underset{\sim}{J}_p = \underset{\sim}{J}_{tot} = \nabla \times \underset{\sim}{B}$$

where:

J_p is the plasma current

$\underset{\sim}{p}_e$ is the pressure tensor of the total hot electrons

p is the scalar pressure of the ions

Now to proceed any further an energy princple must be constructed from which the stability of the system can be analyzed. Fortunately, an energy principle had been studied which is applicable in this type of system. It was first proposed in the late 1950's and studied by many researchers[3,4] and thus was quite well established by the time of rigid ring theory. It can be derived as follows:

Consider the limit m/e small. The energy of an MHD plasma can be written as:

$$\varepsilon = \int d^3 \underset{\sim}{x} \frac{1}{2}(\underset{\sim}{B}^2 + \underset{\sim}{E}^2) + \sum\iiint \frac{\beta}{q} d\mu\, d\varepsilon\, d^3 \underset{\sim}{x}\left[mf\left(\frac{q^2}{2} + \frac{\underset{\sim}{\alpha}^2}{2} + \mu\beta\right)\right] \tag{1}$$

where:

$$\underset{\sim}{v} = \underset{\sim}{\alpha} + \underset{\sim}{v}_\perp + q\hat{n}$$

$$\underset{\sim}{B}(\underset{\sim}{x},t) = |\underset{\sim}{B}(\underset{\sim}{x},t)|\, n(\underset{\sim}{x},t) \equiv \beta\hat{n}$$

$$\underset{\sim}{\alpha} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2}$$

$$q = \underset{\sim}{v}\cdot\hat{n}$$

$$\mu = \frac{\underset{\sim}{v}_\perp^2}{2\beta} = \text{constant}$$

$$\varepsilon = \frac{q^2}{2} + \mu\beta$$

3. M.D. Kruskal and D.P. Oberman Phys. Fluids 1, 275, (1958)

4. M.N. Rosenbluth, N.A. Krall, N. Rostoker Nuclear Fusion Suppl. Part 1, 143, (1962)

f is a rotationally symmetric function in velocity space

The equilibrium condition can be written as:

$$0 = -\nabla \cdot \underset{\sim}{p} \cdot + (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}$$

where $\underset{\sim}{p}$ is the total pressure tensor

Now displace the system by a vector: $\xi(x,t)$. To first order, the change in ε is given by:

$$\varepsilon^{(1)} = \int d^3 \underset{\sim}{x} \left(\frac{1}{2} \underset{\sim}{B}^2 \nabla \cdot \underset{\sim}{\xi} + \underset{\sim}{B} \cdot \underset{\sim}{B}^{(1)}\right) - m \int\int\int d\mu \, d\varepsilon \, d^3 \underset{\sim}{x} \left[(g + \varepsilon g_{\varepsilon})(\beta q \nabla \cdot \underset{\sim}{\xi} + (\beta q)^{(1)} + \beta q (f^{(1)} + \varepsilon f_{\varepsilon}^{(1)})\right] \quad (2)$$

~~where subscripts represent differentiation,~~

superscripts represent quantities of first order change; and,

g is a so far arbitrary function.

Furthermore, if the magnetic field is displaced from $\underset{\sim}{x}$ to a new position $\underset{\sim}{x} + \underset{\sim}{\xi}$, the new value of the magnetic field to second order will be:

$$\underset{\sim}{B}(\underset{\sim}{x}) + \underset{\sim}{B}^{(1)}(\underset{\sim}{x}) + \underset{\sim}{B}^{(2)}(\underset{\sim}{x}) = \underset{\sim}{B} + [\underset{\sim}{B}\cdot\nabla\underset{\sim}{\xi} - \underset{\sim}{B}\nabla\cdot\underset{\sim}{\xi}]$$

$$+ \frac{1}{2}[\underset{\sim}{B}(\nabla\cdot\xi)^2 + \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi} - 2(\nabla\cdot\underset{\sim}{\xi})\underset{\sim}{B}\cdot\nabla\underset{\sim}{\xi}]$$

(3)

Let all the constants of the motion assume their equilibrium values. Summing over all the particles of the system, this boils down to the vanishing of the integral:

$$0 = \iiint (\frac{\beta}{q}) d\mu\, d\varepsilon\, d^3\underset{\sim}{x} G(f,\mu,1)$$

G is an arbitrary function of f,μ, and 1;

where 1 labels a line of magnetic force.

Notice that a consequence of this is that the field lines are frozen into the plasma.

Integration by parts and substitution of the defined quantities yields a more convenient form:

$$0 = -\iiint d\mu\, d\varepsilon\, d^3\underset{\sim}{x} G_f(g(\mu,\varepsilon,1),\mu,1)[\beta q\nabla\cdot\underset{\sim}{\xi} g_\varepsilon + g_\varepsilon(\beta,q)^{(1)} + \frac{\beta}{q} f^{(1)}]$$

(4)

where G_f is an arbitrary function.

Using the identity:

$$\int_T d^3\underset{\sim}{x}\beta a(x) = d\psi \int_1 dl a(x)$$

where integration is over a thin tube of flux T on the LHS,

integration is over a line of force l on the RHS,

and ψ is the flux.

One can rewrite the constraining condition in its final form:

$$0 = \int_T d^3\underset{\sim}{x}\left[\beta q(\nabla\cdot\underset{\sim}{\xi})g_\varepsilon + \left(\frac{\beta}{q}\right)(-\nabla\cdot\underset{\sim}{\xi} + \hat{n}\hat{n}:\nabla\underset{\sim}{\xi})(q^2-\mu\beta)g_\varepsilon - \left(\frac{\beta}{q}\right)f^{(1)}\right]$$

Since $G_f(g(\upsilon,\varepsilon,l),\mu,l)$ is arbitrary let:

$$G_f(g(\upsilon,\varepsilon,l),\mu,l) = -m\varepsilon$$

Eliminating $f^{(1)}$ using equation(4) in equation(2), have:

$$\varepsilon^{(1)} = \int d^3\underset{\sim}{x}\left(\frac{1}{2}\underset{\sim}{B}^2\nabla\cdot\underset{\sim}{\xi} + \underset{\sim}{B}\cdot\underset{\sim}{B}^{(1)}\right) - m\int\int\int d\mu\varepsilon\, d^3\underset{\sim}{x}g\left[\beta q\nabla\cdot\underset{\sim}{\xi} + (\beta q)^{(1)}\right]$$

Substituting into this equation, the equilibrium condition, definitions of pressure and magnetic field variation, $\varepsilon^{(1)}$ can be shown to vanish. It is evident that to obtain an energy principle, one must go to second order.

To this end, the second order variation in energy can be written as:

$$\varepsilon^{(2)} = \int d^3\underset{\sim}{x}\{\frac{1}{4}[(\nabla\cdot\underset{\sim}{\xi})^2 - \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi}]B^2 + (\nabla\cdot\underset{\sim}{\xi})\underset{\sim}{B}\cdot\underset{\sim}{B}^{(1)} + \frac{1}{2}\underset{\sim}{B}^{(1)2}$$

$$+ \underset{\sim}{B}\cdot\underset{\sim}{B}^{(2)} + \frac{1}{2}\underset{\sim}{E}^{(1)2} + \frac{1}{2}\rho\alpha^2\} + \frac{m}{3}\int\int\int d\mu\, d\varepsilon\, d^3\underset{\sim}{x}\{\frac{1}{2}[(\nabla\cdot\underset{\sim}{\xi})^2 - \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi}]\beta q^3$$

$$+ (\nabla\cdot\underset{\sim}{\xi})((\beta q)^3)^{(1)} + (\beta q^3)^{(2)}(\varepsilon g_{\varepsilon\varepsilon} + 2g_\varepsilon) + ((\beta q)^3)^{(1)}$$

$$+ (\nabla\cdot\underset{\sim}{\xi})(\beta q^3)(\varepsilon f_{\varepsilon\varepsilon}{}^{(1)} + 2f_\varepsilon{}^{(1)}) + \beta q^3(\varepsilon f_{\varepsilon\varepsilon}{}^{(2)} + 2f_\varepsilon{}^{(2)})\}$$

where ρ is the mass density

Application of the constraint equation(4), to second order yields:

$$0 = \{\frac{1}{3}\int\int\int \beta q^3\, d\mu\, d\varepsilon\, d^3\underset{\sim}{x}(g' f_\varepsilon^{(2)} + g' f_{\varepsilon\varepsilon})\}^{(2)}$$

primes denoting spatial differentiation

$$= \frac{1}{3}\int\int\int d\mu\, d_{\varepsilon\varepsilon}\, d^3\underset{\sim}{x}\{G''f^{(2)}(\beta q^3)_{\varepsilon\varepsilon}$$

$$+ \frac{1}{2}(G''f^{(1)2}(\beta q^3)_{\varepsilon\varepsilon} + (G'f^{(1)})_{\varepsilon\varepsilon}(\beta q^3)^{(1)}$$

$$+ G'f^{(1)}(\nabla\cdot\underset{\sim}{\xi})(\beta q^3)_{\varepsilon\varepsilon} + \frac{1}{2}[(\nabla\cdot\underset{\sim}{\xi})^2 - (\nabla\underset{\sim}{\xi}):(\nabla\underset{\sim}{\xi})]\beta q^3 + (\nabla\cdot\underset{\sim}{\xi})(\beta q^3)^{(1)}$$

$$+ (\beta q^3)^{(2)})(G''f^{(2)} + G'f_{\varepsilon\varepsilon})\}$$

This can point the way toward the desired energy principle if the second order variation is written in the following manner:

$$\varepsilon^{(2)} = \frac{1}{2}\int d^3\underset{\sim}{x}(e^{(1)2} + \rho\underset{\sim}{\alpha}^2) + \delta w$$

where:

$$\delta w = \frac{1}{2}\int d^3\underset{\sim}{x}\{\underset{\sim}{Q}^2 + (\nabla\times\underset{\sim}{B})\cdot\underset{\sim}{\xi}\times\underset{\sim}{Q} + \underset{\sim}{\xi}\cdot\nabla\xi(\nabla\cdot\underset{\sim}{p})\} - \frac{m}{2}\int\int\int d\mu\,\varepsilon\, d^3\underset{\sim}{x}((\frac{\beta}{q})\frac{f^{(1)2}}{g_\varepsilon})$$

$$+ \frac{m}{3}\int\int\int d\mu\, d\varepsilon\, d^3\underset{\sim}{x}g_\varepsilon\{\frac{1}{2}[(\nabla\cdot\underset{\sim}{\xi})^2 - \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi}]\beta q^3 + (\nabla\cdot\underset{\sim}{\xi})(\beta q^3)^{(1)} + (\beta q^3)^{(2)}\}$$

where: $\underset{\sim}{Q} = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})$

Since δw is quadratic in ξ and f_1, this is the required energy principle for stability analysis given the validity of the constraints. If δw is positive for all allowed values of ξ and f_1, the system will be stable. Otherwise, it will be unstable.

Now to use this equation, note that this calculation is defined in the small m/e limit. Also the hot electron current density must be included to apply this energy principle to ring stabilized plasmas. Substituting the equation of motion for this type of plasma into the energy principle and dropping the last two terms (and adding a term proportional to the growth rate for completeness[5]), then stability can be determined by the sign of:

$$\delta w = \int d\tau \left(Q^2 + (\nabla \times B - J_e) \times \xi \cdot Q + \xi \cdot \nabla p \nabla \cdot \xi + \gamma p (\nabla \cdot \xi)^2\right] \tag{5}$$

If $\nabla \cdot J_e = 0$, this equation can be integrated by parts to yield:

$$\delta w = \frac{1}{2} \int d\tau \left(Q + F X\right)^2 - A X^2 + \gamma p (\nabla \cdot \xi)^2] \tag{6}$$

5. D.B. Nelson and G.O. Spies Phys. Fluids 17, 2133,(1974)

where:

$$\tilde{Q} = \nabla \times (\tilde{\xi} \times \tilde{B})$$

$$X = \tilde{\xi} \cdot \nabla\psi$$

$$\tilde{F} = \tilde{J}_p \times \tilde{e}$$

$$\tilde{e} = \frac{\nabla\psi}{|\nabla\psi|^2}$$

$$\tilde{A} = 2\tilde{F}\cdot(\tilde{B}\cdot\nabla)\tilde{e} - \frac{\tilde{J}_e \cdot \tilde{J}_p}{|\nabla\psi|^2}$$

And $2\pi\psi$ is the magnetic flux enclosed by a surface at constant pressure;

$\tilde{\xi}$ is a small spatial perturbation

~~γ is the growth rate.~~

This equation was studied in 1974 by Nelson when he studied the stabilizing effect of the growth rate term, compressive energy, in the energy principle[6]. He found that by not dropping the growth rate term, a more optimistic stability condition is found if all the field lines are closed. This stability analysis applies in two limiting cases. In

6. Nelson, Spies Phys. Fluids 17, 2133

straight mirrors where the magnetic field has a negligible component in one direction, the energy principle constitutes both a necessary and sufficient condition for stability (if it is positive). Also at low β, equation(6) approaches a sufficient condition for interchange stability in an arbitrary geometry. Thus equation(6) is well applicable to E.B.T. and probably most ring stabilized plasma configurations that will be devised.

Starting, then, from equation(6), using the Schwarz inequality:

$$|\underset{\sim}{Q}|^2 |\nabla\psi|^2 \geqslant (\underset{\sim}{Q}\cdot\nabla\psi)^2$$

And the identity:

$$\underset{\sim}{Q}\cdot\nabla\psi = \underset{\sim}{B}\cdot\nabla X$$

one obtains:

$$\delta w \geqslant \frac{1}{2}\int d\tau\left[\frac{(\underset{\sim}{B}\cdot\nabla X)^2]}{|\nabla\psi|^2]} - AX^2\right] \qquad (7)$$

where $\langle X^2 \rangle$ has been normalized to 1

and $\langle \ldots \rangle = (\frac{\int dl\ldots}{q}$

$q = \int\frac{dl}{B}$

Minimizing X in equation(7), one has the Euler equation:

$$\underset{\sim}{B}\cdot\nabla\left(\frac{\underset{\sim}{B}\cdot\nabla X}{|\nabla\psi|^2}\right)+ (A + \Lambda)X = 0$$

where Λ is the Lagrange multiplier corresponding to the normalization described previously and is stable if on every field line $\Lambda_1 > 0$.

The next step in this stability analysis is to decompose Q in the orthogonal basis (e,j,F). When this is done, the energy principle becomes:

$$\delta w = \frac{1}{2}\int d\underset{\sim}{\tau}\left\{\frac{(\underset{\sim}{B}\cdot\nabla X)^2}{|\nabla\psi|^2} + \frac{\underset{\sim}{J}\cdot\underset{\sim}{Q})^2}{J^2} + \frac{(\underset{\sim}{F}\cdot\underset{\sim}{Q} + \underset{\sim}{F}^2 X)^2}{\underset{\sim}{F}^2} - AX^2 + \frac{\gamma p}{p'^2}[p'\underset{\sim}{B}\cdot\nabla Z + \underset{\sim}{F}\cdot\underset{\sim}{Q} + (\underset{\sim}{F}^2 - A)X]^2\right\} \tag{8}$$

where: $Z = \dfrac{\underset{\sim}{F}\cdot\underset{\sim}{\xi}}{p'}$

$p' = \dfrac{\partial p}{\partial\psi}$

Dropping the $(J\cdot Q)^2$-term, minimizing Z with the constraint:

$$\underset{\sim}{B}\cdot\nabla Z = 0$$

and minimizing the term: $\underset{\sim}{F}\cdot\underset{\sim}{Q}$, one obtains;

$$\delta W \geq \frac{1}{2}\int d\tau \left[\frac{(\underset{\sim}{B}\cdot\nabla X)^2}{|\nabla\psi|^2} - AX^2 + \frac{\langle AX\rangle^2}{\alpha^2} \right]$$

The resulting Euler equation is finally:

$$\underset{\sim}{B}\cdot\nabla\left(\frac{\underset{\sim}{B}\cdot\nabla X}{|\nabla\psi|^2}\right) + (A + \Lambda)X = A\frac{\langle AX\rangle}{\alpha^2} \qquad (9)$$

where:

$$\alpha^2 = \langle F^2 \rangle + \frac{p'^2}{\gamma p}$$

$$\langle F \rangle = \frac{\int F dl/B}{q} \; ; \quad q = \int \frac{dl}{B}$$

Mathematically, this is a non-standard integrodifferential equation. To solve it, note that its lowest eigenvalue Λ_1, can be found from finding the eigenvalues of the equation which results from setting the right hand side of equation(9) equal to zero. Consider the following theorem:

Let λ_1 and λ_2 be the smallest eigenvalues of the reduced equation (right hand side of equation(9) equal to zero).

Then $\lambda_1 \leq \Lambda_1 \leq \lambda_2$ and:

(1) if $\lambda_1 > 0$, then $\Lambda_1 > 0$ (stable)

(2) if $\lambda_2 < 0$, then $\Lambda_1 < 0$ (unstable)

(3) if $\lambda_1 < 0$ and $\lambda_2 > 0$, then $\Lambda_1 > 0$, if and only if $p'(p'q + \gamma pq') < 0$ (Λ_1 stable)

This theorem points out several interesting regions of plasma stability. For example, condition(3) implies that in a region where $q' < 0$ and $p' < 0$, the condition is never satisfied and stability is determined by the sign of λ_1. Additionally, if $q' > 0$ and $p' < 0$, then λ_1 is negative and stability is determined by either condition(2) or (3). Equilibrium is thereby considered to form two regions, depending on the sign of q'. Constant pressure is always stable ($p' = 0$) and $p' < 0$ may be stable in either region, but at the edge of the plasma where p goes to zero, q' must always be negative for there to be stability. It is precisely this function that the hot electrons serve in ring stabilized plasmas. To determine the marginal strength of J_e required to contain a low beta core plasma, one must expand equation(9) in the limit of low beta for arbitrary J_e.

In this limiting case, β can be used as a measure of deviation of B from the vacuum field produced by external currents. Hence $p' \propto J \propto \beta$ in this limit and:

$$\lambda_1 = - \langle A \rangle + O(\beta^2) = \frac{p'q'}{q} + O(\beta^2)$$

Next one can relate $\langle A \rangle$ to the eigenvalues using the reduced equation(9) and the theorem stated above. Specifically:

$$\langle A \rangle \leq 0 \rightarrow \Lambda_1 \gtrless 0 \text{ as } \lambda_1 \gtrless 0$$

$$0 < \langle A \rangle < \alpha^2 \rightarrow \Lambda_1 < 0$$

$$\langle A \rangle \geq \alpha^2 \rightarrow \Lambda \gtrless 0 \text{ as } \lambda_2 \gtrless 0$$

Now using the identity:

$$\langle A - \underset{\sim}{F}^2 \rangle = - \frac{p'q'}{q}$$

One can obtain the simplified condition:

$$\Lambda_1 \geq 0 \leftrightarrow \text{ either } \lambda_1 \geq 0 \text{ or} \lambda_2 \geq 0 \text{ and } p'(p'q + \gamma pq') \leq 0$$

What follows is known as the interchange stabilty condition:

The stability threshold is reached when J_e is strong enough to satisfy:

$$q'(p'q + \gamma pq') \geq 0$$

(10)

Finally, equation(9) must be integrated to see the total stability picture. Due to the intrinsic three-dimensional nature of this problem, numerical techniques on computers are long and costly. The easiest way to do the integration is to assume a large aspect ratio and use perturbation theory. In this limit, equilibrium is assumed to be a perturbation about a periodic bumpy cylinder and one assumes that the J•B term vanishes. To lowest order, then, the eigenvalues are those of a bumpy cylinder and higher order terms are proportional to the inverse aspect ratio.

To fascilitate these calculations, one first expands the coefficient A:

$$A = 2\mathbf{e}\cdot\nabla p\cdot\boldsymbol{\kappa} + \frac{2}{\mathbf{B}^2}(\mathbf{J}_p\cdot\mathbf{B})\mathbf{B}\times\mathbf{e}\cdot(\mathbf{B}\cdot\nabla)\mathbf{e} - \frac{\mathbf{J}_p\cdot\mathbf{J}_e}{|\nabla\psi|^2}$$

Then to lowest order equation(9) becomes:

$$\frac{\partial}{\partial l}\left(\frac{1}{r^2 B}\frac{\partial X}{\partial l}\right) + \frac{\Lambda - p' D}{\mathbf{B}}X = \frac{D}{\mathbf{B}}\,\frac{\gamma p}{1 + \frac{\gamma p I}{q}}\langle DX\rangle \qquad (11)$$

where:

$$D = \frac{-2e_\psi\cdot\boldsymbol{\kappa}}{rB} + \frac{\mathbf{J}_e}{r\mathbf{B}^2} = -\frac{p'}{\mathbf{B}^2} - \frac{\mathbf{J}_{tot}}{r\mathbf{B}^2} - \frac{2\partial\ln B}{\partial\psi}$$

$$I = \int \frac{dl}{B^3}$$

$$e_\psi = \frac{e_\psi}{|\underset{\sim}{e}|}$$

The differential equation(11) was obtained with the use of the identity:

$$\underset{\sim}{e}_\psi \cdot \underset{\sim}{\kappa} = \frac{\underset{\sim}{J}_{tot}}{\underset{\sim}{B}} + \frac{r\partial B}{\partial\psi}$$

Now equation(11) is identical to the case of a bumpy cylinder equilibrium. In this case, Λ_1 being positive is a necessary and sufficient condition for stability. Further, ring stabilized plasma experiments in the past and probably in the future will be on devices with large aspect ratios. In these instances, the corrections to equation(11) are small and in the limit that the aspect ratio goes to infinity, equation(11) is exact. For these reasons one can use equation(11) as the stability condition.

To summarize, it has been shown that equation(11) represents the Euler equation for the form of δw in an axisymmetric configuration (bumpy cylinder) in equilibrium, for large aspect ratio devices as, for example, E.B.T.. This was accomplished by minimizing the displacement in the surface:

$$\delta w = \int \delta w(\psi) d\psi$$

$$\delta w(\psi) = \int \frac{dl}{B}\left[\left(\frac{1}{r}\frac{\partial X}{\partial l}\right)^2 + p' DX^2 \right] + \frac{q^2}{I + \frac{q}{\gamma p}} \langle DX \rangle^2 \qquad (12)$$

The physics in these equations points to the stabilizing effects of the hot electron currents. For equilibrium, p' is negative, which implies that D is negative and δw is positive. Now in the absence of external currents, D is negative and the configuration is always unstable. With a hot electron current, there is a region near the outer edge of the plasma where J_e is negative, and, in this region, D can become negative. However, because J_{total} is negative, D cannot be negative unless $\frac{\partial B}{\partial \psi}$ is negative. Put another way, stabilization is possible only when a local magnetic well is created.

To be more specific, one can insert a test function, X = constant, into equation(12). When this is done, the condition for stability on each flux surface becomes:

$$p' q \langle D \rangle + \frac{q^2 \langle D \rangle^2}{I + \frac{q}{\gamma p}} \geqslant 0$$

But:

$$q \langle D \rangle = q' - p' I$$

Thus for stability:

$$(q' - p' I)(p' q + \gamma p q') \geq 0$$

So depending on the sign of q':

$$p' q + \gamma p q' \geq 0 \text{ if } q' \geq 0$$

$$q' - p' I \leq 0 \text{ if } q' \leq 0$$

The first of these equations is just the interchange condition in the region where q' is positive, but the second equation shows that, except for low beta, making q' negative is not sufficient for stability. $\langle D \rangle$ must also be negative. It is this fact that is the stronger of the two conditions and must be taken as the true guide for finding stable equilibria in this model. Armed with this fact, one can formulate the stable operating regions of ring stabilized plasma in axisymmetric, large aspect ratio devices (like E.B.T.).

To begin, the fundamental equations which apply specifically to an E.B.T.-like magnetic configuration (toroidally connected mirrors, with hot anisotropic electron rings, and a bulk plasma which traverses these regions to form an isotropic torus) can be formed from the equations presented at the beginning of this section:

$$\nabla \times \underset{\sim}{B} = \underset{\sim}{J}_e + \underset{\sim}{J}_p = \underset{\sim}{J}_{tot}$$

$$(\underset{\sim}{J}_e + \underset{\sim}{J}_p) \times \underset{\sim}{B} = \nabla \cdot (\underset{\sim}{\underset{\sim}{p}}_e + p\underset{\sim}{I})$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

Now, assuming azimuthal symmetry implies that $\underset{\sim}{J}_{\parallel} = 0$. In this case, the equations reduce to:

$$\underset{\sim}{B} \cdot \nabla p_{\parallel} = \frac{p_{\perp} - p_{\parallel}}{B} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B} \qquad (13)$$

$$\Delta^* \psi + \nabla \ln \sigma \cdot \nabla \psi = - \frac{r^2}{\sigma} \frac{\partial p_{\parallel}}{\partial \psi} \qquad (14)$$

$$\underset{\sim}{B} = \nabla \psi \times \nabla \theta \qquad (15)$$

where:

$$\Delta^* \psi = \frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial \psi}{\partial r}\right)$$

and ψ is the flux function.

Further, the ellipticity of the assumed equilibrium in E.B.T. requires that the geometrical quantities τ and σ be positive, where:

$$\tau = 1 + \frac{1}{B}\frac{\partial p_{\perp}}{\partial B}$$

$$\sigma = 1 + \frac{p_{\perp} - p_{\parallel}}{B^2}$$

To make the calculations more transparent let:

$$\psi = \psi_c + \psi_p$$

where ψ_c is the flux produced by the external coils and ψ_p is the flux due to the hot electrons and core plasma currents. With this distinction equation(14) becomes:

$$\Delta^* \psi_p + \nabla \ln\sigma \nabla \psi_p = S \qquad (16a)$$

where:

$$S = -\frac{r^2}{\sigma}\frac{\partial p_{\parallel}}{\partial \psi} - \nabla \ln\sigma \cdot \nabla \psi_c \qquad (16b)$$

Making use of this separation of flux, one can solve equation(14) if the perpendicular and parallel components of the pressure have equations (16a) and (16b) as definitions. From this point iterative techniques can be used to find ψ_p. Often a computer can be used in this calculation[7]. First the flux due to the coils alone is used to calculate S and $\nabla \ln\sigma$. Then standard finite differencing techniques for elliptical partial differential equations are used to solve equation(14) for ψ_p. From this ψ_p, S and $\nabla \ln\sigma$ are recomputed for $\psi_p + \psi_c$ and this procedure is repeated until the desired convergence is obtained. The trick here is to find appropriate boundary conditions at the edge of the grid. Again using an infinite aspect ratio approximation, justified in the case of E.B.T.-like machines, the calculation becomes one of finding the flux in a bumpy cylinder. To calculate the boundary conditions for ψ_p then, one merely imposes periodic boundary conditions at the midplane (z = 0) and at an adjacent coil plane (z = 1/2). Further, the radial boundary condition can be obtained via the Biot-Savard law thusly:

$$\underset{\sim}{A} = \frac{1}{4\pi}\int \frac{d^3\underset{\sim}{x}'}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\underset{\sim}{J}$$

$$\psi = rA_\theta$$

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expanding $\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$ in spherical harmonics, obtain:

$$\psi_p = \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{r}{n(n+1)}\frac{P_n^{\,1}(\cos\phi_m)}{R_m^{\,n+1}}\times\int dz'\int r'\,dr'\,(R_0')^n P_n^{\,1}(\cos\phi_0')J_\theta \tag{17}$$

where:

$$R_m = \sqrt{(z-mL)^2 + r^2}$$

$$\cos\phi_m = \frac{(z-mL)}{R_m}$$

Now to solve equation(17) one first recognizes that the problem is azimuthally symmetric, ie.:

$$J_\theta(r,-z) = J_\theta(r,z)$$

$$\psi(r,-z) = \psi(r,z)$$

This implies that even n in equation(17) vanish and only odd n survive. Next, for each iteration, the boundary value of ψ_p is computed from the previous value of J_θ and the value of r_{max} is found (usually about twice the maximum plasma radius).

To complete the analysis, the functional forms of the pressure must be given. Usually, for an E.B.T.-like device, these are given by:

$$p_{\|} = \hat{p}_{\|}(B)g(\psi) + p(\psi)$$

$$p_{\perp} = \hat{p}_{\perp}(B)g(\psi) + p(\psi)$$

where the first expression on the right hand side of the equations describes the hot electrons, and the second, the core plasma. $g(\psi)$ is given by[8]:

$$g(\psi) = \begin{cases} \dfrac{(\psi-\psi_1)^2(\psi-\psi_a)^2}{(\psi_2-\psi_1)^2(\psi_2-\psi_a)^2} & \psi_1 \leqslant \psi \leqslant \psi_2 \\[2ex] \dfrac{(\psi-\psi_b)^2(\psi-\psi_3)^2}{(\psi_2-\psi_b)^2(\psi_2-\psi_3)^2} & \psi_2 \leqslant \psi \leqslant \psi_3 \\[2ex] 0 & \text{otherwise} \end{cases}$$

where:

$$\psi_a = 2\psi_2-\psi_1$$

$$\psi_b = 2\psi_2-\psi_3$$

8. Nelson, Hedrick Nuclear, Fusion, 19, 283

and the coefficients ψ_1, ψ_2, ψ_3 are adjusted to meet the boundary conditions that the peaks and zeroes of the function occur at the correct radii in the midplane (z = 0).

As to the parallel and perpendicular components of the pressure, actually several forms have been investigated. One of these is[9]:

$$\hat{p}_\perp(B) = \frac{1}{2}s(B_c^2 - B^2) \qquad B \leq B_c$$

$$\hat{p}_\parallel(B) = \frac{1}{2}s(B_c - B)^2 \qquad B \leq B_c$$

$$0 \qquad B > B_c$$

Notice that one need only consider $s < 1$ in satisfying the ellipticity requirements, since:

$$\tau = 1 - sg(\psi); \qquad \sigma = 1 + sg(\psi)\left(\frac{B_c}{B} - 1\right)$$

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10. J.B. Taylor Phys. Fluids, 6, 1529 (1963)

will be greater than zero only if $s < 1$. In passing, note that another form of the pressure has been investigated by Taylor[10]:

$$\hat{p}_{\perp} = mcB^2 (B_c - B_m)^{m-1}$$

$$\hat{p}_{\parallel} = cB(B_c - B)^m$$

However, to continue, one can close the equations and make them soluble by introducing a functional form for the scalar pressure of the core plasma. Nelson used a model where the pressure is outward along the magnetic axis up to a specified value of ψ and then falls to zero[11]. Specifically:

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$$p(\psi) = \begin{cases} p0 & \psi < \psi_4 \\ p0\dfrac{(\psi-\psi_c)^2(\psi-\psi_5)^2}{(\psi_4-\psi_c)^2(\psi_4-\psi_5)^2} & \psi_4 \leq \psi \leq \psi_5 \\ 0 & \psi_5 \leq \psi \end{cases}$$

where:

$$\psi_c = 2\psi_4 - \psi_5$$

Now, assume that ψ_p has been found and that the fundamental equations (13-15) apply. The task that remains is to apply these results to a stability analysis. To start, one goes back to the Euler equation derived from δw earlier (equation(9)). In reduced form this equation is:

$$\underset{\sim}{B}\cdot\nabla(\underset{\sim}{B}\cdot\frac{\nabla X}{|\nabla\psi|^2}) + (A + \Lambda)X = 0.$$

Since the equation is solved in cylindrical coordinates with azimuthal symmetry (r,z), one can best solve this equation by changing the variables in equation(9) from 1 to z, with z scaled so that the distance between mirror coils is unity. Using the identity:

$$\partial 1 = \frac{B\partial z}{B_z}$$

The reduced equation becomes:

$$\frac{\partial}{\partial z}\left(f\frac{\partial X}{\partial z}\right)-gX = 0 \qquad (18)$$

$$X(z + N) = X(z)$$

where:

$$f = \frac{B_z}{r^2 B^2}$$

$$g = \frac{p' D-\lambda}{B_z}$$

N = number of bumps of the E.B.T. model under consideration

Equation(18) is a Sturm-Liouville equation, so the number of zeroes increases monotonically with λ and hence λ_1 has no zeroes. Now, in the simplest case f and g are constants. In this case, no periodic solutions exist for $g > 0$ and the lowest eigenvalue occurs where $g = 0$, corresponding to the eigenfunction:

$$X = \text{constant.}$$

In the general case:

$$g = \left(\frac{2\pi n}{N}\right)^2$$

corresponding to:

$$X = \cos(\frac{2\pi nz}{N}) \; ; \quad \sin(\frac{2\pi nz}{N})$$

In actual E.B.T.-like devices, f and g are periodic, with each bump being of unit period and symmetric about each midplane. If one writes the reduced equation as two first order differential equations by making the change of variables:

$$Y = f\frac{\partial X}{\partial z}$$

and neglects the periodic requirements on X, then each λ corresponds to two linearly independent solutions of the system depending only on the choice of initial conditions. The reduced equation(9) is then a matrix differential equation such that:

$$\underset{\sim}{U}_z = \underset{\sim}{A}(z)\underset{\sim}{U}$$

$$\underset{\sim}{A}(z + N) = \underset{\sim}{A}(z)$$

$$\underset{\sim}{A}(-z) = \underset{\sim}{A}(z)$$

$$U(z) = \begin{pmatrix} x_1(z) & x_2(z) \\ y_1(z) & y_2(z) \end{pmatrix}$$

$$\underset{\sim}{A}(z) = \begin{pmatrix} 0 & \frac{1}{f} \\ g & 0 \end{pmatrix}$$

The solution to this equation is:

$$\underset{\sim}{U} = \underset{\sim}{P}(z)e^{z\underset{\sim}{\Gamma}} \qquad (19)$$

where $\underset{\sim}{P}$ carries on the periodic nature of the problem:

$(\underset{\sim}{P}(z + n) = \underset{\sim}{P}(z))$; and, $\underset{\sim}{\Gamma}$ is a constant matrix.

Now assume as an initial condition that U(0) is the identity matrix. Then:

$$\underset{\sim}{U}(n) = e^{n\underset{\sim}{\Gamma}} = \underset{\sim}{C}^n$$

To be a solution for E.B.T.-type devices, for a given λ, the solution must be periodic, with a period equal to the number of bumps (N) in the device under consideration. Consider the eigenvalues of $\underset{\sim}{C}$. If one is an n^{th} root of unity, then there is a solution. To find these eigenvalues of $\underset{\sim}{C}$, one must integrate the reduced equation from the midplane (z = 0) to a mirror coil plane (z = 1/2).with:

$$\underset{\sim}{C} = e^{\underset{\sim}{\Gamma}}$$

Using equation(19),and recognizing that $U(-1/2)\underset{\sim}{C} = U(1/2)$,then:

$$\underset{\sim}{C} = \frac{1}{\underset{\sim}{U}}(-\frac{1}{2})\underset{\sim}{U}(\frac{1}{2})$$

Now U(1/2) can be computed using standard techniques (ie. Runge-Kutta...) starting from $U(0) = \underset{\sim}{I}$. Also U(-1/2) can be calculated from U(1/2) as follows:

If:

$$\underset{\sim}{U}(\frac{1}{2}) = \begin{pmatrix} x_1 & x_2 \\ & \\ y_1 & y_2 \end{pmatrix}$$

Then:

$$\underset{\sim}{U}(-\frac{1}{2}) = \begin{pmatrix} x_1 & -y_1 \\ -x_2 & y_2 \end{pmatrix}$$

with the numbers x_1, x_2, y_1, y_2 being computed numerically. Additionally U^{-1} can be calculated using the following theorem:

$$\det\underset{\sim}{U}(z) = \det\underset{\sim}{U}(0)\exp\left[\int_0^z \mathrm{tr}A(x)dx\right]$$

with $\mathrm{tr}A = 0$ and $\det U(0) = 1$, $\det U(z) = 1$. From this $U^{-1}(1/2)$ can be calculated simply and the result becomes:

$$\underset{\sim}{C} = \begin{pmatrix} x_1y_2 + x_2y_1 & 2x_2y_2 \\ 2x_1y_1 & x_1y_2 + x_2y_1 \end{pmatrix}$$

Using the characteristic equation:

$$\det(\underset{\sim}{C} - \nu\underset{\sim}{I}) = 0$$

The eigenvalues of $\underset{\sim}{C}$ can be calculated from:

$$\nu^2 - \nu \, tr\underset{\sim}{C} + det\underset{\sim}{C} = 0$$

$$tr\underset{\sim}{C} = 4x_2y_1 + 2$$

$$det\underset{\sim}{C} = 1$$

Thus the eigenvalues of $\underset{\sim}{C}$ are:

$$\nu_{\pm} = 1 + 2x_2y_1 \pm \sqrt{(1 + 2x_2y_1)^2 - 1}$$

with the constraint arising:

$$\nu_+\nu_- = 1$$

because $det\underset{\sim}{C} = 1$.

The behavior of the system behind this mathematical analysis is seen through understanding the behavior of $\nu_{\pm}$. For very negative λ, g is positive and all solutions blow up, x_2 and y_1 are large and positive and $\nu_{\pm}$ are real and positive ($\nu_+ > 1$; $\nu_- < 1$). As λ increases, x_2 and y_1 decrease assymptotically and $\nu_{\pm}$ move toward unity, coming together at the point where x_2 and y_1 first vanish. The first eigenvalue thus occurs at $\nu_{\pm} = 1$. Also, because g first becomes negative at the midplane, $x_1(z)$ begins to dip down before blowing up. $y_1(z)$ is thus negative for small z and increases to large positive values as z increases. In addition, because $\nu_{\pm}$ are roots of a quadratic with real coefficient and lie on the unit circle $(\nu_+\nu_-)$ they are complex conjugates of one another, and when their phases take on values of $\frac{2\pi n}{N}$, the corresponding λ has been reached.

Thus the solutions of equation(19) are much like sines and cosines with P(z) adding bumpiness. The lowest eigenvalue is a constant and the next has two nodes over the whole N-bump section. There is even two degeneracies for each eigenvalue as there is for sines and cosines.

One set of results that is derived from calculating the eigenvalues and determining the stability boundaries is presented in figure 1.[12] This graph shows marginally stable β_c as a function of $\beta_\perp$ where:

$$\beta_c = \frac{2p_{max}}{B_0^2}$$

$$\beta_\perp = \frac{2p_{\perp emax}}{B_0^2}$$

Note that until a region is formed by J_e where $q' < 0$, no core plasma can be stably confined. This threshold occurs at $\beta_\perp$ = 10-15%. Above this

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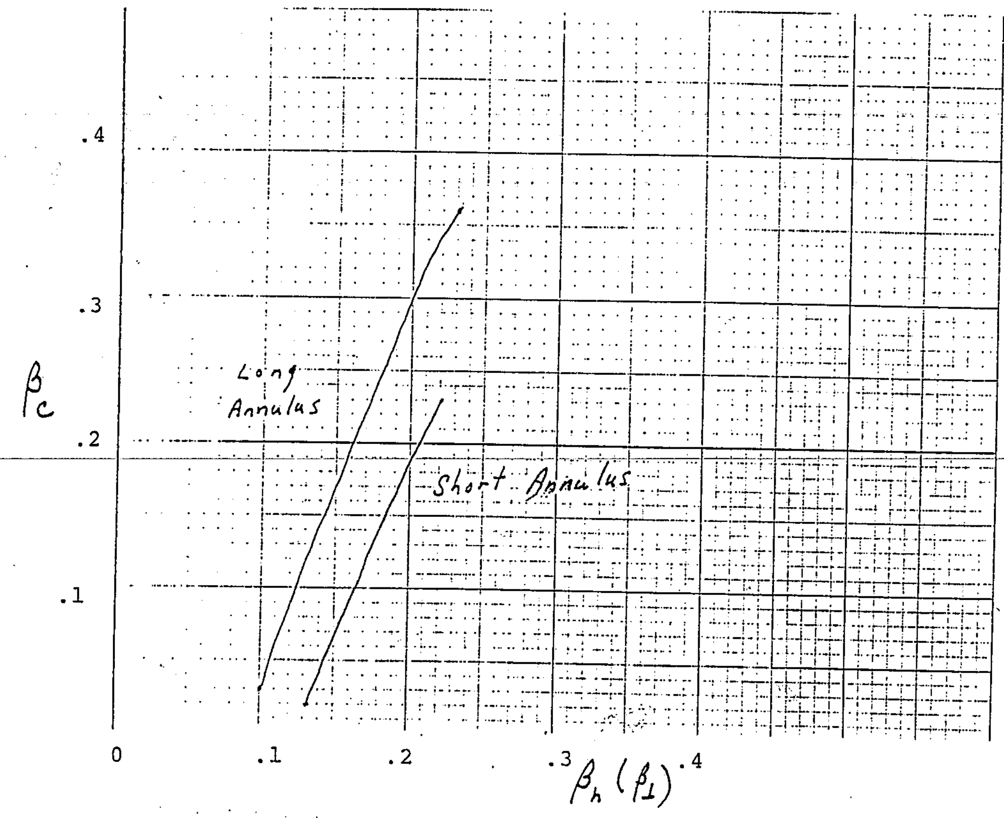


FIGURE #1

Maximum stable core beta vs annulus beta for Taylor's form of the hot electron pressure with m=2 B_c=1.2 Tesla threshold, the marginally stable β_c becomes comparable to $\beta_\perp$. At fixed $\beta_\perp$, if β_c exceeds its marginally stable value, the plasma is unstable to a ballooning mode; below this value, the plasma is stable to all ideal MHD modes. In addition, the shape of the annulus influences the regions of stable confinement. Because the local magnetic well extends further along field lines, longer annuli produce great core plasma stability. Taken together, if one believes this theoretical treatment, these results are extremely encouraging for the success of an eventual E.B.T. fusion reactor.

To understand these results qualitatively, note that the threshold for confining the core plasma occurs when $q' < 0$. Now q' is defined as:

$$\frac{\partial q}{\partial \psi} = -\int \frac{\partial l}{B^2}\left(\frac{\underset{\sim}{e}_\psi \cdot \underset{\sim}{\kappa}}{r} + \frac{\partial B}{\partial \psi}\right)$$

$e_\psi \cdot \kappa$ varies linearly between coil planes from negative at the midplane (destabilizing) to positive at the coil plane (stabilizing) with the factor $1/rB$ giving more weight to the negative contribution. To be stable, then, $\frac{\partial B}{\partial \psi}$ must be positive along most of the field line to balance the curvature term with the threshold point occuring about when $\frac{\partial B}{\partial \psi}$ becomes positive at the midplane. To see where $\frac{\partial B}{\partial \psi}$ vanishes, one can use the equilibrium relation:

$$\nabla_\perp (p_\perp + \frac{B^2}{2}) = \sigma B^2 \underset{\sim}{\kappa}$$

If the derivatives are replaced in the equation by scale lengths, then $\frac{\partial B}{\partial \psi}$ vanishes when:

$$\beta_\perp \simeq \frac{2\sigma r_p}{R_c}$$

where:

r_p is the half width of the annulus

R_c is the radius of curvature of the plasma at the midplane.

~~In figure 1, r_p/R_c = .05-.1. For this ratio, $\beta_\perp$ is approximately 10-20%,~~ which is born out computationally and is further a function of the thickness of the annulus (r_p/R_c).

The rapid rise in the marginal β_c above this threshold value can also be understood qualitatively. In general, the coefficient D, defined previously, is positive (destabilizing) near the midplane and negative near the coil plane. As $\beta_\perp$ is increased, J_e becomes more negative and the positive region of D is smaller. As the field lines bulge out as β_c increases, increasing the bad curvature near the midplane, it also makes the positive region larger. Using the equilibrium relations, this competition can be seen mathematically:

$$\text{if} \qquad \underset{\sim}{J}_{tot} = (p_\parallel - p_\perp)\underset{\sim}{\kappa} + \nabla_\perp p_\perp$$

$$\text{then} \qquad D = \frac{1}{rB^3}\left[-(1+\sigma)B^2 \underset{\sim}{e}_\psi \cdot \underset{\sim}{\kappa} + \underset{\sim}{e}_\psi \cdot \nabla p_{\perp e}\right]$$

where $p_{\perp e}$ is the perpendicular pressure due to the hot electrons only. Now raising β_c makes the first term larger by slowly increasing the curvature. However, β_c goes up rapidly with $\nabla p_{\perp e}$, as $\beta_\perp$ is the integral of $\nabla p_{\perp e}$. This qualitative behavior is shown in figure 1.

THE RIGID RING THEORY OF A GUIDING CENTER PLASMA

The rigid ring stability model paints a rather rosy picture of stability in an E.B.T. fusion reactor. The stable operating regions exist at high enough core beta to provide all the power necessary for an economically viable device. The problem is: Is MHD theory modified as shown in the previous section sufficient to describe a ring stabilized plasma? By the late 1970's, interacting ring theory was born and raised questions which made theorists reconsider rigid ring theory and the whole MHD hypothesis. In 1979, Nelson made a rather convincing effort to save his theory[1]. Using a dispersion relation technique to analyze stability criteria, he used the parameter δ $(\frac{n_a}{n_p})$ to compare his results with that of a single fluid plasma ($\delta = 0$) with success. It is informative to go through this analysis for two reasons:

1. It can best be used to compare with Vam Dam and Lee's interacting ring analysis; and

2. It will forcefully demonstrate a major theme of this paper; namely, that it is the deviation of a ring stabilized plasma from MHD theory which is the basic difference between rigid ring and interacting ring stability

1. D.B. Nelson, Phys. Fluids, 23, 1850, (1980)

models and it is this deviation which leads directly to the different results obtained.

The analysis, from Nelson's point of view follows[2]:

To set up the problem, consider the MHD modes of a ring stabilized plasma using a complete kinetic Maxwell-Vlasov model characterized by three different distribution functions representing the core electrons and ions and the hot ring electrons. For this analysis, one uses a local approximation for the modes of interest, an inhomogeneous slab geometry, simulates the magnetic field curvature by a ficticious gravitational term and assumes that both core plasma species have a finite temperature.

As a means of simplifying the calculation, let there be density and magnetic field in the x-direction only and let the magnetic field in equilibrium be given by the equation:

$$\underset{\sim}{B} = \underset{\sim}{B}(x)\hat{z}$$

Also let the temperature of the three plasma species be constant in space. Further, this analysis will only investigate plasma waves whose extent is

2. Nelson, Phys. Fluids, 23, 1850, (1980)

assumed infinite in space so that $k \cdot B = 0$ and propagation is in the y-direction with the magnetic field in the z-direction. The analysis will be restricted to a local region around $x = 0$ so that the x dependence of the waves can be neglected. The wavelength region to be investigated is such that the wavelength is small compared to a particle's gyroradius ($ka \ll 1$) and the frequencies are small compared to gyrofrequencies:

$$\omega \ll \Omega$$

Finally, the phase velocities are large compared to diamagnetic, grad-B, and gravitational drift velocities of the core plasma, but small compared to the drift velocities of the hot electrons.

With these assumptions and simplifications, the analysis begins with the usual assumption that the waves must obey Maxwell's equations and current densities can be computed from a linearized Vlasov equation for each species. Thus:

$$\underset{\sim}{E} = -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\sum q_s \int d^3\underset{\sim}{v} f_{1s}\right)$$

$$\nabla \times \underset{\sim}{B} = \mu_0(\underset{\sim}{J} + \varepsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}) = \mu_0 \sum_s q_s \int d^3 \underset{\sim}{v}(\underset{\sim}{v} f_{1s}) + \mu_0 \varepsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$

$$M_s f_{1s} = -\sum_s q_s \int dt(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial F_{0s}}{\partial \underset{\sim}{v}}$$

where: q_s, M_s, and f_s are the charge, mass, and distribution functions for each species.

These equations, together with Poisson's equation and Ampere's law:

$$\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$$

$$-\nabla^2 \underset{\sim}{A} + \frac{1}{c^2} \frac{\partial (\frac{\partial \underset{\sim}{A}}{\partial t} + \nabla \phi)}{\partial t} = \mu \cdot \underset{\sim}{J}$$

completely define the system.

Now assume the waves take the usual form and are proportional to:

$$\exp[i(ky - \omega t)]$$

The simplified geometry of this analysis requires only the x-component of A. Thus Poisson's equation and Ampere's law reduce to:

$$\varepsilon_0 k^2 \phi = \rho$$

$$\varepsilon_0 (c^2 k^2 - \omega^2) A_x = J_x$$

Giving ε a real and imaginary part, one must write these equations as:

$$-\left(\frac{\rho_s}{k^2}\right)\varepsilon_0 = \varepsilon_{11}^s \phi^s + i\varepsilon_{12}^s c A_x$$

$$-\left(\frac{J_{xs}}{k^2}\right)\varepsilon_0 c = i\varepsilon_{12}^s - \varepsilon_{22}^s c A_x$$

These equations can be rearranged into a more convenient form and one must solve the equations:

$$\left(1 + \sum_s \varepsilon_{11}^s\right)\phi + i\sum_s \varepsilon_{12}^s c A_x = 0$$

$$i\sum_s \varepsilon_{12}^s \phi + \left(1 - \frac{\omega^2}{c^2 k^2} - \sum_s \varepsilon_{22}^s\right) c A_x = 0$$

For there to be non-trivial solutions, the determinant of these two simultaneous equations must vanish. Thus:

$$D_{es} D_{em} + (CT)^2 = 0$$

where:

$$D_{es} = 1 + \sum_s \varepsilon_{11}^s$$

$$D_{em} = 1 - \frac{\omega^2}{c^2 k^2} - \sum_s \varepsilon_{22}^s$$

$$CT = \sum_s \varepsilon_{12}^s$$

This dispersion relation gives ω in terms of k and can determine whether or not the waves are destabilizing.

Let the equilibrium magnetic field be:

$$\underset{\sim}{B} = B_0(1 + \varepsilon x)\hat{z}$$

where $1/\varepsilon$ is much larger than the particle's gyroradius so that it will be possible later to make an expansion about the guiding center position (x_g). Also let the equilibrium distribution functions of each species be functions of guiding center positions (x_g) and energy ($H = mv^2/2 + mgx$). Then to a good approximation, the distributions for E.B.T.-like configurations can be given by:

$$f_s(x_g,H) = f_0(x_g,H) + N_{0s} f_B (1 + \varepsilon'_s x_g)\exp\left[-\frac{2gx}{\alpha_s^2}\right]$$

$$f_B = \frac{1}{(\alpha_s\sqrt{\pi})^3}\exp\left(-\frac{\underset{\sim}{v}^2}{\alpha_s^2}\right)$$

where the α_s's and ε'_s's can be different for each species.

Consider also a distribution function which is a δ-function in $v_\perp$. Now Poisson's equation and Ampere's law require that:

$$N_i = N_c + N_h$$

$$-\frac{\varepsilon B_0^2}{\mu_0} = \sum_s \frac{N_s M_s \varepsilon'_s \alpha_s^2}{2}$$

where N_i, N_c, and N_h are the density of ions, cold electrons and hot electrons, given by:

$$N_i = N(x)$$

$$N_c = N(x)(1-\delta)$$

$$N_h = \delta N(x)$$

defining:

$$\varepsilon'_s = \frac{1}{r_p} + \frac{2g_s}{\alpha_s^2}$$

(the field curvature is simulated by $g_s = \frac{\alpha_s^2}{R_c}$)
One obtains the equation:

$$2\varepsilon = -\beta\left(\frac{1}{r_p} + \frac{2}{R_c}\right) = -\beta\varepsilon_s'$$

$$\beta = \sum_s \beta_s = \mu_0 \sum_s N_s M_s \frac{\alpha_s^2}{B_0^2}$$

The first order perturbation of the distribution functions f_{1s} are computed from the linearized Vlasov equation by matching orders in the guiding center parameter ε. When this is done, the perturbed distribution function is given by:

$$f_1 = -q\frac{\partial f_0}{\partial H}\phi + \frac{1}{B}\frac{\partial f_0}{\partial x_g}A_y + \frac{iq\omega}{M}D\int_{-\infty}(\phi - \underset{\sim}{v}\cdot\underset{\sim}{A})dt$$

where:

$$D = \frac{M\partial f_0}{\partial H} + \frac{k}{\Omega\omega}\frac{\partial f_0}{\partial x_g}$$

Ω is the gyrofrequency $= \dfrac{qB(x_g)}{M}$

and the integral is taken over a particle's equilibrium trajectory. This integral can best be evaluated using the identity[3]:

3. Nelson, Phys. Fluids, 23, 1850, (1980)

$$\exp(iz)\sin\chi = \sum_n J_n(x)\exp(in\chi)$$

When this is done, the perturbed distribution function can be evaluated and one finds that for a given f_0:

$$f_1 = -\frac{2qN_0f_B}{M\alpha_s^2}\left(1-\frac{\varepsilon_s' v_\perp}{\Omega}\cos\theta\right)\left(\phi + (\omega-\omega_*)\phi\right)\sum_{m,n}\frac{J_mJ_n\exp[i(m-n)\theta]}{\omega_d-\omega-m\Omega}$$

$$+ iv_\perp A_x(\omega-\omega_*)\sum_{m,n}J_m' J_n\frac{\exp[i(m-n)\theta]}{\omega_d-\omega-m\Omega}$$

where:

$$\omega_* = \frac{k\alpha_s^2\varepsilon_s'}{2\Omega}$$

$$\omega_d = kv_d$$

$$v_d = \frac{\varepsilon v_\perp^2}{2\Omega} + \frac{g}{\Omega}$$

$$J_m = J_m\left(\frac{kv_\perp}{\Omega}\right)$$

For MHD modes ($\omega << \Omega$), the perturbed charge and current densities for each species are given by:

$$\frac{\rho}{\varepsilon_0} = -\frac{2\omega_p^2}{\alpha_s^2}\int d^3\underset{\sim}{v}f_B\left[\phi\left(\frac{\omega_*-\omega_d}{\tilde{\omega}} + \frac{k^2v_\perp^2}{2\Omega^2}\,\frac{\omega-\omega_*}{\tilde{\omega}}\right) + iA_x\frac{kv_\perp^2}{2\Omega}\left(\frac{\omega-\omega_*}{\tilde{\omega}}\right)\right]$$

$$\frac{J_x}{\varepsilon_0} = -\frac{2\omega_p^2}{\alpha_s^2}\int d^3\underset{\sim}{v} f_B \int [\, i\phi \frac{kv_\perp^2}{2\Omega}\left(\frac{\omega-\omega_*}{\tilde{\omega}}\right) + A_x \frac{v_\perp^2}{2\Omega^2}\left(\frac{k^2 v_\perp^2}{2} - \tilde{\omega}^2\right)\left(\frac{\omega - \omega_*}{\tilde{\omega}}\right)]$$

where:

$$\tilde{\omega} = \omega - \omega_d$$

$$\omega_{ps}^2 = \frac{q_s^2 N_s}{\varepsilon_0 M_s}$$

Now for the cold species of the core plasma, $\omega >> \omega_*, \omega_d$ and thus the integral can be evaluated by a Taylor series expansion. The result is:

$$\varepsilon_{11}^{i,c} = \frac{2\omega_p^2}{k^2\alpha_{i,c}^2}\left[\left(\frac{\omega_* - \omega_d}{\tilde{\omega}}\right) + \frac{k^2\alpha_{i,c}^2}{2\Omega^2}\left(\frac{\omega-\omega_*}{\tilde{\omega}-\omega_B}\right)\right]$$

$$\varepsilon_{12}^{i,c} = \frac{\omega_p^2}{kc\Omega}\left(\frac{\omega-\omega_*}{\tilde{\omega} - \omega_B}\right)$$

$$\varepsilon_{22}^{i,c} = \frac{\omega_p^2}{k^2c^2\Omega^2}\left[-k^2\alpha_{i,c}^2\frac{\omega-\omega_*}{\tilde{\omega} - \omega_B} + (\omega-\omega_*)(\tilde{\omega} - \omega_B)\right]$$

where:

$$\omega_B = \frac{k\varepsilon\alpha_{i,c}^2}{2\omega}$$

$$\omega_d = \omega_B + \frac{kg}{\Omega} \quad \text{(evaluated at the thermal velocity)}$$

and terms arising from the resonant denominator are considered negligible and dropped.

The hot electrons cannot be evaluated in this limit for the frequency domain is different than it is for the core plasma. For them, one must use the fact $\omega < \omega_{*h}$, but account for the fact that the thermal frequency ω_{dh} can vanish[4]. To accomplish this, consider both a δ-function and a Maxwellian in $v_\perp$ as their distribution function. A Taylor series expansion assuming $\omega < \omega_{*h}$ is then performed to evaluate the integrals for J_x and ρ. The result is:

$$\varepsilon^h_{11} = \frac{2\omega_p^2}{k^2 \alpha_h^2} \frac{\omega_*}{\tilde{\omega}} C_1$$

$$\varepsilon^h_{12} = \frac{-2\omega_p^2}{kc\Omega} \frac{\omega_*}{\tilde{\omega}} C_2$$

$$\varepsilon^h_{22} = \frac{\omega_p^2 \alpha_h^2}{2c^2 \Omega^2} \frac{\omega_*}{\omega} C_3$$

The coefficients C_1, C_2, and C_3 depend on whether a δ-function or a

4. Nelson, Phys. Fluids, 23, 1850, (1980)

Maxwellian was used as the distribution function. For the δ-function all the coefficients are 1, but for the Maxwellian:

$$C_1 = \frac{\tilde{\omega}}{\omega_*}\left(1 - \frac{\omega-\omega_*}{\omega_B}F(a)\right)$$

$$C_2 = -\frac{\tilde{\omega}}{\omega_B}(1-aF(a))$$

$$C_3 = -\frac{\tilde{\omega}}{\omega_B}(1 + a-a^2F(a))$$

where:

$$a = \frac{\omega-kg/\Omega}{\omega_B}$$

$$F(a) = \exp(-a)P\int_{-\infty}^{a} dt\frac{e^t}{t} \qquad (a > 0 \quad \text{for } \varepsilon < 0)$$

Finally, the three terms can be assembled in the form of the dispersion relation:

$$D_{es}D_{em} + (CT)^2 = 0$$

where:

$$D_{es} = \frac{1}{k\lambda_{Di}^2}\left[(\omega_*-\omega_{di})\left(\frac{\omega_{di}}{\omega^2} + \frac{\delta}{\omega}\right) + \frac{k^2\alpha_i^2}{2\Omega_i^2}\right]$$

$$D_{em} = 1 + \beta_c + C_3\beta_h\frac{\omega_{*h}}{2\omega_{dh}} - \frac{\omega^2}{(kv_A)^2}$$

$$CT = -\frac{1}{k\lambda_{Di}}\left(\frac{\beta_c}{2}\right)^{1/2}\left(\frac{\omega_{*i}-\omega_{di}-\omega_{Bi}}{\omega} + \delta\frac{C_2\omega_{*h}-\omega_{dh}}{\omega_{dh}}\right)$$

where:

$$\delta = \frac{n_h}{n_i}$$

$\lambda_D = \dfrac{\varepsilon_0 M\alpha^2}{2q^2 N}$ is the Debye Length

$v_A^2 = \dfrac{B^2}{\mu_0 MN}$ is the Alfven velocity.

and the electrostatic dispersion coefficient D_{es} is independent of any contribution by the hot electrons.

Now, for purposes of this report, comes the important part of the analysis. Setting $\delta = 0$, $\beta_h = 0$ corresponds to a plasma with a single fluid component. This was done as early as 1961 independently by two physicists, Newcomb[5] and Tserkovnikov[6] and the result is well established.

5. W.A. Newcomb, Phys. Fluids, 4, 391, (1961) 6. Y.A. Tserkovnikov Dokl. Akad. Nauk. 130, 295, (1960)

Thus if their result can be duplicated via this analysis it would justify to a certain extent guiding center plasma analysis and by implication MHD analysis in computing the stable operating regions in a ring stabilized plasma. This can be done in just a few steps.

To accomplish this comparison, one sets $\delta = 0$ and $\beta_h = 0$ and considers frequencies $\omega \ll kv_A$ in the dispersion relation previously derived. When this is done, the dispersion relation becomes:

$$D_{es} = \frac{1}{\omega^2} \frac{1}{(k\lambda_{Di})^2} \frac{1}{N_i M_i} \frac{k^2 \alpha_i^2}{2\Omega_i^2} \times$$

$$\left[\rho_m \omega^2 + \frac{\partial p}{\partial x} \frac{1}{B} \frac{\partial B}{\partial x} + g \frac{\partial \rho_m}{\partial x} - \frac{\rho_m g}{B} \frac{\partial B}{\partial x} - \frac{p}{B^2} \left(\frac{\partial B}{\partial x}\right)^2\right]$$

$$D_{em} = 1 + \beta$$

$$CT = -\frac{1}{\omega k \lambda_{Di}} \left(\frac{\beta_c}{2}\right)^{\frac{1}{2}} \frac{k}{\Omega_i} \frac{1}{N_i M_i} \left(\frac{\partial p}{\partial x} - \frac{2p}{B} \frac{\partial B}{\partial x}\right)$$

where:

$$\rho_m = \sum_s N_s M_s \quad ; \quad p = \sum_s \frac{N_s M_s \alpha_s^2}{2}$$

To compare this to the results obtained by Newcomb and Tserkovnikov, one can write the dispersion relation as a quadratic in ω:

$$\rho_m\omega^2 + g\frac{\partial \rho_m}{\partial x} + \frac{\rho_m^2 g^2}{\frac{B^2}{\mu_0} + 2p} + \frac{p}{B^2}\left(\frac{\partial B}{\partial x}\right)^2 = 0$$

The result obtained by Newcomb and Tserhovnikov for $k_{\parallel} = 0$ can be derived in much the same way as was done in the previous chapter,ie. finding the Lagrangian equation and minimizing it. If one goes a step further and creates a dispersion relation out of the derived equation, the result is:

$$\rho_m\omega^2 + g\frac{\partial \rho_m}{\partial x} + \frac{\rho_m^2 g^2}{B^2 + \gamma p} = 0$$

This differs slightly from the guiding center treatment because MHD allows only one value of γ, but in fact the difference is quite small and the two theories can be said to agree. Indeed, rigid ring theory appears to be justified at least in the $\delta = 0$, $\beta_h = 0$ case. But the analysis cannot stop here. For stability in ring stabilized plasmas there must be hot electrons and in that case one must treat $\delta \neq 0$ and $\beta_h \neq 0$. Herein lies the downfall of rigid ring theory.

In principle, it is possible to determine the stability criteria of a ring stabilized plasma by setting $\delta \neq 0$, $\beta_h \neq 0$ and solving for ω in the dispersion relation. To this end, one can write the dispersion relation in the form of a quadratic in ω. Assuming $\omega \ll kv_A$ one has:

$$A\omega^2 + 2B\omega + C = 0$$

where:

$$A = \frac{1}{2}k^2 a_i^2 + \delta^2 \frac{\beta_c}{2} \frac{1}{D_{em}} \left(\frac{C_2\omega_{*h}}{\omega_{dh}} - 1\right)^2 \quad ; \quad a_i = \frac{\alpha_i}{\Omega_i}$$

$$B = \frac{1}{2}\delta\left[\omega_{*i} - \omega_{di} \frac{\beta_c}{D_{em}}(\omega_{*i} - \omega_{di} - \omega_{Bi})\left(\frac{C_2\omega_{*h}}{\omega_{dh}} - 1\right)\right]$$

$$C = (\omega_{*i} - \omega_{di})\omega_{di} + \frac{1}{2}\frac{\beta_c}{D_{em}}(\omega_{*i} - \omega_{di} - \omega_{Bi})^2$$

$$D_{em} = 1 + \beta_c + \frac{1}{2} C_3 \beta_h \frac{\omega_{*h}}{\omega_{dh}}$$

Now the condition for stability is that ω be real which means that $B^2 - AC > 0$ is the stability requirement. For small β, this is approximately equivalent to:

$$D_{em}^2(\omega_{*i} - \omega_{di})\left[\frac{1}{2}\delta^2 (\omega_{*i} - \omega_{di}) - k^2 a_i^2 \omega_{di}\right] > 0$$

But $\partial n/\partial x > 0$ so $\omega_{*i} - \omega_{di} > 0$. Also with small β_h, ω_{di} and D_{em} are positive which implies that the plasma is always unstable as δ goes to zero. Notice also that as β_h and β_c increase ω_{di} becomes negative. Specifically:

$$\omega_{di} = \frac{k\alpha_i^2}{2\Omega_i r_p}\left[2\xi - \frac{1}{2}\beta(1 + 2\xi)\right]$$

$$\xi = \frac{r_p}{R_c}$$

$$\beta = \beta_c + \beta_h$$

And the condition that ω_{di} change sign is that:

$$\beta_c + \beta_h > \frac{4\xi}{1 + 2\xi}$$

In contemporary E.B.T. devices $r_p \sim 3$ cm., $R_c \sim 30$ cm. So the sign change occurs around $\beta \geqslant .3$.

However, both physically and mathematically, it is impossible to stabilize the plasma by increasing β_c. Physically, hot electrons are necessary for stability. This is reflected by the fact that as β_c increases, electromagnetic terms must be incorporated into the dispersion relation. This destabilizing effect cancels out the stabilizing effect of changing the sign of ω_{di}. In short, a plasma cannot dig its own well. With hot electrons a plasma can be stabilized. By keeping δ small and increasing β_h by increasing the hot electron temperature, the sign of ω_{di}

will change and the result is a prediction of the stable operating regions of the plasma.

By now it should be clear that the critical parameter in the analysis is δ (n_a/n_p). For finite δ with $D_{em} > 0$, the stability analysis requirement is:

$$\beta_h > \frac{1}{(1 + 2\xi)(1 + \hat{\delta}^2)}(4\xi + 2\hat{\delta}^2) \qquad (1)$$

where:

$$\hat{\delta}^2 = \frac{1}{2}\left(\frac{\delta}{ka_i}\right)^2$$

At the lower limit as δ goes to zero, the stability requirement reduces to the requirement that ω_{di} change sign, as before, for small β_c. But for short wavelength, finite gyroradius destabilization results one must take into account the full stability requirement above. Taking $ka_i = .1$ as being the largest possible value allowed by the expansion techniques used to derive the dispersion relation, equation(1) shows that for $\delta = .05$, $\xi = .1$, β_c 0, modes of $ka_i < .1$ are stable for $\beta_h > .11$.

For finite β_c, the stability requirement can be written as a function of δ, ξ, κa_i, β_h, β_c, and the hot electron distribution function (either δ-function or Maxwellian). Written in the form $\hat{B}^2 - \hat{A}\hat{C} \geq 0$, the stability requirement can be given as before, with:

$$\hat{A} = \frac{1}{2}k^2 a_i^2 \hat{D}\hat{\omega}_d + \frac{1}{2}\delta^2 \beta_c (C_2\hat{\omega}_* - \hat{\omega}_d)^2$$

$$\hat{B} = (\frac{\delta}{2})[(\hat{\omega}_* - \hat{\omega}_d)\hat{D} + \beta_c(\hat{\omega}_* - \hat{\omega}_d - \hat{\omega}_B)(C_2\hat{\omega}_* - \hat{\omega}_d)]$$

$$\hat{C} = (\hat{\omega}_* - \hat{\omega}_d)\hat{D} + \frac{1}{2}\beta_c(\hat{\omega}_* - \hat{\omega}_d - \hat{\omega}_B)^2$$

$$\hat{D} = (1 + \beta_c)\hat{\omega}_d + \frac{1}{2}C_3\beta_h\hat{\omega}_*$$

$$\hat{\omega}_* = 1 + 2\xi$$

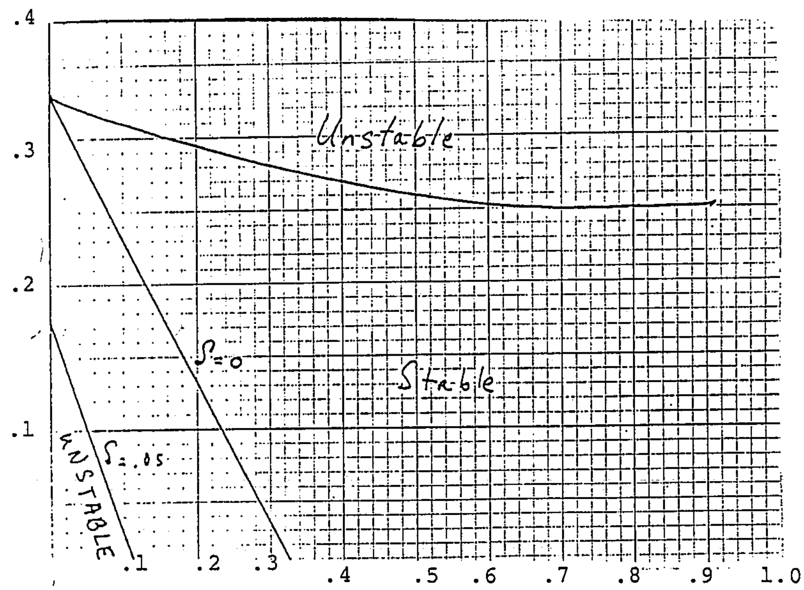
$$\hat{\omega}_B = -\frac{1}{2}\beta\hat{\omega}_*$$

$$\hat{\omega}_d = 2\xi + \hat{\omega}_B$$

The region of stability of β_c as a function of β_h are plotted for both the δ-function and the Maxwellian hot electron distribution functions, the Maxwellian being the more pessimistic of the two.(See figures1,2,3).[8] But the main point of this analysis is clear. For E.B.T. parameters, MHD treatment and its consequent rigid ring theory can be generalized by allowing for electron ring perturbations assuming a guiding center plasma. But can it? In a subsequent chapter, Vam Dam and Lee perform a similar analysis using a non-guiding center, non-MHD plasma by changing a basic

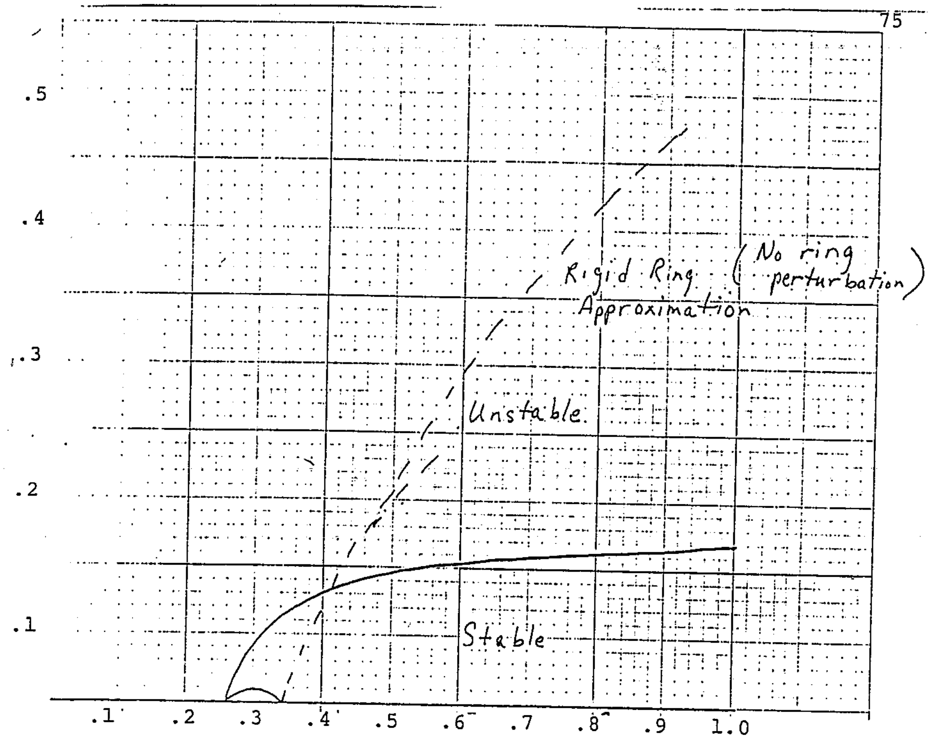
8. D.B. Nelson, Phys. Fluids, 23, 1850, (1980)

FIGURE #1



Regions of stable core beta as a function of ting for ∫-function, with ξ =0.1.

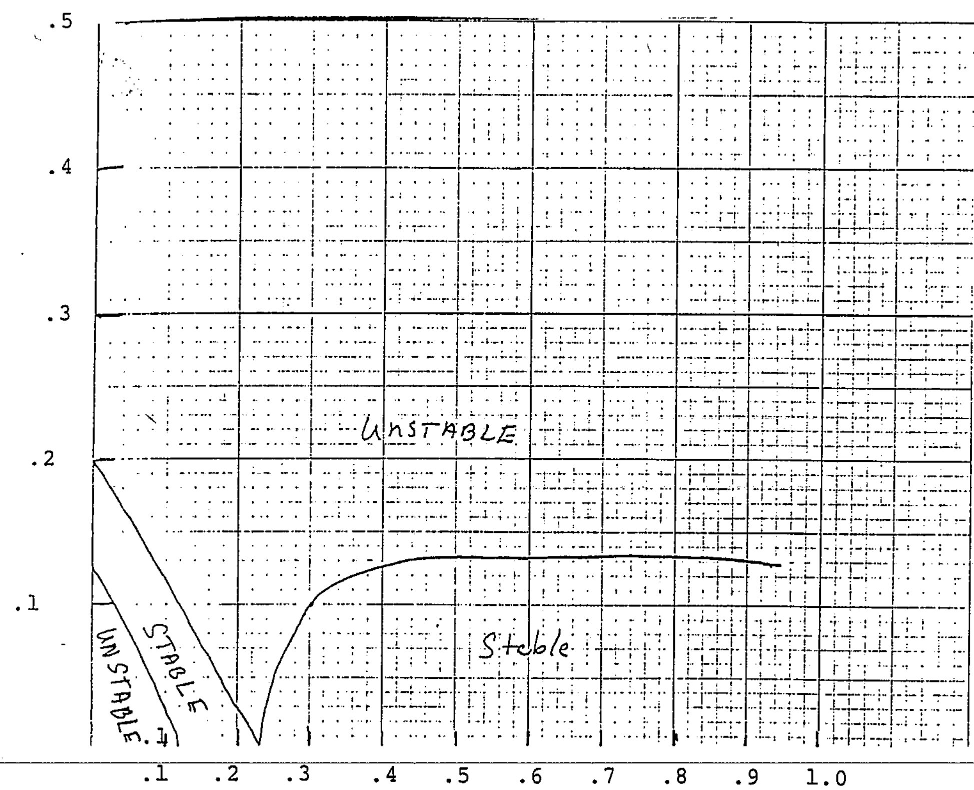
FIGURE #2



Regions of stable core beta vs. beta of the annulus for Maxwellian hot electron distribution ($\zeta = 1$)
=0 ka arbitrary

Dotted line shows corresponding stable region for a non-perturbed hot electron approximation.

FIGURE #3



Same as figure#2 with
δ =0.05 ka=0.1

constraint used in MHD theory (frozen-in field lines) with a very different result. Surely in the case $\delta = 0$, the comparison of this dispersion relation with that of Tserkovnikov and Newcomb gives confidence that rigid ring theory gives accurate results. The key is surely the parameters δ and β. δ defines the amount of non-MHD-like hot component there is in the plasma and β the effect of this hot component (in terms of pressure). It shall be shown that the divergence of rigid ring theory from interacting ring theory is exactly this, the divergence of ring stabilized plasmas from MHD and guiding center plasmas and further that this divergence can be isolated from the MHD part of the plasma by considering these very parameters δ and β.

INTRODUCTION TO INTERACTING RING THEORY

Many problems in physics have been solved by the method of continually improving approximations. First a zeroth order approximation, which satisfies the intuition and produces a kind of smoothed-over curve or function which qualitatively describes the behavior of the phenomenon being studied, is given. Soon, however, unanswered questions about the phenomenon arise, and paradoxes crop up which put the theory in jeopardy. Finally, the questions and paradoxes are resolved by going to the next highest order approximation, and so on. The stability of a ring stabilized plasma seems to be a case in point. Rigid ring models which treat the hot electrons and core plasma as two non-interacting components of a basically MHD plasma seems very much like a zeroth order approximation. However, close scrutiny reveals problems which cannot be overlooked. The drift frequency of the hot electrons is comparable to the ion cyclotron frequency of the core plasma, an area where MHD theory is not valid. Defining a quantity δ, the ratio of the densities of the hot and cold species of the plasma and demanding that $(n_h/n_c) \ll 1$ appears like the well-worn road of previous physics, where a disturbing part of a theory is treated as a perturbation on what is already established in the old theory. In this case of the stability of ring stabilized plasmas, however, this appears to be not quite right. The appropriate approximating technique appears to be to look at the analysis rather than the results of the analysis; and, to improve the results, the whole approximation of MHD analysis must be brought to "next highest order" in a much more subtle way than did perturbation theory of older physical problems.

The first step in this analysis improvement scheme is to figure out exactly what needs to be modified in the analysis in order to obtain self-consistent results. As early as the mid 1970's when rigid ring theory was first being proposed, H. L. Berk, Dominquez and others realized the non-applicability of MHD theory to the hot electron rings of a ring stabilized plasma[1]. In thinking about the hot electrons, the line of reasoning might have gone something like this:

Question: What exactly about MHD theory is inapplicable to the hot electron rings?

Answer: The field lines are not frozen into the plasma.

Question: What can be done to improve MHD analysis with this in mind?

Answer: Modify the analysis to exclude the frozen-in field line contraint.

1. H.L. Berk, Phys. Fluids, 19, 1255, (1976)

Now to begin this procedure, it must be established that the frozen-in field line condition is the offending part of MHD analysis. In fact, it had been known for many years that line-tying could physically explain the stability of a plasma that would normally be MHD unstable in simple mirror machines[2]. If the magnetic field lines are all frozen into fixed positions, two flux tubes cannot change positions without twisting and thereby increasing the magnetic energy of the system. Hence the stability of the system against interchange modes is enhanced. Since line-tying requires the frozen-in field line constraint, such an occurence is suitably explained by MHD analysis. In ring stabilized plasmas the presence of a cold plasma, which is a very good conductor, could act to keep the ends of the field lines fixed. However, the major drawback to this idea is that the frequency domain in which this concept is valid is limited to frequencies much less than the ion cyclotron frequency, not the case in ring stabilized plasmas, and, in fact, the very heart of the problem. If a stability analysis could be performed without the cold component of the plasma and then cold electrons added to attempt to stabilize the plasma (in which instabilities with growth rates comparable to the ion gyrofrequency

2. H.L. Berk, Phys. Fluids, 19, 1255, (1976)

3. Berk, Phys. Fluids, 19, 1255

arise), results could be obtained which would exclude the effects of the frozen-in field line constraint from the analysis[3].

As befitting a first attempt to correct the theory, the most simple model possible should be used. Therefore, consider a local model in slab geometry, with density and magnetic field gradients in the x-direction only and with:

$$\underset{\sim}{B} = B_0(x)\hat{z}$$

The cold plasma is approximated by fluid equations and the hot electrons with Vlasov's equation. To further simplify the analysis consider only electrostatic flute modes.

Now in a ring stabilized plasma, Poisson's equation can be written as:

$$\nabla^2\phi = -4\pi(q_e n_e + q_e n_h + q_i n_i)$$

where:

$q_{i,e}$ are the electronic charges of electrons and ions

$n_{c,h,i}$ are the densities of the three plasma species in equilibrium, such that:

$$n_i = \frac{n_0(x)}{Zi} \qquad (Z = 1 \text{ for simplicity})$$

$$n_c = (1-\alpha)n_0(x)$$

$$n_h = \alpha n_0(x)$$

For the hot electrons one can use the distribution function:

$$f_h = \alpha F_h(E,\mu,x + \frac{v_y}{\omega_{ce}})$$

where:

$$E = \frac{1}{2}v^2$$

$$\mu = \frac{1}{2B}(v_\perp - v_D)^2$$

$$v_D = [\frac{\frac{v_\parallel^2 + v_\perp^2}{2}}{\omega_{ce}} R_b]\hat{y}$$

$$R_b^{-1} = \frac{\partial}{\partial x}\ln B << \frac{\partial}{\partial x}\ln n_0(x) = r_p^{-1}$$

Consider a perturbation $\propto \phi e^{i(ky-\omega t)}$ in which $ky >> \frac{\frac{\partial\phi}{\partial x}}{\phi}$

Using the continuity equation and E × B drift for the electron drift velocity, one finds:

$$4\pi q_e \delta n_c = \frac{\alpha\delta\omega_{pi}^2 k_y^2 \phi}{\omega\omega_{ci}}$$

where:

$$\delta = \frac{1}{k_y r_p}$$

The perturbation in ion density is found from the continuity equation and Newton's law in the form:

$$\frac{\partial v_i}{\partial t} + \omega_{ci} v_i \times \hat{b} = -q_i \frac{\nabla\phi}{M_i}$$

One obtains the equation:

$$4\pi q_i \delta n_i = \frac{\omega_{pi}^2}{\omega - \omega_{ci}^2}\left[1 + \frac{\omega_{ci}}{\omega}\delta\right] k_y^2 \phi$$

As for the hot electrons, one solves the Vlasov equation in the limit $\omega \ll \omega_{ce}$. The result is:

$$4\pi q_e \delta n_h = \alpha \frac{M_i}{m_e} \frac{\omega_{pi}^2}{v_h^2} I_h(\omega)\phi$$

where:

$$I_h(\omega) = -\frac{v_h^2}{n_0(x)}\int d^3v\left[\frac{\left(k_y v_d\frac{\partial F_h}{\partial E} - \frac{k_y}{\omega_{ce}}\frac{\partial F_h}{\partial x}\right)J_0^2\left(\frac{k_y(\mu B)^{\frac{1}{2}}}{\omega_{ce}}\right)}{\omega - k_y v_D}\right]$$

and v_h is a typical hot electron velocity. Combining these two results and substituting them into Poisson's equation yields the dispersion relation:

$$1 - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}\left(1 + \frac{\delta\omega_{ci}}{\omega} - \frac{\delta(1-\alpha)\omega_{pi}^2}{\omega\omega_{ci}} - \frac{\alpha\omega_{pi}^2 M_i}{k_y^2 v_h^2 M_e}\right)I_h(\omega) = 0$$

To complete the analysis, one must evaluate the integral $I_h(\omega)$. To do this, note that the hot electrons exist in the frequency range:

$$k_y v_D \simeq \omega_{ci} \simeq \omega$$

Further, assume $k_y v_D$ is a constant. With these provisos, the integral can be estimated to be:

$$I_h(\omega) = \frac{k_y v_D R_b}{(\omega - k_y v_D) r_p}$$

where it has been assumed $v_D(\frac{\partial F_h}{\partial E}) << \frac{1}{\omega_{ce}} (\frac{\partial F_h}{\partial x})$ based on the assumption $\frac{r_p}{R_b} << 1$. Taken together, this leads to the dispersion relation:

$$\frac{\omega_{ci}^2}{\omega_{pi}^2} - \frac{(1 + \frac{\delta}{x})}{x^2 - 1} - \frac{\delta(1-\alpha))}{x} - \frac{\delta\alpha}{x + q} = 0$$

where:

$$x = \frac{\omega}{\omega_{ci}}$$

$$q = - \frac{v_D}{r_p \omega_{ci} \delta} \quad \text{for } \frac{R_b}{r_p} > 0 \; ; \quad q > 0$$

Now what this means is that for $R_b/r_p > 0$, there is always instability. Actually, one can solve the dispersion relation to find a specific stability criteria. Taking $\omega^2_{ci}/\omega^2_{pi} << 1$ and setting $\delta = 1$, the dispersion relation can be solved in terms of x, such that:

$$x = - \frac{q(1-\alpha)}{2} \pm (\frac{q^2(1-\alpha)^2}{4} - \alpha q)^{\frac{1}{2}}$$

Clearly, for $q >> 1$, only a small fraction of the cold electrons are needed for stability.

In experiments, these results have been verified and the conclusion is that cold electrons can stabilize interchange modes. Now this phenomenon is not line-tying. The basic mechanism of stability, it would seem is not the frozen-in field line constraint as it is in MHD plasmas. This is an important point. To summarize, the basic flaw in rigid ring theory is that an MHD analysis does not apply in the hot electrons' frequency domain. This flaw can be traced to the frozen-in field line constraint. And now, the frozen-in field line constraint has been shown to be not the mechanism which stabilizes a ring stabilized plasma against interchange modes. The stage has been set for a new theory based on the concept of an interacting hot electron annulus.

To conclude this section, it must be pointed out that in actuality there was not this one to one series of events from MHD analysis to rigid ring theory to interacting ring theory. Many researchers tried to find self-consistent ways to analyze the hot electron component of a ring stabilized plasma without MHD theory. Variational methods and WKB approximations were tried with insightful results which will be accounted for subsequently. However, the motivation for the formulation of interacting ring theory might very well have run along these lines, and the next chapter will follow this one as the next step in this evolution of knowledge.

A δ, β PARAMETER ANALYSIS BASED ON INTERACTING RING THEORY

In researching the problems inherent in the rigid ring theory approach to ring stabilized plasmas, the strongest clue that something is wrong is that in the hot electron plasma, the electron's drift velocity is on the order of or greater than the ion gyrofrequency of the core plasma. This leads to the need to alter the methodology of the stability analysis and to different predictions about the regions of stability in the plasma. In general, there are three stability analysis methods one could chose from[1]:

$$\delta w_{MHD} \leqslant \delta w_{GCP} \leqslant \delta w_{DA}$$

The first two cannot be applied to a hot electron plasma's frequency domain and therefore must be ruled out. The third is satisfactory and can provide the basis for a new approach to the stability of ring stabilized plasmas.

To begin, one starts with the drift kinetic equation:

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1. Lecture by J.W. Van Dam at The University of Texas, April,1981

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f + \frac{\partial E}{\partial t}\frac{\partial f}{\partial E} = 0$$

One next gyro-averages this equation using the spatial coordinates (ψ,θ,1) and assumes the temporal dependence:

$$\frac{\partial E}{\partial t} = mv_{\parallel} E_{\parallel} + E\cdot v_{\prime\prime} + \mu\frac{\partial B}{\partial t}$$

Now this equation can be analyzed in three limits, but for ring stabilized plasmas with their mildly relativistic rings, one must use the limit:

$$\omega < \omega_d \leqslant \omega_{bounce}$$

In this limit, bounce averaging implies:

$$\langle\langle \; .... \; \rangle\rangle = \int d\theta \; \langle \; ... \; \rangle$$

With this averaging, the perturbed distribution function becomes:

$$\tilde{f} = \frac{\partial f}{\partial \psi}\frac{1}{\langle v_{de} \rangle}\left[\langle H \rangle - \langle\langle H \rangle\rangle \right] - \frac{\partial f}{\partial \varepsilon} \langle\langle H \rangle\rangle$$

where the Hamiltonian is:

$$H = \mu Q_{\parallel} + \frac{1}{2}v_D^2$$

which produces the energy equation:

$$\delta w_{kinetic} = \frac{1}{2}\int dv_x \int \frac{\partial f}{\partial \psi} \frac{1}{\langle v_{De} \rangle} \{ [\langle H \rangle^2 - \langle\langle H \rangle\rangle^2] - \frac{\partial f}{\partial \varepsilon} \langle\langle H \rangle\rangle^2 \}$$

Further, without line-tying, field lines cannot be said to be frozen into the plasma as has been pointed out in the previous chapter. For ring stabilized plasmas, then, one cannot use the constraints of previous MHD analysis. Instead one uses:

$$\mu = \frac{mv_\perp^2}{2B} = \text{constant}$$

$$J_\parallel = \int v_\parallel \, dl = \text{constant}$$

$$\phi = \int B \cdot ds = \text{constant} = \int \alpha \, d\beta = \text{flux}$$

where:

$$B = \nabla\alpha \times \nabla\beta \text{ and}$$

$$\alpha = \alpha(\beta, \mu, \varepsilon, J = \text{constant})$$

Now at this time it will not be necessary to rigorously derive an energy principle. This will be done in the next section. These are only to be taken as general ground rules upon which the interacting ring model is based. From here, without a rigorous energy principle, one can set out to make predictions about the stability regions of ring stabilized plasmas.

To carry out this analysis, consider a δ, β parameter analysis using the interacting ring assumption that the frozen-in field line constraint is replaced by flux conservation. In order to make a direct comparison with rigid ring theory and, utilizing as closely as possible the parameters of actual E.B.T. machines, similar approximations to those of the δ, β analysis of chapter 3 will be made when building a model of the plasma. Since the region of interest exists around the annulus where the radial gradient of the magnetic field is positive, using the same density profile is justified. Also assume that the temperatures of the three plasma species have no spatial dependence in this region. To further enhance the comparison use:

$$\frac{T_h}{T_c} >> 1 \qquad (10^3)$$

$$\frac{n_h}{n_c} << 1 \qquad (.1)$$

With these stipulations, one can describe the plasma by its characteristic frequencies: The core plasma which can be described conventionally in terms of the MHD equations, has a diamagnetic frequency:

$$\omega_* = -i\left(\frac{v_{th}^2}{\Omega}\right)(\hat{b} \times \nabla \ln N)\cdot\nabla_\perp$$

and a magnetic drift frequency:

$$\omega_d = -\frac{1}{i\Omega}\left[v_\parallel^2(\hat{b} \times \hat{b}\cdot\nabla\hat{b}) + \frac{v_\perp^2}{2}(\hat{b} \times \nabla \ln B)\right]\cdot\nabla_\perp$$

where:

$$\hat{b} = \frac{\underset{\sim}{B}}{B}$$

Note that both of these frequencies are smaller than the fluctuation frequency of the plasma, ω, in the MHD limit.

Equipped with these frequencies, one can now explore the equations of motion of the plasma. Assuming that the core plasma is influenced essentially by $E \times B$ effects, one can write down the MHD expression for its pressure perturbation (for an isotropic distribution) as:

$$\tilde{p} = \underset{\sim}{\xi}_\perp\cdot\nabla p + \left(\frac{\gamma p}{B^2}\right)\left[\underset{\sim}{\xi}_\perp\cdot(\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}) + \xi_\perp\cdot\nabla\left(\frac{B^2}{2}\right) + \underset{\sim}{B}\cdot\underset{\sim}{B}\right]$$

This expression can be transformed via kinetic theory in the limit $\omega >> w_*,\omega_d$ to give:

$$\tilde{p} = -eN_i(\frac{\omega_{*i}}{\omega})(1 + \tau)\tilde{\phi} + eN_i(\frac{\omega_{di}}{\omega})(1 + \tau)\tilde{\phi} + (\frac{\beta}{4\pi})B\tilde{B}_{\parallel}$$

where:

$$\beta = \frac{8\pi N_i T_i(1 + \tau)}{B^2}$$

$$\tau = \frac{T_e}{T_i}$$

Notice, preliminarily, that the terms of this kinetic representation describe two separate contributions to the pressure. The first term is convective pressure change while the second and third terms represent compressional change[2]. This will prove of importance later in tracing the differences of the two theories. For the present, the pressure perturbations, obtained from considering the appropriate frequency domains can be used to form a pressure balance equation. Specifically, in the perpendicular direction, pressure balance is of the form:

$$B\tilde{B}_{\parallel} + 4\tilde{p}_{\perp} = 0$$

2. Y.C. Lee and J.W. Van Dam EBT Ring Physics: Proceedings of the Workshop, Dec. 3-5, 1979, Oak Ridge, Tennessee, p.471

The justification for using this form of pressure balance is that it is good in magnetic configurations like E.B.T. (although it can be thought of as eliminating fast Alfven waves from consideration). Its domain of validity applies to low frequency modes whose perpendicular wavelength is much smaller than its parallel wavelength and any scale length in the plasma. Basically, the magnetics of E.B.T. make this desirable since particles with large components of perpendicular velocity are trapped in its mirrors. This simplification is also desirable from a mathematical point of view, because the longitudinal magnetic field perturbation is now proportional to the electrostatic contribution to pressure fluctuations divided by $1 + \beta$, the finite gyroradius effect

As for the hot electrons, $E \times B$ motion is negligible because of their large perpendicular energies and thus their motion is largely adiabatic with:

$$\omega \ll \omega_{*h}, \omega_{dh}$$

To model this motion in non-MHD terms, then, one can use the fact that

3. Lee and Van Dam, EBT Ring Physics: Proceedings of the Workshop, Dec. 3-5, 1979, p.471

their drift motion follows the field strength surfaces[3]. This can be written as:

$$\underset{\sim}{v}_{dh} \cdot \nabla \tilde{p}_{\perp h} + \underset{\sim}{v}_{dh} \cdot \nabla p_{\perp h} = 0$$

With this characterization of the motion of the hot electrons, when the magnetic field is perturbed, the trajectories of the electrons are perturbed and this critical difference between rigid ring and interacting ring models is built into the equations.

Now the ring electrons' motion are perturbed mainly by ∇B drift, so the condition prevails:

$$\nabla \cdot \underset{\sim}{J}_{\perp h} = 0$$

One can relate the magnetic field to the pressure, then, by noting that the surfaces of constant pressure are identical to the constant magnetic field contours (in the adiabatic limit) and with this relationship one can construct an equation for the perturbed hot electron pressure analogous to that for the core plasma:

$$\tilde{p}_h = \left(\frac{\partial p_{\perp h}}{\partial B}\right)\tilde{B} = \left(\frac{\beta_h}{2}\right)\left(\frac{\omega_{*h}}{\omega_{bh}}\right)\tilde{B}B_{\parallel} \qquad (1)$$

where:

ω_{bh} is the ∇B drift frequency for the hot electrons (velocity averaged)

Since the hot electrons are anisotropically heated, $t_{\perp h} > t_{\parallel}$, so the analysis is consistent.

Having described the pressure perturbations for both hot electrons and core plasma, the next step is to combine them in a quasi-static pressure balance:

$$B\tilde{B}_{\parallel} = - \frac{4\pi \tilde{p}_{\perp}^{e.s.}}{\left[1 + \beta_c + (\frac{\beta_h}{2})(\frac{\omega_{*h}}{\omega_{bh}})\right]} \tag{3}$$

where:

$\beta_c =$ the total beta of the core plasma $(\beta_i + \beta_e)$

$p_{\perp}^{e.s.}$ = the electrostatic part of the total perpendicular pressure fluctuation.

Consider a slab geometry (also used in the previous rigid ring δ,β-parameter analysis). In this geometry:

$$\frac{\omega_{*h}}{\omega_{bh}} = \frac{\omega_{*i}}{\omega_{bi}} = - \frac{2}{\beta}$$

And further, since there is no curvature, there is much cancellation between the first and last terms in equation(3). Thus a dispersion relation can be constructed from equation(2) such that:

$$D = 1 + \beta_c + (\frac{\beta_h}{2})(\frac{\omega_{*h}}{\omega_{bh}}) \rightarrow\rightarrow\rightarrow \frac{\beta_c}{\beta}(1 + \beta) \tag{4}$$

which is very small for E.B.T. parameters. Notice by way of comparison that this result implies that magnetic field perturbations are affected by the hot electron rings in a non-trivial fashion. This is a major distinction between interacting and rigid ring models which describes basically a compressional effect not before seen using MHD analysis.

To get closer to a direct comparison, consider flute-type interchange modes in the low frequency limit ($\omega \ll \Omega_i$). These modes can be described by the equation:

$$(\frac{c}{c_A})^2(\omega-\omega_{*i})\nabla_\perp\tilde{\phi}-(\frac{4\pi ic}{B})[\hat{b} \times (\hat{b}\cdot\nabla\hat{b})]\cdot\nabla_\perp(\tilde{p}_\parallel + \tilde{p}_\perp) = 0 \tag{5}$$

This is a good starting point because it is equivalent to charge neutrality and is generally valid for modes with small transverse wavelength, consistent with previous analysis[4]. Now this equation consists of a contribution due to the ionization polarization drift (first term) and the field line curvature drift of a species in the plasma (second term). The first term also has remnants of finite gyroradius effects of the old E × B term from the ion species. Introducing the parameter $\delta = n_h/n_c$, one can consider the cases of $\delta = 0$ and δ finite as in the rigid ring analysis.

CASE 1: $\delta = 0$

For simplicity, ignore the azimuthal variation of local equilibrium quantities and use a large poloidal mode number analysis. This is applicable to strongly ballooning modes in the bad curvature regions and flute modes in a bumpy cylinder similar to the previous δ,β-parameter analysis of the rigid ring model. In this approximation, in the interacting ring model, the compressional pressure effect implies that:

$$\hat{p}_{\perp} = \tilde{p}_{\perp c} + \tilde{p}_{\perp h} = \frac{\tilde{p}_{\perp c}}{D}$$

Now perturb the core plasma by an isotropic, completely convective pressure perturbation:

4. Lee and Van Dam, EBT Ring Physics: Proceedings of the Workshop, Dec. 3-5, 1979, p.471

$$\tilde{p}_{\parallel c} = \tilde{p}_{\perp c} = -eN_i\left(\frac{\omega_{*i}}{\omega}\right)(1 + \tau)\tilde{\phi}$$

Using the equation for low frequency flute-type interchange modes, the dispersion relation can be written as:

$$\omega(\omega-\omega_{*i})(k_{\perp}\rho_i)^2 = -\omega_{\kappa i}\omega_{*i}(1 + \tau)(1 + D^{-1})$$

where:

$\omega_{\kappa i}$ is the curvature drift velocity (velocity averaged)

From this equation, one can see the stabilizing effect of the hot electrons. Take the core beta to be zero. When the annulus is absent $D \simeq 1$, which implies an interchange instability which is purely growing such that:

$$\omega^2 = -\frac{C_s^2}{r_p R_c} \quad \text{(neglecting finite Larmour effects)}$$

where:

r_p is a density scale length

R_c is the radius of curvature

Clearly, ω^2 is negative definite. Once again it is obvious that a plasma cannot dig its own well.

Next put in the annulus. The dispersion relation becomes:

$$D \simeq \frac{\omega_{\kappa i}}{\omega_{bi}} \quad \text{(neglecting anisotropy)}$$

This implies:

$$\omega^2 = -\omega_{\kappa i}\omega_{*i}(1 + \tau)(1 + \frac{\omega_{bi}}{\omega_{\kappa i}}) = -\omega_{*i}\omega_{di}(1 + \tau)$$

Here ω^2 has some positive values. Specifically, when the magnetic drift velocity ω_{di} has the opposite sign to ω_{*i}, the plasma is stable. To accomplish this, a local minimum in the magnetic field must be created and its gradient must be approximately $1/R_c$. Said another way, the ring beta must be at least as large as the threshold value:

$$\beta_h = 4(\frac{r_p}{R_c}) \simeq 20\%$$

This is in agreement with rigid ring theory. In fact, the hot electrons only enter the problem through the equilibrium pressure balance for $\delta = 0$ and $\beta_c \ll \beta_h$, assumptions entirely consistent with rigid ring theory.

The difference between the two theories arises in the region where an E.B.T. device becomes a fusion reactor. In this region of β-space, one desires to have a large core beta and/or reduce the ring input power by narrowing the annulus. In this case, if one ignores curvature in the equilibrium pressure balance relation (ie. $\frac{\beta_c}{2} |\frac{\nabla p}{p}| > \frac{1}{R_c}$), other interchange modes are no longer stable.

Consider the dispersion relation given previously. Enhanced compressional pressure can only stabilize the plasma when D is negative, ie., when $\beta_c \ll 1$. In a reactor, though, β_c is increased and D becomes positive:

$$D = \frac{\omega_{\kappa i}}{\omega_{bi}} + \beta_c\left(1 - \frac{\omega_{*i}}{2\omega_{bi}}\right)$$

$$\propto \left(\varepsilon - \frac{\beta_c}{2}\right)(1 + \beta_c) - \frac{\beta_c \beta_h}{2}$$

where:

$$\varepsilon = \frac{r_p}{R_c}$$

This implies that for stability:

$$\beta_c \leqslant \frac{2\varepsilon}{1 + \beta_h}$$

Now in a reactor, β_h can be in the range 20-40%, so β_c is limited to 7 or 8%, a very different result than obtained from rigid ring theory. In fact, the difference between the two model exists in this region. The rigid ring model predicts stability for substantially higher values of β_c than does interacting ring theory. To be perfectly precise about the distinction, rigid ring and interacting ring theories agree when $\delta = 0$, $\beta_c << \beta_h$. This can be looked upon as the single fluid limit, for the annulus enters the analysis as an incompressible, non-perturbative species in the plasma. As β_c is increased, the annulus assumes less and less of its MHD or guiding center plasma characteristics as its compressional effects are increased and consequently the results of the two theories diverge.

CASE 2: $\delta \neq 0$

Next let $\delta \neq 0$. Starting from the flute-like interchange mode equation, equation(4), one need only consider the perturbed electrostatic potential ϕ and the perturbed parallel component of the magnetic field, $B_{\parallel}$, to describe the perturbed electromagnetic field. Using charge neutrality:

$$\sum q\tilde{N} = 0$$

and the quasi-static transverse pressure balance equation, equation(1), one can write down the partial pressure density:

$$q\tilde{N} = -\frac{k_D^2}{4\pi}\left\{\tilde{\phi}-(\omega-\omega_*)\left<\frac{1}{\omega-\omega_d}\left(1-\frac{k_\perp^2 v_\perp^2}{2\Omega^2}\right)\left(\tilde{\phi}+\frac{v_\perp^2}{2\Omega}\tilde{B}_\parallel\right)\right>\right\}$$

where $\langle \ldots \rangle$ is velocity averaging.

The transverse pressure perturbation is:

$$\tilde{p}_\perp = \sum\left(\frac{qMN}{T}\right)(\omega-\omega_*)\left<\frac{v_\perp^2/2}{\omega-\omega_d}\left(1-\frac{k_\perp^2 v_\perp^2}{2\Omega^2}\right)\left(\tilde{\phi}+\frac{v_\perp^2}{2\Omega}\tilde{B}_\parallel\right)\right>$$

Additionally, assume $\omega \gg \omega_{*i,e}$ for species of the core plasma and $\omega \ll \omega_{*h}$ for the hot electrons. A local dispersion relation can be formed analogously with that for the $\delta \neq 0$ case in rigid ring theory by casting it into the quadratic form:

$$A\omega^2 + B\omega + C = 0$$

This time the coefficients are:

$$A = (k_\perp \rho_i)^2 + \delta^2 \frac{\beta_c}{2D}\left(1-\frac{\omega_{*h}}{\omega_{bh}}\right)^2$$

$$B = \delta\left[(\omega_{*i}-\omega_{di}-(\frac{\beta_c}{D})(1-\frac{\omega_{*h}}{\omega_{bh}})(\omega_{*i}-\omega_{di}-\omega_{bi})\right]$$

$$C = \left[(\omega_{*i}-\omega_{di})\omega_{di}-\omega_{bi}^2 + \frac{\beta_c}{2D}(\omega_{*i}-\omega_{di}-\omega_{bi})^2\right](1 + \tau)$$

where:

$$\tau = (1-\delta)\frac{T_e}{T_i}$$

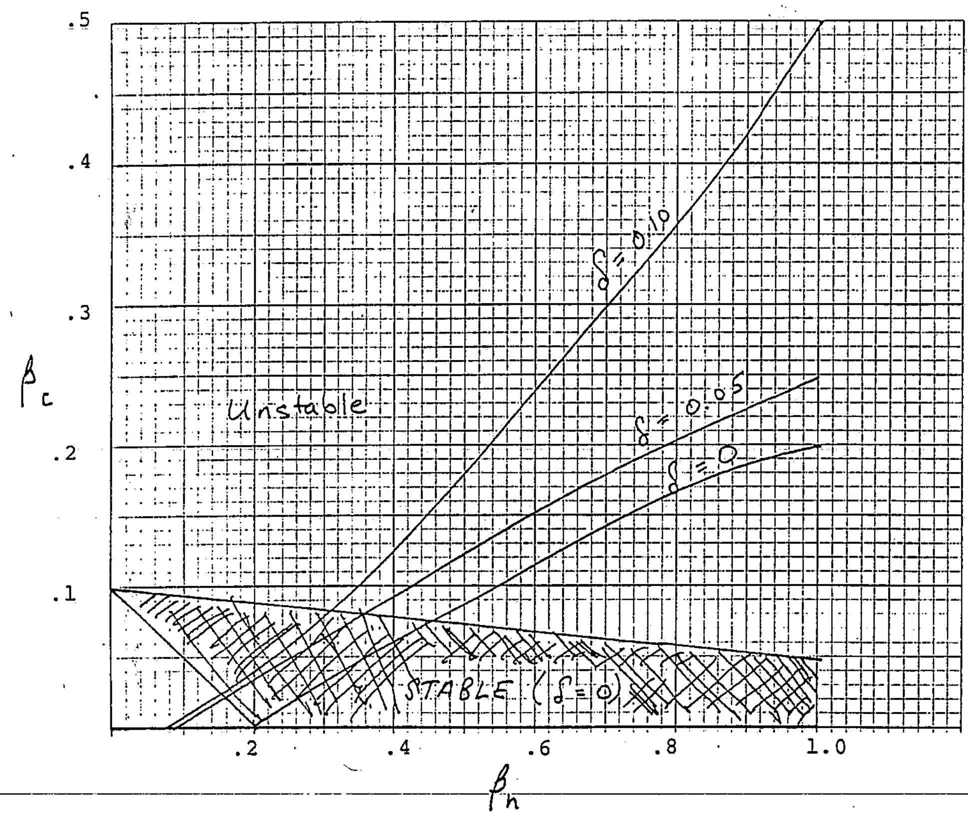
Compare the figures derived in the MHD, guiding center plasma analysis of rigid ring theory to those derived using the assumption of interacting rings (See figures 1,2,3).[5] Using the slab geometry common to both calculations, $D \propto \beta_c$ in equation(4). However, there is no longer a distinction between β_h and β_c, they only appear as a total β. The other free parameter is δ. Now for $\delta = 0$:

$$B^2-4AC \geqslant 0$$

is never satisfied and hence there is no stability. At high β, δ must exceed 5% for stability. The analysis of the case $\delta = 0$ and the case $\delta \neq 0$ are therefore nearly identical.

5. Y.C. Lee and J.W. Van Dam EBT Ring Physics: Proceedings of the Workshop, Dec. 3-5, 1979 Oak Ridge, Tennessee p.471

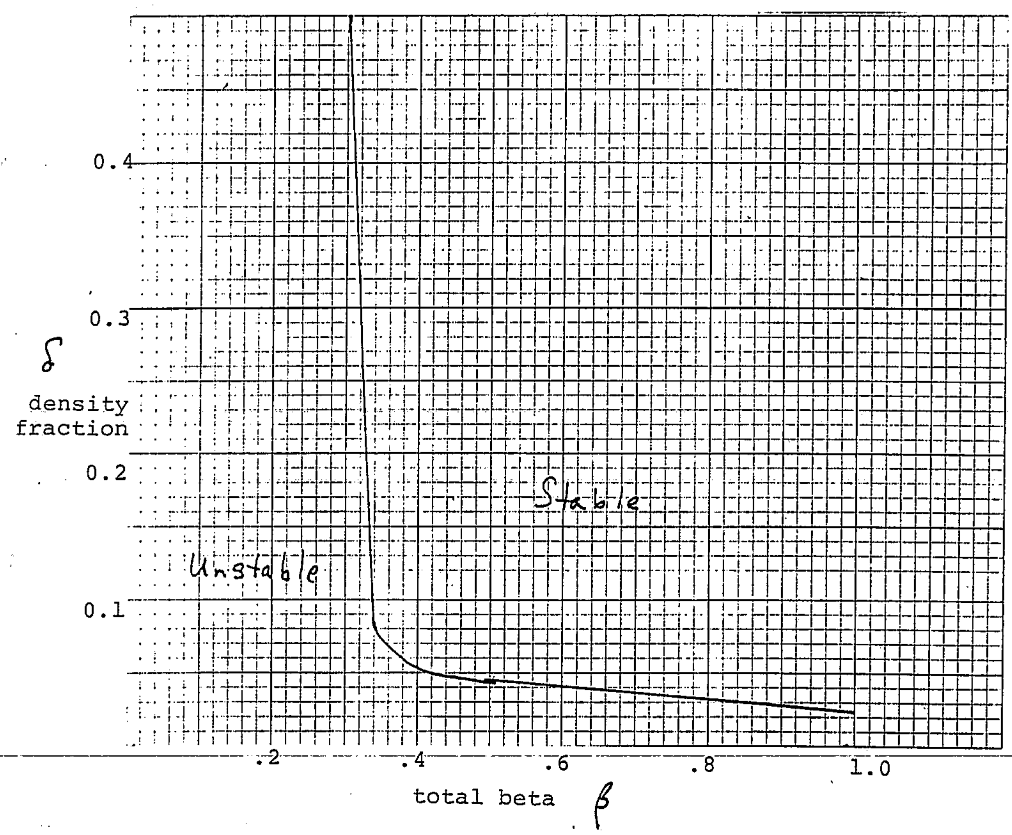
FIGURE #1



Core beta vs. annulus beta for interchange stability when $\delta = 0$ (interacting ring theory)

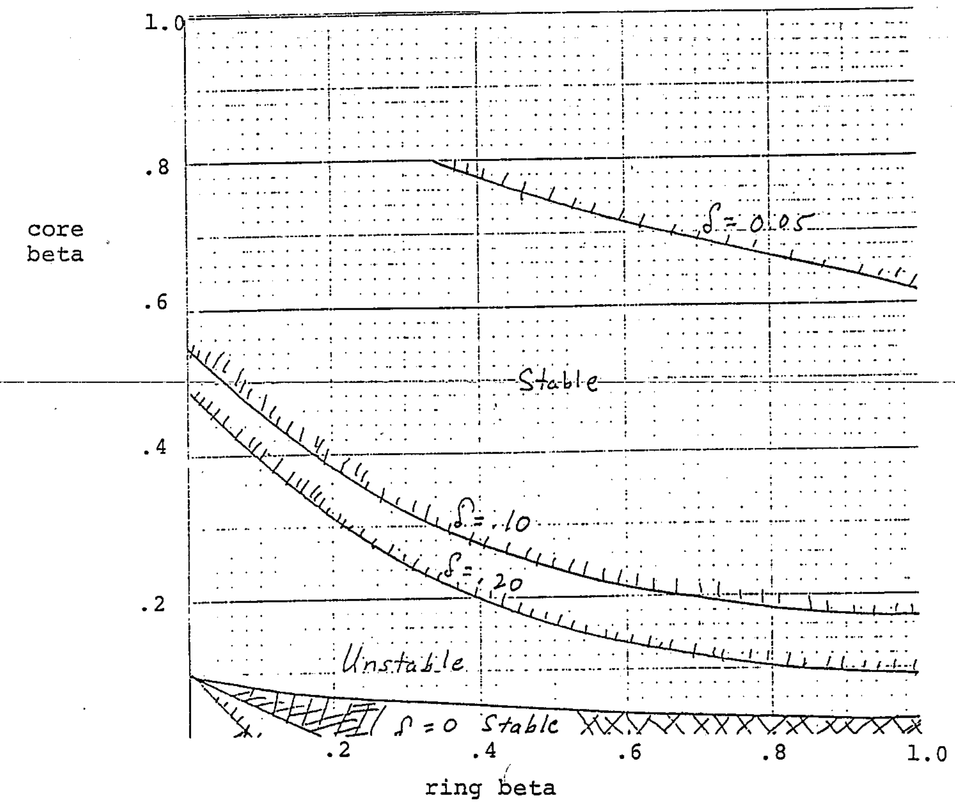
Predicted stability regions $\delta \neq 0$ (rigid ring theory)

FIGURE # 2



Hot electron density fraction as a function of total beta $\beta = \beta_i + \beta_c$ using a slab model, with $k_\perp \rho_i = 0.1$

FIGURE #3



Marginal stability boundary for core beta vs. ring beta with $K_{\perp}\rho_i = 0.1$. Interchange stable below the common lower boundary (for =0, shaded). Also stable above the appropriately labeled curve.

Using a different model which includes the effects of curvature such that:

$$\frac{\omega_{*i}}{\omega_{bi}} = (-\frac{2}{\beta})(1-\frac{\omega_{\kappa i}}{\omega_{bi}})$$

and replacing the curvature and the various gradients by their local values, β_c becomes a function of β_h as in the $\delta = 0$ case. For finite β, however, there exists two stability regions. The lower one resembles the $\delta = 0$ region. The major difference is that the stability threshold for β_h, when β_c approaches zero is shifted downward as δ is increased, approaching for $\delta \geqslant .05$, the line above which mod-B has no minimum. The upper boundary is strongly influenced by the singular behavior of the dispersion relation at D = 0. The upper stability region, on the other hand, appears to be unattainable, unless the ion temperature range can be greatly increased[6].

What appears to be coming out of this comparison is this: the higher the annulus density (n_h), the larger the difference between interacting and rigid ring theory. Clearly, the annulus is having an effect on the

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6. Lee and Van Dam, EBT Ring Physics: Proceedings of the Workshop, Dec. 3-5, 1979, p.471

stability of the core plasma not defined when using a rigid ring model. And that effect is purely non-MHD.

THE ENERGY PRINCIPLE

If one were to put as precisely as possible the crux of the difference between rigid ring theory and interacting ring theory, it would be this: The trouble in MHD or guiding center analysis of ring stabilized plasmas is the frozen-in field line constraint. This does not represent their actual behavior. With this fact in mind, the next step in the theoretical development of interacting ring theory is obvious. The heart of the theory must be an energy principle which will accurately predict stability regions without invoking the frozen-in field line constraint.

To begin, the energy principle from which rigid ring theory originates is well-known (see chapter 2). Based on MHD analysis it requires the adiabatic conservation of single particle magnetic moments and the longitudinal action, as well as that the magnetic field lines are frozen into the plasma. In ring stabilized plasmas, the electrons are highly energetic so it would seem questionable that a displacement in the hot electron ring plasma would follow exactly a displacement of the magnetic field. One can make this argument more plausible when one considers that the $E \times B$ displacement is inversely proportional to the frequency. Now, in the case of the hot electron rings, the frequency is Doppler shifted by a large precessional magnetic drift. Clearly the normal MHD response of the hot electrons is negligible to a perturbation in the magnetic field. The frozen-in field line constraint must go.

Basically, to replace this third condition in MHD analysis, Vam Dam and Lee used the condition that the magnetic flux passing through a precessional drift orbit is conserved[1]. This new condition is guaranteed if the hot electrons complete several orbits in a typical plasma oscillation. This serves to restrict the energy principle to low frequency regions while relating particle behavior to magnetic field perturbations thus quite naturally serving as a good replacement for the frozen-in field line condition. And it does work well in ring stabilized plasmas where the region of the hot electron drift frequency is comparable to the ion cyclotron frequency.

Consider an isotropic guiding center plasma whose behavior is completely magnetohydrodynamic and thus the field lines are frozen into the plasma. There are two adiabatic invariants:

$$\mu = \frac{Mv_{\perp}^{2}}{2B} = \text{the magnetic moment}$$

where:

M is the mass of the particle and $v_{\perp}$ is its transverse velocity

1. J.W. Van Dam, M.N. Rosenbluth, and Y.C. Lee, IFS Report 12 March, 1981

$J = M\int dl v_{\parallel}$ = the longitudinal action

where:

l is the arc length along a field line and $v_{\parallel}$ is a particle's parallel velocity.

Now the existence of these two invariants implies that the gyration and bounce periods of the particles occur in much less time than a characteristic fluctuation of the plasma. Because of this, the potential energy of the plasma can be written as:

$$W_p = \int d^3\underset{\sim}{x}\int \frac{d\varepsilon\, d\upsilon B}{v_{\parallel}}(\varepsilon F) = \int d\alpha\, d\beta\, dJ(\varepsilon F)$$

where:

F is the gyro- and bounce- averaged guiding center distribution function;

ε is a particle's energy $= \frac{Mv_{\parallel}^2}{2} + \mu B$ and;

α, β are the flux and azimuthal angles respectively for axisymmetric systems and define the magnetic field such that:

$$\underset{\sim}{B} = \nabla\underset{\sim}{\alpha} \times \nabla\underset{\sim}{\beta}$$

Also note that the velocity space integration includes a summation over the species in the plasma; one which, if taken over parallel motion, would vanish for trapped particles. In addition, finite Larmour effects, the electrostatic potential, and relativistic effects are neglected.

The procedure is to take this integral, give it a displacement $\underset{\sim}{\xi}$ in the magnetic field, and calculate the variation in plasma energy[2]. Now if a Lagrangian viewpoint is adopted (coordinate system moving with the local velocity of the plasma), the number of particles in an infinitesimal phase space volume:

$$F d\alpha d\beta d\mu dJ \text{ in } (\alpha,\beta,\mu,J)\text{-space}$$

is conserved by Liouville's theorem. Thus one need only calculate a change in a single particle's energy (ε) and integrate it in order to find the total change of energy in the system. This can be accomplished by using the adiabatic invariants μ and J.

With this in mind, displace the magnetic field while keeping ε constant. The action integral J would change to first order, by:

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2. Van Dam, Rosenbluth, and Lee, IFS Report 12, March, 1980

$$J^{(1)} = M\int(\frac{\partial l}{B})^{(1)}[Bv_{\parallel} + \frac{\partial l}{B}(Bv_{\parallel})^{(1)}] = \int\frac{\partial l}{v_{\parallel}}(Mv_{\parallel}^2 q - \mu B') \qquad (1)$$

$$= \int\frac{\partial l}{v_{\parallel}}(Mv_{\parallel}^2 q - \mu B')$$

where: the subscript 1 indicates first order change and B' is the first order Lagrangian change in field strength:

$$B' = -B(\nabla\cdot\underset{\sim}{\xi} - q)$$

$$q = \hat{b}\hat{b}:\nabla\underset{\sim}{\xi}$$

$$\hat{b} = \frac{\underset{\sim}{B}}{B}$$

In order to calculate the arc length, notice that in a displacement, the quantity $\partial l/B$ has the same transformation properties as the Jacobian relation $\underset{\sim}{x}$ to $\underset{\sim}{x} + \underset{\sim}{\xi}$, ie.:

$$\frac{\partial l}{B} \rightarrow \frac{\partial l}{B}\{1 + \nabla\cdot\underset{\sim}{\xi} + \frac{1}{2}[(\nabla\cdot\underset{\sim}{\xi})^2 - \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi}] + ...\}$$

Also the invariance of J implies that there must be a change $\varepsilon^{(1)}$ to balance the change $J^{(1)}$. Thus the first order change is:

$$J_1 = 0 = J^{(1)} + \varepsilon^{(1)}(\frac{\partial J}{\partial\varepsilon})_{\alpha,\beta,\mu}$$

With this, one can easily invert this equation to solve for $\varepsilon^{(1)}$.

Now, defining the bounce average as:

$$< \ldots. > \; = (\frac{\partial J}{\partial \varepsilon})^{-1} \int \frac{\partial l}{v_{\parallel}} (\ldots.)$$

where: $\frac{\partial J}{\partial \varepsilon}$ is a bounce period

Then:

$$\varepsilon^{(1)} = \; < H > \qquad (2)$$

where:

$H = -Mv_{\parallel}^{2} q \; + \; \mu B^{(1)}$ is the gyro-averaged increment to a particle's kinetic energy.

Substituting equation(2) into equation(1) yields:

$$W_{p1} = \int d\alpha \, d\beta \, d\mu \, dJ F < H > \; = \int d^3 \underset{\sim}{x} (-p_{\parallel} q + \frac{p_{\perp} B^{(1)}}{B})$$

where:

$p_{\parallel}$ is the longitudinal component of the pressure

$p_{\perp}$ is the transverse component of the pressure

To complete the analysis, one needs the change in magnetic field energy:

$$W_{m1} = \int d^3\underset{\sim}{v} B(B^{(1)} - \underset{\sim}{\xi}\cdot\nabla B)$$

However, when this is added to the plasma energy variation subject to the boundary conditions and when one considers the equilibrium relation:

$$\underset{\sim}{B} \times (\nabla \times \underset{\sim}{B}) + \nabla\cdot[(p_\parallel - p_\perp)\hat{b}\hat{b} + p_\perp \underset{\approx}{I}] = 0$$

the change in potential energy vanishes to first order. Apparently the energy variation is second order as in chapter 2.

To do the analysis, second order change in action and energy are needed. These are calculated in the same fashion as first order change:

$$J_2 = 0 = J^{(2)} + \varepsilon^{(2)}(\frac{\partial}{\partial\varepsilon}) + \frac{1}{2}\varepsilon^{(1)^2}(\frac{\partial^2 J}{\partial\varepsilon^2}) + \varepsilon^{(1)}(\frac{\partial J}{\partial\varepsilon})$$

or:

$$\varepsilon^{(2)} = -(\frac{\partial J}{\partial\varepsilon})^{-1}[J^{(2)} - \frac{\partial}{\partial\varepsilon}(\frac{1}{2}\varepsilon^{(1)2}\frac{\partial J}{\partial\varepsilon}] \qquad (3)$$

where $J^{(2)}$ is:

$$J^{(2)} = \frac{1}{2}(\frac{\partial J}{\partial\varepsilon} < Mv_\parallel^2[(\hat{b}\cdot\nabla\underset{\sim}{\xi})^2 - q^2]$$

$$-\mu B[(\hat{b}\cdot\nabla\underset{\sim}{\xi})^2 - q^2 - (\nabla\cdot\underset{\sim}{\xi})^2 + \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi} + 2(\frac{B^{(1)}}{B})^2] - \frac{1}{M} < (\frac{\mu B^{(1)}}{v_\parallel})^2 >$$

This comes from the second Lagrangian field change:

$$B^{(2)} = \frac{1}{2}B[(\nabla\cdot\underset{\sim}{\xi})^2 - 2q(\nabla\cdot\underset{\sim}{\xi}) - q^2 + \nabla\underset{\sim}{\xi}:\nabla\underset{\sim}{\xi} + (\hat{b}\cdot\nabla\xi)^2]$$

and yields the change in magnetic energy:

$$W_{m2} = \frac{1}{2}\int d^3\underset{\sim}{x}\{\underset{\sim}{Q}^2 - (\underset{\sim}{\xi}\cdot\nabla\underset{\sim}{\xi})\cdot[\underset{\sim}{B}\times(\nabla\times\underset{\sim}{B})] + (\underset{\sim}{\xi}\times\underset{\sim}{Q})(\nabla\times\underset{\sim}{B})\}$$

where:

$$\underset{\sim}{Q} = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})$$

The total change in energy comes, once again, by combining the plasma and field energies:

$$W = \frac{1}{2}\int d^3\underset{\sim}{x}[\sigma Q_\perp^2 + \tau Q_\parallel^2 + \sigma J_\parallel \hat{b}\cdot(\underset{\sim}{\xi}\times Q_\perp) + q\underset{\sim}{\xi}\cdot\nabla' p_\parallel - \frac{1}{B}(2Q_\parallel + \underset{\sim}{\xi}\cdot\nabla\underset{\sim}{B})\underset{\sim}{\xi}\cdot\nabla' p_\perp] + \delta w_k$$

where:

$$\sigma = 1 - \frac{1}{B}\frac{\partial p_\parallel}{\partial B}$$

$$\tau = 1 - (\frac{\partial^2 p_\parallel}{\partial B^2})$$

are measures of stability against firehose and mirror anisotropy modes;

$J_{\parallel}$ is the parallel equilibrium current

$\nabla' = \nabla - \nabla B\frac{\partial}{\partial B}$

and δw_k is the kinetic energy variation which arises from the second term in equation(3). Performing an integration by parts on the distribution function F in this term yields:

$$\delta w_k = -\frac{1}{2}\int d\alpha\, d\beta\, d\mu\, dJ\left(\frac{\partial F}{\partial \varepsilon}\right)\langle H \rangle^2$$, a standard result.

Now these results are consistent with those of a guiding center plasma with μ,J invariance. Its consistency establishes that the methods here yield results well-known in MHD plasmas. The next step is to carry out this analysis in the region of interest, to non-MHD plasmas.

Consider a plasma that is entirely non-MHD. According to this new energy principle, this consists of replacing the frozen-in field line constraint with the conservation of magnetic flux through a precessional drift orbit. The calculation begins as in the MHD case; namely, the invariance of J implies that a particle's energy must change as the magnetic field is displaced, where α and β label the same field line. The difference occurs when it is noted that particles are not constrained to follow field lines. This causes the surfaces of J to move in (α,β)-space and the original contours:

$$\alpha = \alpha_0(\beta,\mu,\varepsilon,J)$$

obtained from J-invariance, are displaced into new contours:

$$\alpha = \alpha_0 + \alpha_1(\alpha_0,\beta,\mu,\varepsilon,J)$$

The first order change in J is thus different than in the MHD case:

$$J_1 = 0 = J^{(1)} + \varepsilon^{(1)}\left(\frac{\partial J}{\partial \varepsilon}\right)_{\alpha,\beta,\mu} + \alpha^{(1)}\left(\frac{\partial J}{\partial \alpha}\right)_{\varepsilon,\beta,\mu} \qquad (4)$$

where:

$\frac{\partial J}{\partial \alpha}$ indicates the drifting away of the J-contours from the lines of force.

One must add to this calculation a new step: the calculation of the secondary displacement $\alpha^{(1)}$. This is obtained from the condition of flux invariance:

$$\phi^{(1)} = 0 = \int \alpha^{(1)} d\beta$$

where integration is around a constant J-contour. Now defining a double average over both bounce and drift motion:

$$\langle\langle \ldots \rangle\rangle = \left(\frac{\partial \phi}{\partial \varepsilon}\right)^{-1} \int d\beta \left(\frac{\partial \varepsilon}{\partial \alpha}\right)^{-1} \langle \ldots \rangle$$

where:

$\frac{\partial \phi}{\partial \varepsilon}$ is the precessional drift period

$\frac{\partial \varepsilon}{\partial \alpha} = \langle \frac{\partial B}{\partial t} \rangle$ is the rate of precession

and solving for $\alpha^{(1)}$ and $\varepsilon^{(1)}$ using equations (4) and (5), as in the MHD case, one obtains:

$$\varepsilon^{(1)} = \langle\langle H \rangle\rangle \tag{6a}$$

$$\alpha^{(1)} = \left(\frac{\partial \varepsilon}{\partial \alpha}\right)^{-1} \left[\langle\langle H \rangle\rangle - \langle H \rangle \right] \tag{6b}$$

where H has the same form as the MHD case. Again, utilizing the equilibrium relation of the MHD case and the boundary conditions for the magnetic energy, the variation of the total energy vanishes to first order. Once more second order variations $J^{(2)}, \phi^{(2)}, \varepsilon^{(2)}$ are needed.

Thus following the same procedure as in the MHD case, the second order invariance conditions imply that:

$$J_2 = 0 = J^{(2)} + \varepsilon^{(2)}(\frac{\partial J}{\partial \varepsilon}) + \alpha^{(2)}(\frac{\partial J}{\partial \alpha})$$

$$+ \frac{1}{2}\varepsilon^{(1)^2}(\frac{\partial^2 J}{\partial \varepsilon^2}) + \frac{1}{2}\alpha^{(1)^2}(\frac{\partial^2 J}{\partial \alpha^2}) + \alpha^{(1)}\varepsilon^{(1)}(\frac{\partial^2 J}{\partial \alpha \partial \varepsilon}$$

$$+ (\varepsilon^{(1)}\frac{\partial}{\partial \varepsilon} + \alpha^{(1)}\frac{\partial}{\partial \alpha})J^{(1)} \quad (7a)$$

$$\phi^{(2)} = 0 = \int \alpha^{(2)} d\beta \quad (7b)$$

Using $(\frac{\partial \phi^{(1)}}{\partial \varepsilon}) = 0$, one can solve equations (6a and 6b) and (7a and7b) for $\varepsilon^{(2)}$. The result is:

$$\varepsilon^{(2)} = (\frac{\partial \phi}{\partial \varepsilon})^{-1} \int \frac{\partial \beta}{\partial F/\partial \alpha} \{ J^{(2)} + \frac{\partial}{\partial \alpha}[\frac{1}{2}\alpha^{(1)}(\varepsilon^{(1)}\frac{\partial J}{\partial \varepsilon} - J^{(1)})] - \frac{\partial}{\partial \varepsilon}(\frac{1}{2}$$

$$\times \varepsilon^{(1)^2}(\frac{\partial J}{\partial \varepsilon})\} \quad (8)$$

The plasma energy can now be calculated from the equation:

$$W_{p2} = \int d\alpha \, d\beta \, d\mu \, dJ (F\varepsilon^{(2)})$$

Note that attention must be given to maintaining the correctness of the coordinates. Specifically, α must be transformed from a coordinate which describes a drift orbit to one which describes a field line. The transformation is:

$$(\frac{\partial}{\partial \varepsilon})_{\alpha_0} \rightarrow\rightarrow\rightarrow\rightarrow (\frac{\partial}{\partial \varepsilon})_J$$

$$(\frac{\partial J}{\partial \alpha_0})^{-1} (\frac{\partial}{\partial \alpha_0})_{\varepsilon} \rightarrow\rightarrow\rightarrow\rightarrow (\frac{\partial}{\partial J})_{\varepsilon}$$

But to continue, the variation in total energy is formed again by combining the change in field and particle energies. Since $J^{(2)}$ makes the same contribution to the total energy and δw_k can be derived from the last two terms of equation(8), one merely follows the procedure outlined earlier. Integrating equation(6), the result is a new generalized kinetic energy principle:

$$W = \frac{1}{2}\int d^3 x [\sigma Q_\perp^2 + \tau Q_\parallel^2 + \sigma_\parallel J_\parallel \hat{b}\cdot(\xi \times Q) + q\xi\cdot\nabla' p - \frac{1}{B}(2Q_\parallel + \xi\cdot\nabla B)\xi\cdot\nabla' p_\perp]$$

$$+ \frac{1}{2}\int d\alpha\, d\beta\, d\mu\, dJ \{ (\frac{\partial F}{\partial \varepsilon}) \langle\langle H \rangle\rangle^2 + \frac{\partial F}{\partial J}\frac{\partial J}{\partial \varepsilon}[\langle H \rangle^2 - \langle\langle H \rangle\rangle^2]\} \qquad (9)$$

It is instructive to compare this generalized energy principle in the MHD and non-MHD limits. The local part in the two cases are the same, but the kinetic part is quite different. While the first term resembles the MHD result, the second term, dominant in E.B.T. applications is proportional to $\partial f/\partial J$ (although it vanishes when the distribution function makes the components of the pressure functions of the magnetic field only).

Now the δw of equation(8) conforms to all the mathematical properties of stability analysis and hence $\delta w \geqslant 0$ is both a necessary and sufficient condition for stability. Because of this, predictions of this new energy principle offer a departure from rigid ring theory. According to these results, because the rings of a ring stabilized plasma are not MHD plasmas, they will have different stability properties than they would if they were MHD plasmas.

Consider as E.B.T. configuration. First of all, it is axisymmetric. In this case, the drift averaged quantities in equation(9) vanish when $m = 0$ and the equation becomes identical to that of the MHD case. Employment of the Schwarz inequality, much in the same way it was used when discussing rigid ring theory, leads to the fact that the kinetic part of δw in equation(9) is positive definite provided the local conditions are such that:

$$\left(\frac{\partial F}{\partial \varepsilon}\right)_{J,\mu} < 0$$

$$\left(\frac{\partial F}{\partial J}\right)_{\varepsilon,\mu} < 0$$

Now in a ring stabilized plasma, the core plasma can be described by the MHD case, but the hot electron rings must utilize the non-MHD analysis. Using both the MHD energy equation and equation(9), the new δw becomes:

$$\delta w = \delta w^{G.C.} + \frac{1}{2}\int d\alpha\, d\beta\, d\mu\, dJ \left\{ \left(\frac{\partial F_h}{\partial \varepsilon}\right)\left(1 - \frac{\omega_{*h}}{\langle \omega_{dh} \rangle}\left[\langle H\rangle^2 - \langle\langle H\rangle\rangle^2\right]\right\} \quad (10)$$

For ease of interpretation, equation(10) has been broken into two parts[3]. $\delta w^{G.C.}$ is exactly what would have been found if the entire ring stabilized plasma (core and rings) had been an MHD plasma. Only the second term involves the non-MHD hot electrons (with ω_{*h} and $\langle w_{dh} \rangle$ representing the diamagnetic and bounce averaged drift velocities). Now the purpose of the rings in a ring stabilized plasma is to cause a local magnetic well as

3. Van Dam, Rosenbluth, Lee, IFS Report 12, March, 1981

discussed earlier. This in turn creates a local minimum B and reverses the magnetic drifts, since the plasma pressure decreases steadily outward. Hence ω_{*h} and $\langle \omega_{dh} \rangle$ are of opposite signs. Now given the local conditions for positive definiteness defined earlier, the difference term in equation(10) is negative. This predicts that the plasma will be less stable than it would given purely MHD assumptions.

One can specifically compare the two energy principles by finding the stable operating region of β_h as a function of β_c[4]. Consider the perturbed parallel field $Q_\parallel$, and assume that the magnetic well produced by the rings is sufficient to reverse a particle's drift. The energy principle describing this component of the energy is:

$$U=\int \frac{\partial 1}{B}(\tau Q_\parallel)^2 - \int d\varepsilon\, d\mu \left(\frac{\partial F_c}{\partial \varepsilon}\right) \frac{\left[\int \frac{dl}{v_\parallel}(\mu Q_\parallel)^2\right]}{\int \frac{dl}{v_\parallel}} + \int d\varepsilon\, d\mu \left(\frac{\partial F_h}{\partial \alpha}\right) \frac{\left[\int \frac{dl}{v_\parallel}(\mu Q_\parallel)\right]^2}{\int \frac{dl}{v_\parallel}(v_\parallel^2 \kappa + \mu B')} \qquad (11)$$

where:

c,h are ring and core plasma quantities

$\kappa = (\hat{b}\cdot\nabla b)\cdot\frac{\nabla\alpha}{|\nabla\alpha|^2}$ is the curvature

$' = \frac{\partial}{\partial\alpha}$

4. Van Dam, Rosenbluth, Lee, IFS Report 12, March, 1981

For large mode number m in axisymmetric systems δw may minimized on each field line. By the Schwarz inequality:

$$U \leqslant \int \frac{dl}{B}\left(1 + \frac{\partial p_{\perp h}}{B \partial B} + \frac{2p_{\perp c}}{B^2}\right)Q_{\parallel}^2$$

$$+ \frac{\left[\int \frac{dl}{B^2} Q_{\parallel}\left(p'_{\perp h} - B' \frac{\partial p_{\perp h}}{\partial B}\right)\right]^2}{\int \frac{dl}{B}\left[\kappa\left(p'_{\parallel h} - B' \frac{\partial p_{\parallel h}}{\partial B}\right) + \frac{B'}{B}\left(p'_{\perp h} - B' \frac{\partial p_{\perp h}}{\partial B}\right)\right]}$$

Now one would like to choose a perturbation which would make the sign of U determine the overall stability of the plasma. One case is when $Q_{\parallel} \propto 1/B$. In this instance, the condition for stability (at least for an isotropic plasma) is such that:

$$-p'_c \int \frac{dl}{B^3} + 2\int \frac{dl}{B}\kappa + \frac{\int \frac{dl}{B^3}\left(\frac{2p_c}{B^2}\right)\int \frac{dl}{B}\left(2\kappa - \frac{(p_h + p_c)'}{B^2}\right)}{\int \frac{dl}{B^3}} \geqslant 0$$

If this condition is not satisfied, what will occur is a type of mirror instability. On the other hand, if equation(12) is positive, the quadratic terms in the field perturbation are positive definite and δw can also be wth respect to $BQ_{\parallel}$.

Next, take a flute-like displacement:

$\underset{\sim}{\xi}\cdot\nabla\alpha$

Using the Schwarz inequality, stability along each field line is determined by the sign of:

$$V(\alpha) = -p_c'\int\frac{dl}{B}(2\kappa - \frac{p'}{B^2}) + 2p_c\int\frac{dl}{B}[\frac{7}{2}\kappa^2 - 3\kappa(\frac{p'}{B^2}) + (\frac{p'}{B^2})^2]$$

$$- \frac{[p_c'\int\frac{dl}{B^3} - p_c\int\frac{dl}{B^3}(2\kappa - \frac{p'}{B^2})]^2}{\int\frac{dl}{B^3}(1 + \frac{2p_c}{B^2}) + [\frac{p_h'(\int\frac{dl}{B^3})^2}{\int\frac{dl}{B}(2\kappa - \frac{p'}{B^2})}]}$$

Further, $V(\alpha)$ can be interpreted such that the first term is the driving term for electrostatic interchange modes and the second represents compressional modes. Now the first term has been calculated previously using rigid ring assumptions and has been traditionally associated with the threshold for stable operation in E.B.T.; namely:

$$\beta_h = \frac{2p_h}{B^2} > \frac{4\int\frac{dl}{B}\kappa}{B^2\int\frac{dl}{B^3}(\frac{p_h'}{p_h})}$$

But the last term produces additional instability not seen before coming out of MHD or guiding center plasma analysis. It is not a trivial distinction. If equation(12) is satisfied, the last terms of $V(\alpha)$ overwhelm the first two and the interchange mode is unstable. For this case, the upper limit corresponds to a β_c three of four times lower than that predicted by MHD analysis. It is clear that this new energy prnciple will lead to a whole new model of stable operating regions in ring stabilized plasmas[5].

5. Van Dam, Rosenbluth, Lee, IFS Report 12, March, 1981

THE FUTURE OF STABILITY THEORY OF RING STABILIZED PLASMAS

The direction of theoretical research into the modeling of hot electron rings in a warm, guiding center plasma can be broken into two categories. The first is to make a more realistic model of the plasma. Geometry is the leading edge of this research. Slab geometry has no curvature and thus neglects the driving terms of some destabilizing modes, weakening the realism of the analysis. The density structure of the plasma, charcterized by the parameter δ, also lacks realism in past models by being simplified to handle certain domains of reactor operation. Further, analysis in the past has assumed a local view which may lead to a lack of cohesiveness one senses when reading various papers on interacting ring theory. The key to this problem seems to be, basically, the concept of a mode spectrum.

The second direction of research is tied to the concept of evolution in science. If an MHD plasma has a small non-MHD component, it would seem that a modified MHD analysis, treating the non-MHD component as a perturbation would produce the same results as an analysis in which both components of the plasma are treated self-consistently. As has been shown this has not been the case. Yet perturbation theory has worked on many physical theories in the past and the recognition that interacting ring theory may be the next higher order approximation holds the intriguing possibility of this second direction in theoretical research.

Overall the two directions are tied to one another. One would have great difficulty making a more advanced theory from one whose full geometrical implications are not yet understood. Thus it is reasonable to start at this point. Recently, efforts have been made using interacting ring theory to treat the radial aspects of stability in ring stabilized plasmas by modeling the plasma as a z-pinched core plasma surrounded by hot electron rings[1]. The hope of this model is to allow for curvature in a more realstic way than an artificial gravity term. This would lead to finding a mode structure and allow construction of a mode spectrum which would include modes from diverse calculational methods (such as W.K.B. approximation methods) and long wavelength modes excluded to this point.

Without going into the details of the calculation in great depth, the procedure consists of:

(1) Finding a radial differential equation which describes the behavior of the system based on pressure balance relations and the Vlasov-Maxwell equations familiar from previous chapters.

(2) Analysing the resultant equations in the appropriate frequency limits to determine its mode structure.

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1. H.L. Berk, M.N. Rosenbluth, J.W. Van Dam, and D. Spong, Z-pinch Model For A Hot Electron Annulus (to be published)

The results have been encouraging. Not only have modes obtained from W.K.B. analysis been recovered, but long wavelength modes, with wavelengths stretching over the entire annulus have been calculated, thus creating a more unified mode structure.

As to the density problem, researchers using interacting ring theory have discovered a new mode in the plasma if the backround density is too high. Recent work has shown that when this new mode is included in the analysis, the energy principle obtained in the preceding chapter is both a necessary and sufficient condition for stability[2]. However, a more detailed model is needed to understand this phenomenon in the context of the entire density picture. Recent work points to a density threshold around which the work with radial models of the plasma and density parameter analysis can be merged, but once again a more detailed model of the plasma is needed in order to undertake the task of such a merger. In the end, though, the picture of interacting ring theory is gradually becoming more complete, organized into a single unified form, and self-consistent. The major question that remains is: Is there something beyond interacting ring theory and if there is, how accurate are the results obtained via its use?

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2. H.L. Berk, M.N. Rosenbluth, J.W. Van Dam, and D. Spong, Z-pinch Model For A Hot Electron Annulus (to be published).

One way to answer this question is to consider the distinction between rigid ring theory and interacting ring theory. When $\delta = 0$, $\beta_c << \beta_h$, both theories show identical results. But this limit implies that the annulus is a negligible part of the plasma with its effect being minimized also. This is just begging the question. If there is no non-MHD component to a plasma, the plasma can be handled by MHD-analysis. A more meaningful question is to ask in what limit the analysis methods of the two theories agree. Said another way, when does the frozen-in field line constraint lead to flux conservation (or vica versa). In a recent paper, Nelson let the hot electrons be perturbed in discussing a different formulation of rigid ring theory[3]. His analysis still showed basic differences in stability regions between the two approaches, although the slab model used clouded the issue by its lack of realism. Even discounting this, it can be hypothesized that using a form of perturbation theory to describe the effects of hot rings on core plasmas is not enough as long as the frozen-in field line constraint is utilized to close the equations. One must somehow perturb the method of analysis itself.

Now the work in the first direction of improving the realism of plasma modeling provides insight into work on the second by clarifying the distinction between the two results. In the limit of high backround

3. C.L. Hedrick and D.B. Nelson Nuclear Fusion 19, 283, (1979)

density, anyway, the more pessimistic predictions of interacting ring theory are tied to the compressional effects of the annulus on the core plasma. These new modes come out of interacting ring modeling quite naturally, but cannot be accounted for using rigid ring theory. Having the frozen-in field line constraint seems directly to lead to a lack of knowledge about compressional effects. To avoid this, it might be possible to use a perturbation technique on this constraint. Let the field lines be perturbed, but by only a small amount, from their original position. Another approach might be to analyze the functional relation between flux conservation and having the field lines frozen into the plasma. If one could understand the parameters on which this relationship is based, one could do a parametric study on exactly when the two constraining conditions are the same and how they diverge parametrically. This information then could be used to compare the stability predictions of the two theories.

Compressional effects of the annulus are established as facts in ring stabilized plasmas. This implies that the natural behavior of the system is not well described by the frozen-in field line assumption. The flux conservation of interacting ring theory is a more physically valid assumption about the plasma. How far it goes in describing the real nature of the system is subject to speculation. Could it be that flux conservation is a less rigorous constraint than the frozen-in field line condition, but that they represent two levels of approximating a more general physical condition? If so, is interacting ring theory nothing more than a modified perturbation theory analysis which acts on the method of analysis rather than the behavior of the system? These and other questions will be answered in the years ahead as the problem of stability in ring stabilized plasmas is studied ever more accurately in the future. In all this, one goal is manifestly present, to discover a unified and complete picture of stability in ring stabilized plasmas. With all the recent progress in this area, this goal is realistic and close at hand.

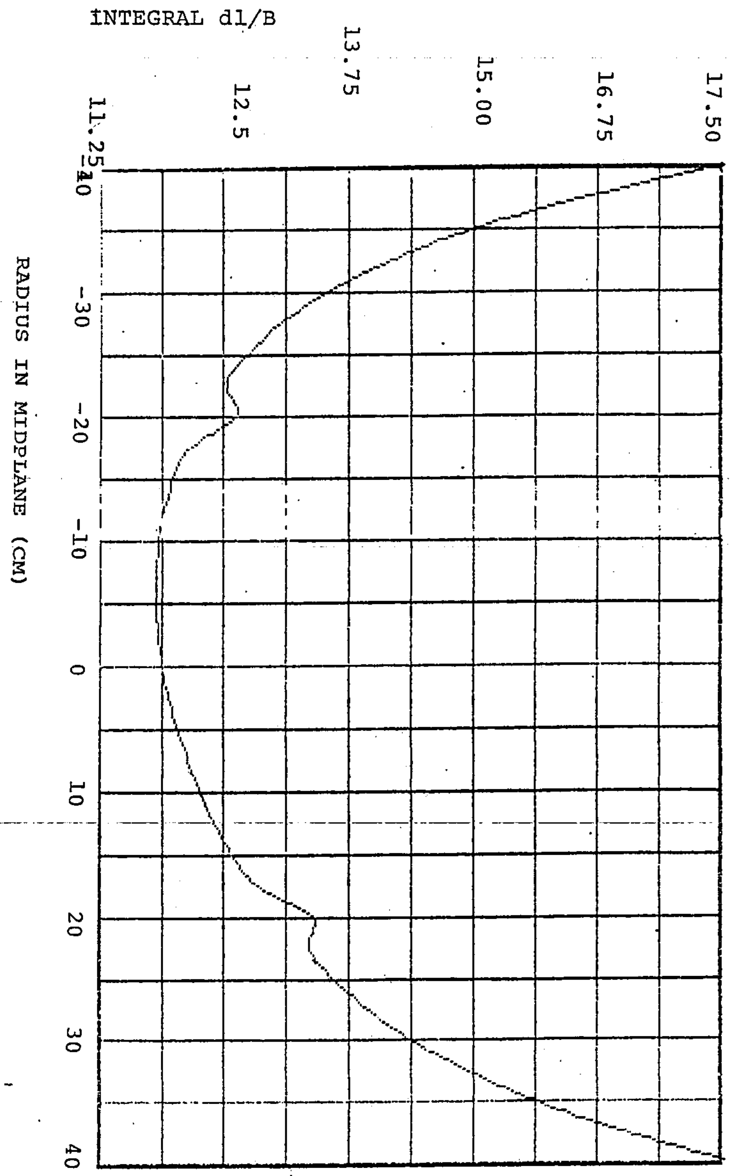
FUSION POWER POSSIBILITIES

The evolution of stability theory herein discussed points to the necessity of using the more pessimistic predictions of interacting ring theory in attempting to build a successful fusion reactor out of the E.B.T. design (See figures 1 and 2). Even allowing for the possibility of a new breakthrough in theory, the bright picture of stability presented by the rigid ring model seems to be a thing of the past. There is just too much theoretical basis for the compressional effects of interacting ring theory to deny that such destabilizing effects exist. Given this fact, the beta of the core plasma must be limited to around 10% for the plasma to be stable against interchange modes.

Among the recent developments in hot electron ring stabilization is an attempt to utilize a tandem mirror-like device in a polygonal configuration. The motivation for this research is an article by J. W. Vam Dam applying the interacting ring model to a tandem mirror device[1]. Due to reduced radial transport, no need to recircularize flux surfaces, and simplicity of coil construction, electron rings have on occasion been proposed to act as end plugs on tandem mirrors. The only difference in this scheme is that a polygon configuration would incorporate these

1. J.W. Van Dam (inter-IFS note)

FIGURE #1



dl/B vs. radius in the midplane for a 6 Tandem mirror configuration with a major radius of 21.6 meters

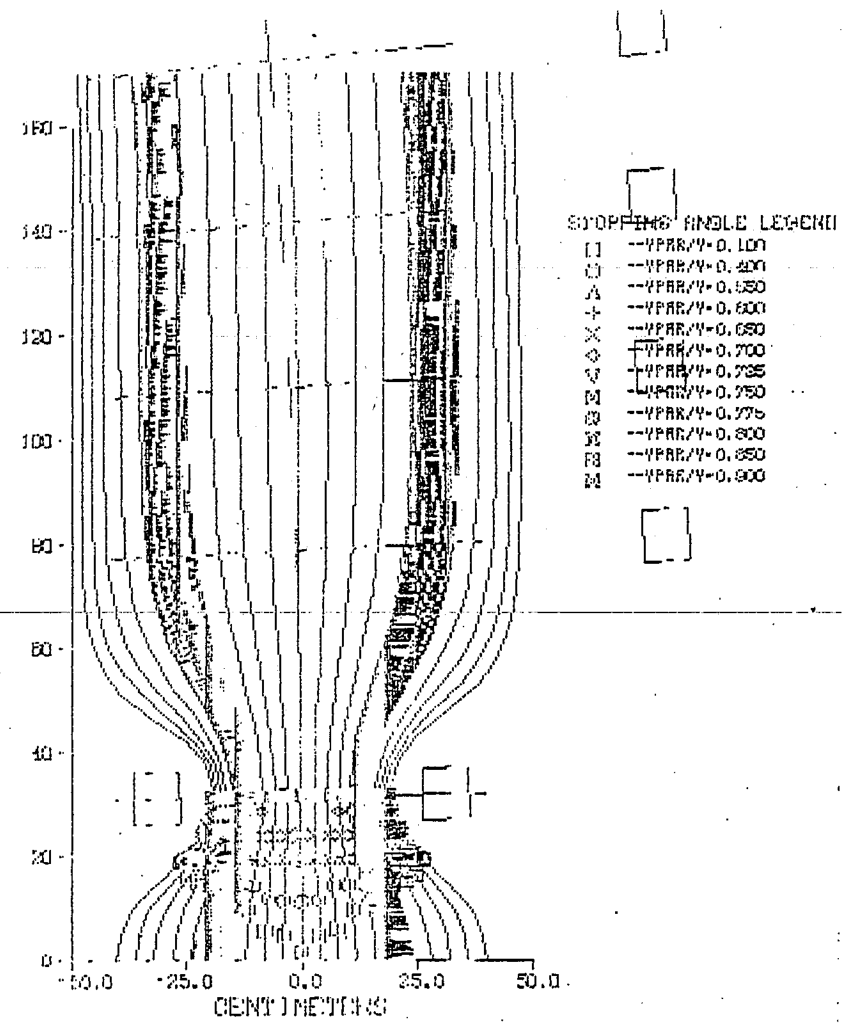


FIGURE #2

Field line and stopping angle diagram for E-ring tandem mirror described in previous figure.

advantages with the toroidal flow of plasma inherent in the E.B.T. design.

To begin, in an E-ring tandem mirror device, the stability condition is determined by the field average of the driving term introduced previously using pressure balance:

$$2\kappa p_c'\left(1-\frac{\frac{1}{2}p_{\perp h}'}{B^2\kappa-p_c'}\right)$$

where:

κ is the curvature

$p_c, p_{\perp h}$ are the core and ring pressures; and,

primes are radial derivatives.

Now, in the plug region, β_c must be low enough to prevent the denominator of equation(1) from blowing up and causing destabilization. Advantage can be gained by widening the annulus through appropriate microwave heating and increasing the curvature of the field lines, allowing β_c to have a fairly high value (10-15%) yet avoiding the singularity. Further, β_c could be increased in the straight sections, where there is no curvature, for fusion purposes.

The purpose of an ongoing computer study is to study the extent to which the rings would serve as line-tying agents. For this study, a modified version of the ballooning eigenmode equation must be constructed. It starts from the generalized kinetic energy principle derived in the previous chapter. For large mode number perturbations (flute-like in the vicinity of the rings), the ballooning eigenmode equation is given by:

$$\underset{\sim}{B}\cdot\nabla\left(\frac{\sigma}{|\nabla\psi|^2}\underset{\sim}{B}\cdot\nabla X\right) + \frac{\rho\omega^2}{|\nabla\psi|^2}X + \left\{ p_c'\left(\kappa + \frac{B'}{B}\right) - p_c\left[2\left(\frac{B'}{B}\right)^2 + 3\kappa\frac{B'}{B} + 3\kappa^2\right]\right.$$

$$\left. + \int\frac{Bd\mu\, d\varepsilon}{v_\parallel}\left(\frac{\partial f_h}{\partial\psi}\right)\left[v_\parallel^2\kappa + \mu B' - \langle v_\parallel^2\kappa + \mu B'\rangle\right] - \frac{\frac{1}{B^2}\left(p_c' - \frac{2p_c B'}{B} - p_c\kappa\right)^2}{\left[\tau + \frac{2p_c}{B^2} + \frac{1}{B^2}\int\frac{Bd\mu\, d\varepsilon}{v^\parallel}\left(\frac{\partial f_h}{\partial\psi}\right)\frac{(\mu B)^2}{\langle v_\parallel^2\kappa + \mu B'\rangle}\right]}\right\} X = 0$$

$X = \underset{\sim}{\xi}\cdot\nabla\psi$; the radial component of the displacement $\underset{\sim}{\xi}$

$\kappa = (\underset{\sim}{B}\cdot\nabla B)\frac{\nabla\psi}{B^2}|\nabla\psi|^2$; the curvature

$\sigma = 1 + \frac{p_{\perp h} - p_{\parallel h}}{B^2}$ and $\tau = \frac{1 + \frac{\partial p_{\perp h}}{\partial B}}{B}$

are the measures of anisotropy and must be positive.

f_h = the hot electron distribution function which is a function of flux (ψ), magnetic moment (μ) and energy ($\varepsilon = \frac{1}{2}Mv_\parallel^2 + \mu B$)

$\langle \ldots \rangle$ = bounce averaging (Note that when this is done the ring component of the plasma is entirely trapped in the mirror field)

$' = \frac{\partial}{\partial\psi}$

Now the fact that the hot electron rings must act as end plugs in being the agents which produce line-tying is the only difference between this and previous stability analyses. To accomplish this, at marginal stability, field line averaging must be performed on equation(2) and the result must be negative (subject to the boundary condition: $\partial X/\partial l = 0$ at the ends). Since equation(2) is an interchange stability condition, the idea of line-tying will be justified.

Equally important as the prospect of stability in this proposed configuration is its promise as a possible reactor. To determine this facet of its feasibility, one must perform a calculation on this type of device, in the Lee-Vam Dam limit, that has been applied to a standard E.B.T. reactor by many researchers[2]; namely, a Q-value calculation. It is useful, for the sake of clarity to review this calculation and then to apply it to the E-ring tandem mirror.

For ring stabilized plasmas, the economic feasibility boils down to the fact that if a reactor is heated to ignition, the core plasma heating could be turned off, but the ring heating must continue for the life of the reactor. Thus the power required to sustain the rings defines the so-called "figure of merit" for the reactor and is the major factor contributing to its economic merit. To be more specific, the power supplied to the annulus is dissipated via three dominant mechanisms: drag

2. N.A. Uckan, EBT Ring Workshop: Proceedings Of The Workshop December 3-5, 1979, Oak Ridge , Tennessee, p.507

loss, scattering, and synchrotron radiation (with brehmstrahlung radiation and non-classical power losses playing a much smaller role in the temperature regions of a reactor plasma) and this power must be constantly replenished. In contemporary experiments, drag loss (energy exchange between ring electrons and core electrons through Coulomb collisions) proves to be the main mechanism for power loss. However, when dealing with reactors, these experimental results must be scaled upwards, so it is important to be as general as possible in order to avoid scaling errors. These facts motivate the subsequent analysis:

One begins by defining a Q-value for the reactor ($Q_E = Q_{electric}$):

$$Q_E = \frac{P_{net}}{P_M} = \frac{P_e - P_M}{P_M} \qquad (1)$$

where:

P_{net} = net electrical power

P_e = gross electrical power produced

P_M = input maintenance power

This equation may be rewritten in terms of reactor parameters such that:

$$P_e = \eta_{th}(P_{th} + P_\mu) \qquad (2a)$$

$$P_M = \frac{P_\mu}{\eta_\mu + P_{aux}} \qquad (2b)$$

where:

P_{th} = thermal fusion power

P_{μ} = microwave power required to sustain the rings

η_{th} = thermal conversion efficiency coefficient

η_{μ} = microwave conversion efficiency coefficient

P_{aux} = auxillary power requirements (pumping, refrigeration...)

Substitution of the definitions of P_e and P_{μ} into equation(1) yields (See figure3)[3]:

$$Q_E = \frac{\eta_{th}(P_{th}+P_{\mu})-(P_{aux}+\frac{P_{\mu}}{\eta_{\mu}})}{P_{aux}+\frac{P_{\mu}}{\eta_{\mu}}} \qquad (3)$$

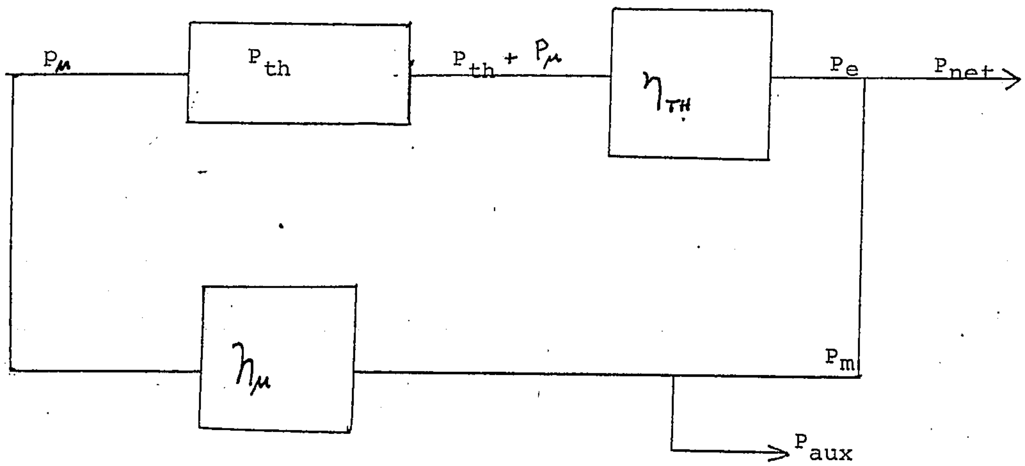
If one assumes that $P_{aux} < P_{\mu}/\eta_{\mu}$, one can simplify equation(3) to read:

$$Q_E = \eta_{th}\eta_{\mu}\frac{P_{th}}{P_{\mu}} -(1-\eta_{th}\eta_{\mu}) \qquad (4)$$

For large Q_E, the reactor would have a high overall efficiency, while for small Q_E, the reactor costs would rise. In fact, for $Q_E > 15$, the cost per kilowatt would remain nearly constant. This is because for $Q_E \geqslant 15$,

3. N.A. Uckan, EBT Ring Physics: Proceedings of The Workshop Dec. 3-5, 1979, Oak Ridge, Tennessee, p.507

FIGURE # 3



Flow Diagram

$$n_i^2 = 1.56 \times 10^{42} \frac{\beta_c^2 B^4}{T^2}$$

where one uses the assumption:

$$T_i \sim T_e \sim T$$

Substituting this back into equation(5), one obtains:

$$\frac{P_{th}}{V_p}\left(\frac{MW}{M^3}\right) = 1.33 \times 10^{24} < \sigma v >_{DT} \frac{\beta_c^2 B^4}{T^2}$$

Next one can calculate the microwave power necessary to sustain the rings. In its most basic form, the annulus power loss is given by the equation:

$$P_{\mu A} = P_{synch} + P_{brehms} + P_{scatt} + P_{drag} + P_{non\text{-}classical} \qquad (6)$$

where:

P_{synch} = power loss due to synchrotron radiation

P_{brehms} = power loss due to brehmstrahlung radiation

P_{scatt} = 90^o pitch angle scattering of ring electrons on core ions.

P_{drag} = power loss due to Coulomb interactions of hot electrons on cold core electrons

$P_{\mu A}$ = microwave power to sustain the annuli

$P_{non-classical}$ = non-classical power loss mechanisms assumed negligible here (and verified negligible in experiments)

These power loss terms can be expressed in terms of plasma and machine parameters. When this is done and substituted into equation(6), the result is:

$$\frac{P_{\mu A}}{V_A} = 3.2 \times 10^{-20} B_A^2 n_A(\gamma-1)[1 + 0.5(\gamma-1)] \quad (P_{synch})$$

$$+ 8.5 \times 10^{-42} n_A n_{ep} \sqrt{\gamma^2-1} \quad (P_{brehms})$$

$$+ 1.0 \times 10^{-39} n_A n_{ip} \frac{\gamma(\gamma-1)}{(\gamma^2-1)^{3/2}} \ln\Lambda_{ei} \quad (P_{scatt})$$

$$+ 2.5 \times 10^{-39} n_A n_{ep} \frac{\gamma \ln\Lambda}{\sqrt{\gamma^2-1}} \quad (P_{drag})$$

where:

$$\gamma = 1 + \frac{T_A}{511}$$

$$\ln\Lambda_{ei} \simeq 38-\ln(\frac{\sqrt{n_A}}{T_A} \simeq 20\text{-}25$$

n_A, T_A, B_A = density, kinetic energy, and magnetic field strength of the annulus

n_p, n_{ip}, n_{ep} = core plasma density near the annulus

$\ln\Lambda$ = Coulomb Logarithm ~ 20

$V_A = 2\pi a \delta_A L_A$ = annulus volume

δ_A = total annulus thickness

L_A = total annulus length = $l_A N = \frac{\text{length}}{\text{section}} \times$ number of sections.

One can neglect brehmstrahlung radiation since it is more than two orders of magnitude less than the other loss mechanisms. When this is done, equation(6) becomes:

$$\frac{P_{\mu A}}{V_A} \simeq 3.2\times 10^{-20} N_A \{ B_A^2(\gamma-1)[1.0+0.5(\gamma-1)]+1.56\times 10^{-18} n_p \frac{\gamma}{\sqrt{\gamma^2-1}}(1+\frac{0.5}{\gamma+1}\}$$

But this equation still contains unmeasureable, or rather, hard to measure quantities. To put this into an equation formed with easily measured parameters one must first find $P_{\mu a}/V_A$ in terms of the magnetic field and the beta of the annulus. To accomplish this, one can use the fact that a cutoff ($\omega_{ce} \leqslant \omega_{pe}$) exists for microwave propagation which imposes a cutoff on the core density (n_p) near the annulus which, basically scales as the square of the magnetic field.

Let f_c = microwave cutoff,

Then $n_p = f_c n_{cutoff} = f_c n_c \quad ; \quad n_c = 10^{19} B_A^2$

Now, defining the annulus density in terms of β_h and B_A:

$$n_A = 2.5 \times 10^{21} \frac{\beta_A B_A^2}{T_A} = \frac{5 \times 10^{18} \beta_h B_A^2}{T_A/511}$$

And substituting these values into equation(6), one obtains:

$$\frac{P_{\mu A}}{V_A}\left(\frac{MW}{M^3}\right) = 3.2 \times 10^{-20} n_A B_A^2 \{(\gamma-1)[1.0 + 0.5(\gamma-1)]$$

$$+ 15.6 f_c \frac{\gamma}{\sqrt{\gamma^2 - 1}}\left(1 + \frac{0.5}{\gamma + 1}\right)\}$$

$$\simeq 3.2 \times 10^{-20} n_A B_A^2 F_c(\gamma, f_c)$$

$$= \frac{.16\beta_h^2 B_A^4}{T_A/511} F_c(\gamma, f_c)$$

Assuming $T_A < 1$ Mev, this becomes:

$$= 2.5 F_c \frac{\gamma}{\sqrt{\gamma^2 - 1}} \frac{\beta_h^2 B_A^4}{T_A/511}$$

where:

$$F_c = (\gamma-1)[1.0 + 0.5(\gamma-1)] + 15.6 f_c \frac{\gamma}{\sqrt{\gamma^2 - 1}}\left(1 + \frac{0.5}{\gamma + 1}\right)$$

$$\simeq 16 f_c \frac{\gamma}{\sqrt{\gamma^2 - 1}} \qquad \text{when } T_A < 1 \text{ Mev}$$

There is one further fact that needs consideration. In the introduction it was pointed out that the stability of the core plasma in ring stabilized plasmas depends on the ring to core plasma densities. To be more specific, for stability:

$$\frac{n_p}{n_A} > 1$$

Now defining this density ratio:

$$f_r = \frac{n_p}{n_A} = \frac{n_{ep}}{n_A} \equiv \frac{1}{\delta}$$

Equation(6) can be written as:

$$\frac{P_{\mu A}}{V_A} \frac{MW}{M^3} \simeq 3.2\times 10^{-20} B_A^2 n_A(\gamma-1)[1+0.5(\gamma-1)]+5\times 10^{-38} f_r n_A^2 \frac{\gamma}{\sqrt{\gamma^2-1}}(1+\frac{0.5}{\gamma+1})$$

$$\simeq 1.2 f_r \frac{\gamma}{\sqrt{\gamma^2 - 1}} \frac{\beta_h^2 B_A^4}{(T_A/511)^2} \qquad T_A < 1 \text{ Mev}$$

$$\simeq \frac{1.2}{f_r} \frac{\gamma}{\sqrt{\gamma^2 -1}} \frac{\beta_{ep}^2 B_A^4}{(T_{ep}/511)^2}$$

where:

β_{ep} = core electron plasma beta near the annulus

T_{ep} = temperature near the annulus

It is evident that these equations confirm the experimental finding that drag losses dominate a large range of ring energies; but, as the temperature of the annulus increases, drag losses decrease, falling off as $T_A^{-1/2}$. At energies around 2 Mev, ring cooling is accelerated by synchrotron radiation. Globally, the total loss rate has a broad minimum bounded by the extremes of these two loss mechanisms. Further, from these two equations, the important parameters for loss rate seem to be: ring beta, the ratio of hot to cold plasma densities required for stability, the magnetic field strength near the annulus, the fraction of microwave propagation, and the ring volume. It follows that determination of microwave heating requirements depends on a knowledge of all these parameters.

Experiments with the E.B.T. series (E.B.T.-I and E.B.T.-S) have been conducted at Oak Ridge to determine these parameters[4]. The results are in reasonable agreement with theoretical predictions and there is some confidence that these figures, though done in a low parameter range, can

4. N.A. Uckan, EBT Ring Physics: Proceedings Of The Workshop December 3-5, 1979 Oak Ridge, Tennessee,p.507

be scaled upwards to reactor parameters. This is attempted in figure4.[5] Assuming this to be accurate, one can pick a hot to cold plasma density ratio that is in a stable region and a corresponding ring beta. Also the magnetic field strength near the annulus and a microwave cutoff frequency can be determined readily from machine parameters. All that is left to do is derive an equation for the annulus volume in order to calculate the microwave heating requirements and further, the Q-value for any hypothetical ring stabilized fusion device.

The annulus volume, based on experimental measurements and theoretical estimates, appears to have a thickness that ranges between a few and several relativistic gyrodiameters of the hot electrons[6]. Also as seen in these experiments, the annulus length appears to be of the same order as the plasma size (in E.B.T. type reactors), and seems limited to a length of not more than 10 centimeters. Theoretically, ring length is associated with the heating anisotropy ($T_{\parallel}/T_{\perp}$) because the size of the annulus depends on its stability which in turn depends on its drift

5. Uckan, EBT Ring Physics: Proceedings of the Workshop, Dec. 3-5,1979 p.507

6. N.A. Uckan, EBT Ring Physics: Proceedings Of The Workshop December 3-5, 1979 Oak Ridge, Tennessee, p.507

surface characteristics. From these considerations, two formulae and scaling laws for the annulus volume have been derived[7].

1. The ring thickness $\delta_a \simeq$ a few times the ring gyroradius (ρ_{ea}) and the ring length $l_a n \leqslant$ the plasma radius a.

2. The relativistic and dimensionless ring and core parameters remain constant from device to device (ie. $\delta a/a$, l_a/a)

These two scaling laws correspond to the following two separate calculations for the annulus volume:

CASE 1:

Hot electrons have a gyroradius:

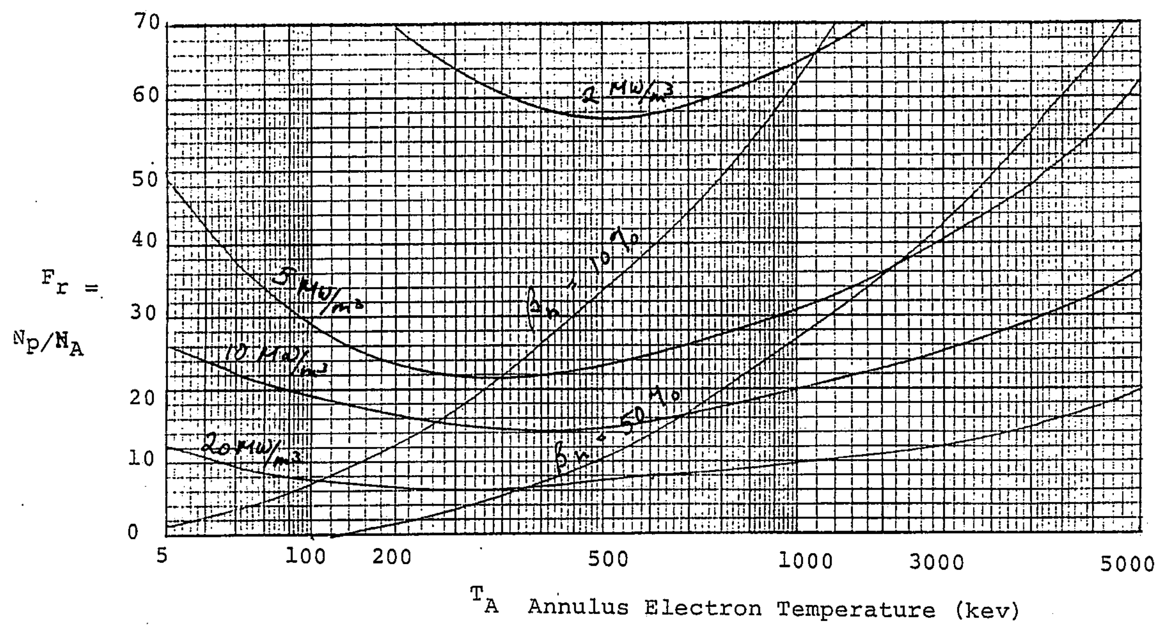
$$\rho_{eA}(m) = 1.7 \times 10^{-3} \frac{\sqrt{\gamma^2 - 1}}{B_A}$$

This implies that the annulus volume is calculated from the formula:

$$V_A = 1.07 \times 10^{-2} \frac{\sqrt{\gamma^2 - 1}}{B_A} f_E f_L a^2 N$$

7. Uckan, EBT Ring Physics: Proceedings Of The Workshop, p.507

FIGURE # 4



$P_{\mu A}/V_A$ = 2, 5, 10 and 20 MW/m³

where:

f_E = annulus thickness enhancement factor ($\delta_A = f_E \rho_{eA}$)

f_L = annulus length enhancement factor ($l_A = f_L a$)

Experimentally, $f_E \simeq 4$-6 ; $f_L \simeq .5$-1.0.

CASE 2:

To make the parameters dimensionless, the ratio V_A/V_p must be calculated. With this in mind, the formula for the annulus volume can be written as:

$$\frac{V_A}{V_p} = \frac{2\pi a \delta_A l_A N}{\pi a^2 2\pi R} = \frac{\delta_A}{a} \frac{l_A}{a} \frac{a}{R} \frac{N}{\pi}$$

Finally, Q_E can be calculated from plasma and machine parameters. Assuming in the energy range of interest, drag losses dominate (ie. $T_A \simeq$ 300-1000 kev) and $\eta_{th} \simeq .35$, $\eta_\mu \simeq .5$, one can write:

$$Q_E = \eta_{th}\eta_\mu \frac{P_{th}}{P_{\mu A}}$$

$$\simeq 3 \times 10^{18} \frac{V_p}{V_A} \frac{n_i^2}{n_A n_p} \langle\sigma v\rangle_{DT} \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

Using the following definitions and approximations:

$$n_i \simeq \left(\frac{B}{B_A}\right)^2 n_p = f_r \left(\frac{B}{B_A}\right)^2 n_A$$

$$\beta \simeq \frac{n_i(T_e + T_i)}{B^2} \simeq 2 f_c T \quad [\text{if} \quad T_e \sim T_i \sim T\,]$$

$$\beta_{ep} \simeq \frac{n_{ep} T_{ep}}{B_A^2} \simeq f_c T_{ep}$$

$$Q_E = 7.5 \times 10^{17} \frac{V_p}{V_A} f_r \left(\frac{\beta}{\beta_{ep}}\right)^2 \left(\frac{T_{ep}}{T}\right)^2 \left(\frac{B}{B_A}\right)^4 \langle\sigma v\rangle_{DT} \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

$$\simeq 3 \times 10^{18} \frac{V_p}{V_A} f_r \left(\frac{B}{B_A}\right)^4 \langle\sigma v\rangle_{DT} \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

$$\simeq 4.8 \times 10^{19} \frac{V_p}{V_A} \langle\sigma v\rangle_{DT} \frac{\sqrt{\gamma^2 - 1}}{\gamma}$$

For core plasma temperatures $T_e \sim T_i \sim$ 15-25 Kev

$\langle\sigma v\rangle_{DT} \sim (3\text{-}6) \times 10^{-22}$

and for a range of $T_A \sim$ 700 - 1500 Kev

$$\frac{\sqrt{\gamma^2 - 1}}{\gamma} \sim .9\text{-}.97 \qquad (0(1))$$

$$Q_E = (1.4\text{-}2.8) \times 10^{-2} f_r \frac{V_p}{V_A}$$

One can use figure 4 to find f_r. It is based on a magnetic field of 2 Tesla and carries with it the stability profile of rigid ring theory (a tenuous assumption). If one uses this threshold value of f_r on which the plasma is marginally stable then $f_r \simeq 20$. Finally one can use the formula derived previously for $\frac{V_p}{V_A}$. In the first case:

$$\frac{V_p}{V_A} \simeq 1.85 \times 10^3 \, \frac{RB_A}{N} \, \frac{1}{f_E f_L} \, \frac{1}{\sqrt{\gamma^2 - 1}}$$

If, for example the machine parameters are:

$T_A \simeq 700 - 1500$ Kev; $f_E \geqslant 10$, $f_L \simeq 1.0$, $R = 30$ meters,

$N = 24$, $B_A = 2$ T

Then:

$$Q_E > 30$$

In the second case, $V_p/V_A > 40$ which means that $Q_E > 10$. Both of these cases predict the feasibility of ring stabilized fusion reactors for sufficiently large parameters of temperature and major radius.

For use in this paper, this calculation is general enough to apply to any reactor that utilizes hot electron rings as a method of stabilizing the core plasma. Indeed, research done by the author into a new type of reactor which uses semi-toroidally connected tandem mirrors will be proposed as a future extension of E.B.T.. There the assumptions are somewhat different from the ones used here (specifically, the parameter f_r and β_c are chosen from the stable operating regions of interacting ring theory [the Lee-Vam Dam limit] and the annulus exists only in a small part of the plasma). However, this analysis is general enough to include these features and applies quite well to ring stabilized plasmas in general.

In all fairness, it would be improper to neglect entirely the more basic ingredients of ring power balance. Briefly, the equation for $P_{\mu a}$ is derived from first principles as follows: the energy lost per unit distance by a fast electron moving through matter (including plasma) is:[8]

$$\frac{\partial E}{\partial x} = \frac{4\pi e^4 n_e}{mv^2}\left[\ln\Lambda - \frac{v^2}{c^2}\right]$$

where Λ is the ratio of maximum energy loss per collision to the minimum energy loss

The energy lost per unit time is:[9]

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8. W.B. Ard and R.J. Kashuba EBT Ring Physics: Proceedings Of The Workshop, Dec. 3-5, 1979, p.333

$$\frac{\partial E}{\partial t}\Big)_{drag} = \frac{4\pi e^4 n_e}{mv}\left[\ln\Lambda - \frac{v^2}{c^2}\right]$$

The other terms in the power balance equation come similarly from basic equations in physics:[10]

$$\frac{\partial E}{\partial t}\Big)_{scatt} = \frac{\nu_{90}}{2}(\gamma-1)mc^2 = \frac{16e^4 n_i}{\pi mc}\frac{\gamma(\gamma-1)}{(\gamma^2-1)^{3/2}}\ln\Lambda$$

where:

$$\nu_{90} = \frac{32e^4 n_i}{\pi p^2 v}\ln\Lambda$$

Additionally:[11]

$$\frac{\partial E}{\partial t}\Big)_{brehms} = vn_i\gamma mc^2\phi_{rad}$$

9. Ard, Kashuba EBT Ring Physics: Proceedings Of The Workshop, p.333

10. J.D. Jackson, Classical Electrodynamics, John Wiley and Sons, Inc., New York et. al., 1975

11. Jackson, Classical Electrodynamics, pp.701,738

where ϕ_{rad} varies in different velocity limits:

$$\phi_{rad} = \frac{16}{3}\bar{\phi}$$

in the non-relativistic limit

$$\phi_{rad} = 4(\ln 2\gamma - \frac{1}{3})\bar{\phi}$$

in the extreme relativistic limit

and:

$$\bar{\phi} = \frac{r_0^2}{137} = \frac{e^4}{137m^2c^4}$$

Finally:[12]

$$\frac{\partial E}{\partial t})_{synch} = \frac{2}{3}\frac{e^2\gamma^2\omega^2p^2}{m^2c^3}$$

The actual figures used in this section can easily be shown to arise from these equations.

One last point is that many different methods of the calculation of ring power balance have been used, the most basic of which start directly with the Fokker-Planck equation. Fortunately they all produce close to the same result. Consequently, coming from basic physics and varying methods of calculation, the problem of ring power balance is fairly well understood and the results of these calculations can be viewed with some measure of confidence.

12. Jackson, Classical Electrodynamics, pp.672,679

For the purposes of this paper, the previous calculations must be applied in the Lee-Van Dam limit of interacting ring theory; namely, that the beta of the core plasma can be no larger than .1. To accomplish this, assume that the temperature range of the plasma is such that drag losses and synchrotron radiation losses dominate the microwave heating balance equation. Thus it is necessary to resurrect a few equations from the previous calculation. For easy reference, they are listed below:

$$Q_E = \eta_{th}\eta_{\mu} \frac{P_{th}}{P_{\mu A}} ; \qquad \frac{P_{\mu A}}{V_A} = \frac{P_{synch}}{V_A} + \frac{P_{drag}}{V_A}$$

$$\frac{P_{synch}}{V_A} = 3.2 \times 10^{-20} \; B_A^2 n_A(\gamma - 1)[1 + 0.5(\gamma - 1)]$$

$$\frac{P_{drag}}{V_A} = 2.5 \times 10^{-39} \; n_A n_{ep} \frac{\gamma \ln\Lambda}{\sqrt{\gamma^2 - 1}}$$

$$\frac{P_{th}}{V_p} = 1.33 \times 10^{24} \; \langle\sigma v\rangle_{DT} \frac{\beta_c^2 B^4}{T^2}$$

Direct substitution yields:

$$Q_E = \eta_{th}\eta_{\mu}\frac{V_p}{V_A}\,\frac{1.33\times 10^{24}\ \langle\sigma v\rangle_{DT}\,\frac{\beta_c^2 B^4}{T^2}}{3.2\times 10^{-20} B_A^2 n_A(\gamma-1)\left[1 + 0.5(\gamma-1) + 2.5\times 10^{-39}\, n_A n_{ep}\frac{\gamma \ln\Lambda}{\sqrt{\gamma^2-1}}\right.}$$

This can be simplified using the previous assumptions that:

$$n_A = \frac{1}{f_r}\left(\frac{B_A}{B}\right)^2 n_i \qquad n_i = 1.56\times 10^{42}\frac{\beta_c^2 B^4}{T^2}$$

$\ln\Lambda \sim 20$

$\eta_{th} = .5$

$\eta_{\mu} = .35$

to yield:

$$Q_E = 3\times 10^{18}\,\frac{V_p}{V_A}\,f_r\left(\frac{B}{B_A}\right)^4 \langle\sigma v\rangle_{DT} k^2\, F(\gamma,\beta_c)$$

$$\sim 4\times 10^{19}\,\frac{V_p}{V_A}\,f_r\,\langle\sigma v\rangle_{DT} k^2\, F(\gamma,\beta_c)$$

where:

$$F(\gamma,\beta_c) = [5.13 \times 10^{-4} (\frac{B_A}{B})^2 (\gamma^2 - 1)\frac{T_c}{\beta_c} + \frac{\gamma}{\sqrt{\gamma^2 -1}}]^{-1}$$

$$\simeq [1.3 \times 10^{-4} (\gamma^2 - 1)\frac{T_c}{\beta_c} + \frac{\gamma}{\sqrt{\gamma^2 - 1}}]^{-1}$$

and the previous assumptions have been used that:

$$\beta_c \simeq \frac{n_c T_c}{B_A^2}$$

$$\beta \simeq \frac{\beta_c k^2}{2}$$

The only difference is that a density profile $k = \frac{\bar{n}}{n_c}$ has been added to account for the very real radial fluctuations such that the density is greater at the center of the plasma than it is at the vicinity of the rings.

From here, for the purposes of greater accuracy, the analysis will diverge from the previous case. To begin, the ratio $\frac{V_p}{V_A}$ can be obtained in the following manner:

The radius of curvature $R_c \simeq 2a$ and the length of a bump is about 4a for a mirror ratio of ~2. Hence:

$$\frac{V_p}{V_A} \sim \frac{R_c}{2\Delta} \sim \frac{.5}{\varepsilon} \qquad \text{where } \varepsilon \equiv \frac{\Delta}{R_c}$$

Actually, researchers have been as optimistic as:[13]

$$\frac{V_p}{V_A} \simeq \frac{1.3}{\varepsilon}$$

So there is an uncertainty as to the exact value of this volume ratio. (See figure 6)

Secondly, $f_r \sim \frac{n_c}{n_A} \sim \frac{T_A}{T_c}$ if $\beta_h \sim .2$ $\beta_c \sim .1$.

If $T_A \sim 1000$ kev $T_c \sim 10$ kev

Then $f_r = 0(100)$

(See figure 5)

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13. N.A. Uckan, D.A. Spong, and D.B. Nelson, Beta Limits In EBT And Their Implications For A Reactor, Trieste, Italy, June2-9,1981

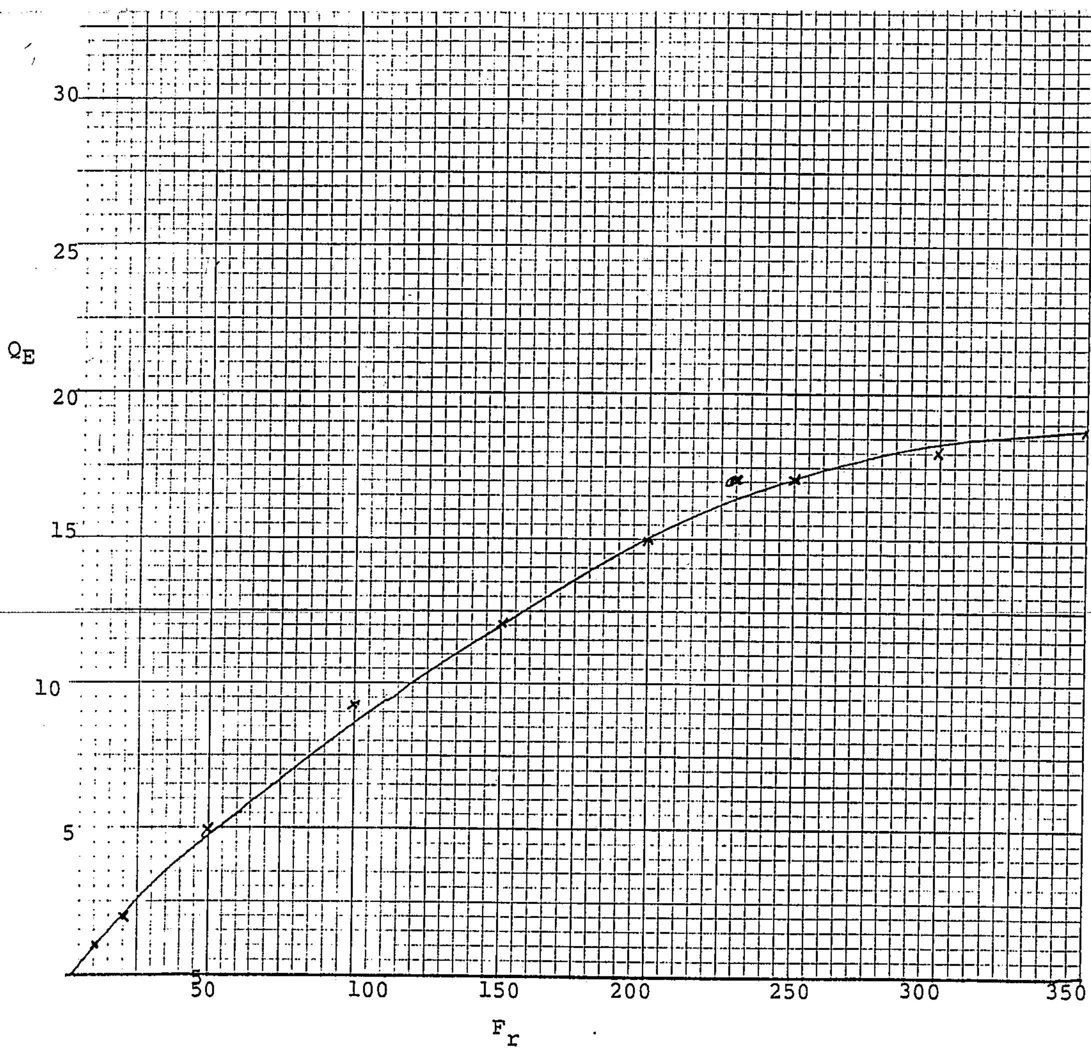
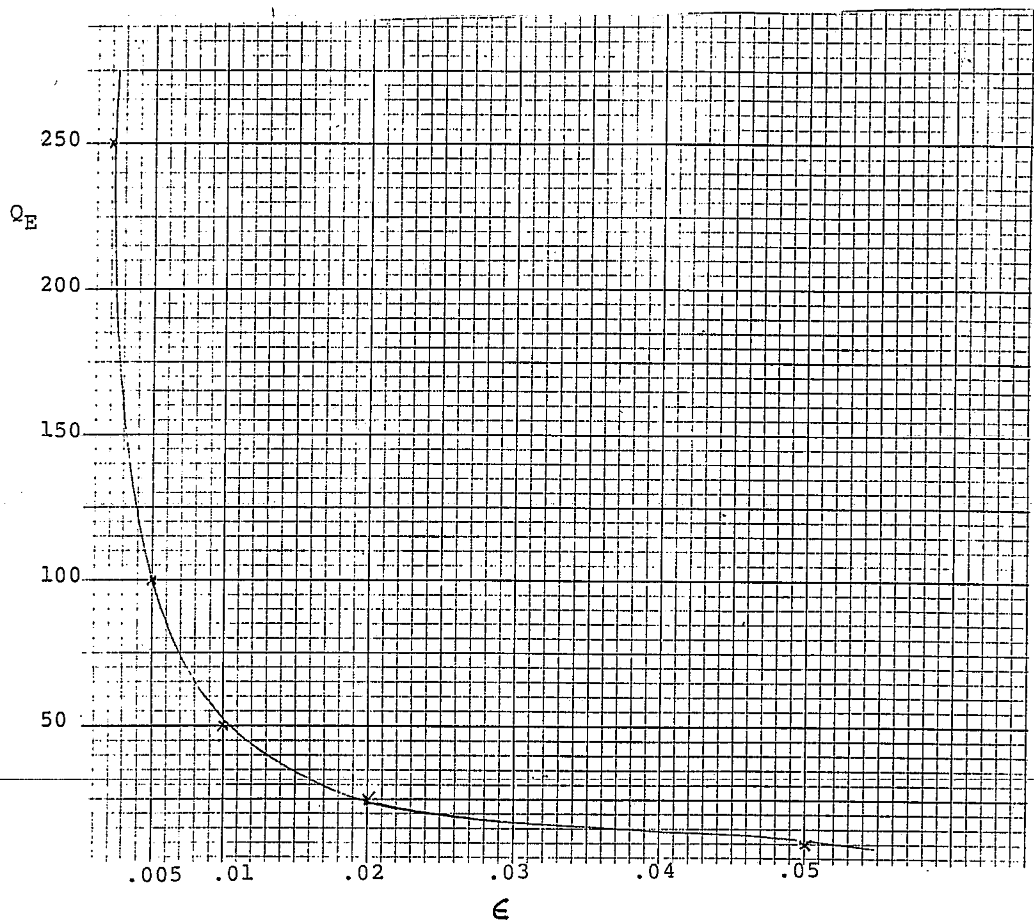


FIGURE #5

Q_E vs. F_r with k=1.5, $\epsilon = 5 \times 10^{-2}$

FIGURE # 6



$-Q_E$ vs. ϵ with $k=1.5$, $F_r=100$

Further, at 10Kev, $\langle\sigma v\rangle_{DT} = 1.1 \times 10^{-22}$

Thus

$$Q_E = \frac{(1.5\text{-}4.4) \times 10^{-1}}{\varepsilon} k^2 F(\gamma, \beta_c)$$

Additionally, for ring energies of $T_A \sim (700\text{-}1500)$ Kev

$$\frac{\gamma}{\sqrt{\gamma^2 - 1}} \sim .9 - .97 \sim 0(1)$$

so $\quad G(\gamma, \beta_c) = \left[0(1) + (3\text{-}4) \times 10^{-2}\right]^{-1} = 0(1) \qquad$ for $\dfrac{T_c}{\beta_c} \sim 100$

Also $\quad k = \dfrac{\bar{n}}{n_c}$

Using the relation:

$\dfrac{\partial p}{\partial r} = \dfrac{-p}{R_c} 2\gamma$ where p is the pressure; R_c the radius of curvature;

$p \propto$ Gaussian

so that the density profile also ressembles a Gaussian especially as $t \to \infty$ as it would in a microwave heating loss calculation. Now:

$$\text{Power} \propto \int_0^{r_{ring}} \bar{n}^2 \, r dr$$

For the density profile chose the Gaussian:

$$\bar{n} = n_0 \frac{\exp[-\frac{r^2}{r_0^2}]}{\exp[-\frac{r_1^2}{r_0^2}]}$$

Thus

$$\text{Power} \propto [(\exp-[\frac{2r_{ring}^2}{r_0^2}]/4 - 1)n_0^2]\pi r_0^2$$

It is a safe assumption that k ranges between 1-2 with k = 0(1.5) being a reasonable value. (see figure7 and table) Thus:

r_{edge}	k
r_{ring}	1.033
$2r_{ring}$	1.37
$3r_{ring}$	1.51
$4r_{ring}$	1.61
$5r_{ring}$	1.67
$10r_{ring}$	1.81

$$Q_E \simeq \frac{.4 - 1}{\varepsilon}$$

From present experiments, ε has been scaled to a reactor configuration as:[14]

$$\varepsilon \simeq 10^{-2} - 5 \times 10^{-2}$$

So

$$Q_E \sim 10 - 100$$

But using $\beta = \alpha\varepsilon$ with α depending on the distribution function of the ring electrons, in the Lee-Van Dam limit $\varepsilon \simeq 5 \times 10^{-2}$. In this limit

$$Q_E \sim 8 - 20$$

With an order of magnitude enhancement factor of the E-ring tandem mirror configuration coming from the fact that V_A is reduced with respect to V_p due to the fact that the annulus is confined to a small portion of the plasma where the tandem mirror sections meet (in this configuration of tandem mirrors joined to form a polygon, a tandem mirror is analogous to a bump in the E.B.T. concept), Q_E is obviously satisfactory for a reactor.

14. N.A. Uckan, D.A. Spong and D.B. Nelson, Beta Limits In EBT And Their Implications For A Reactor, Trieste, Italy, June 2-9, 1981

However, there is another criterion for determining the economic feasibility of a fusion reactor, a phenomenon known as wall-loading. According to this idea, in order to have an economically viable energy source, the power output per square meter on the surface of the reactor must be between 1 - 3 megawatts/m^2. This necessity of minimum power output can be translated into a new criterion for feasibility studies via the following mathematical analysis:

One starts with the equation for fusion power per volume:

$$\frac{P_{th}}{V_p} = 1.33 \times 10^{24} \frac{\beta_c^2 B^4 \langle \sigma v \rangle_{DT}}{T^2}$$

Now $V_p = S_p \frac{a}{2}$

where:

S_p = surface area of the plasma

a = average minor radius

Thus:

$$\frac{P_{th}}{S_p} = \text{Load} = 1.33 \times 10^{24} \frac{\beta_c^2 B^4 \langle \sigma v \rangle_{DT}}{T^2} \frac{a}{2}$$

or, solving for a:

$$a = \frac{2(\text{Load})T^2}{1.33 \times 10^{24} \langle\sigma v\rangle_{DT}\beta_c^2 B^4}$$

To use this result, it must be realized that a reactor must produce between (1-10) × 10^3 MW to be of much impact as a power source since a reactor which produces more than 10^4 MW would be too large for most energy production needs. Assuming this, one can write down the formula for total power using (1-10) × 10^3 MW as a second boundary condition on economic feasibility. Thus:

$$P_T = (2\pi R)\pi a^2 \times \frac{1.33 \times 10^{24} \langle\sigma v\rangle_{DT}\beta_c^2 B^4}{T^2}$$

Substituting for a, have:

$$P_T = \frac{8\pi^2 (\text{Load})^2 T^2 R}{1.33 \times 10^{24} \langle\sigma v\rangle_{DT}\beta_c^2 B^4}$$

$$= \frac{32\pi (\text{Load})^3 T^4 N}{(1.33 \times 10^{24})^2 \langle\sigma v\rangle_{DT}\beta_c^2 B^8}$$

What this equation is saying is that a problem arises in that too much total power is produced. Using the equation:

$$f = \frac{1.8 \times 10^7 B}{2\pi}$$

the proposed experiment E.B.T.R. will only produce about a 2 Tesla field[15]. Using a wall loading of 1 Mw/m^2 implies that in the Lee-Van Dam limit:

$$P_T \sim O(10^5 \text{ mw}) \quad (N = 24)$$

clearly well beyond what must be obtained in a reactor. The key parameter is the magnetic field. Even a 2.5 Tesla field will yield a marginally acceptable value for the total power.

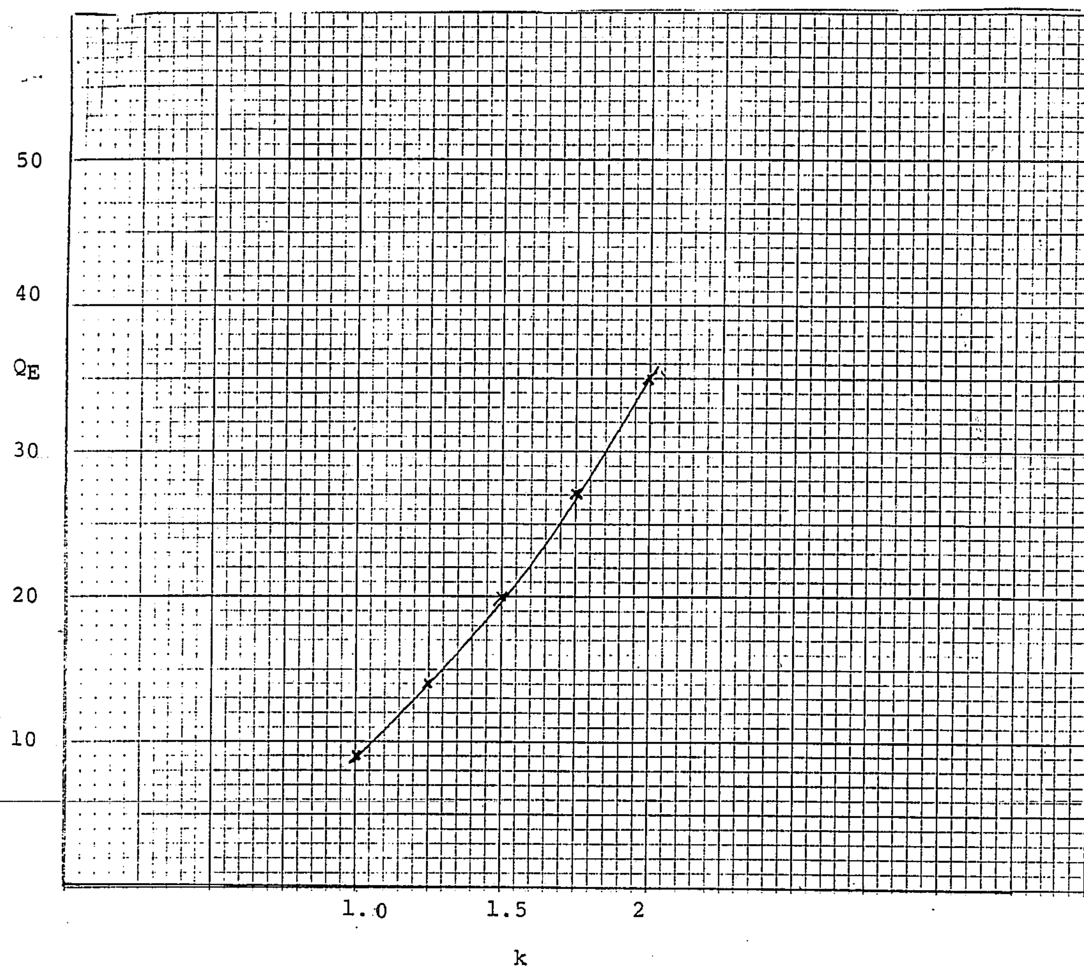
In addition, the minor radius is also a problem. For a 2-Tesla field, in the Lee-Van Dam limit, the equations demand a minor radius in excess of 7 meters for a $1MW/m^2$ power output at the walls. However a

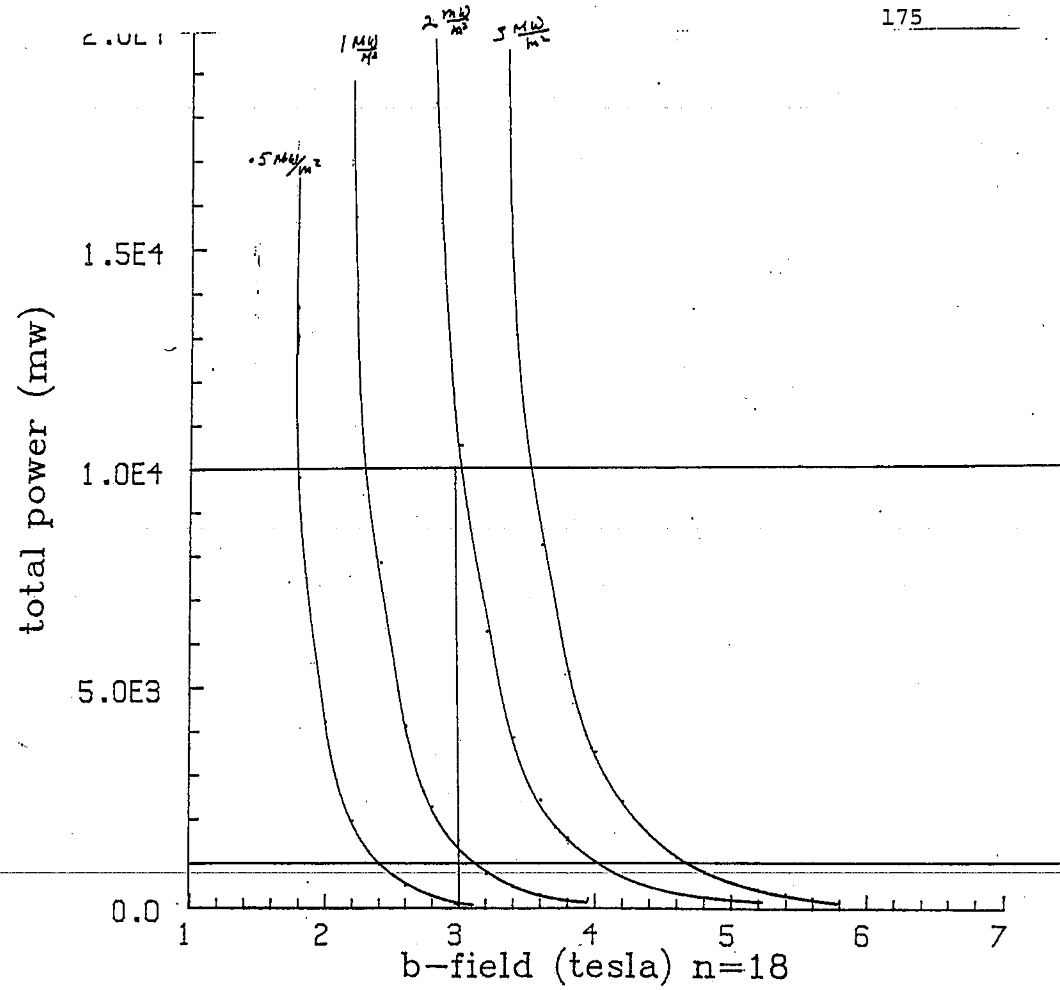
15. EBT "Checkpoint" Review LA-8882-MS, Elmo Bumpy Torus Reactor and Power Plant- Concept and Design Study, DOE, Germantown, Aug. 4-7, 1981

magnetic field of 3 Tesla would reduce the size of the minor radius by a factor of 5.

For a detailed description of the effect of the wall loading and total power requirements see figures 8-10. What is clear is that a small window of operation exists at the extreme limits of what present day magnetic field technology can produce. This window, though small, does provide a representation of where reactor technology exists today. The Lee-Van Dam limit on β is much more demanding than the older rigid ring stability requirement. It thus pushes at the limits of present day technology. Yet for a given β, a suitable magnetic field can always be found. The future of fusion reactors rests with this technology and given an infinite capacity for creating magnetic fields, any desired reactor capacity can be obtained. Clearly, then, a fusion reactor can be built in the future and modeled as a ring stabilized plasma device.

FIGURE #7





total power (mw)
2.0E4
1.5E4
1.0E4
5.0E3
0.0
1
2
3
4
5
6
7
b−field (tesla) n=18
.5 MW/m²
1 MW/m²
2 MW/m²
5 MW/m²

FIGURE #8

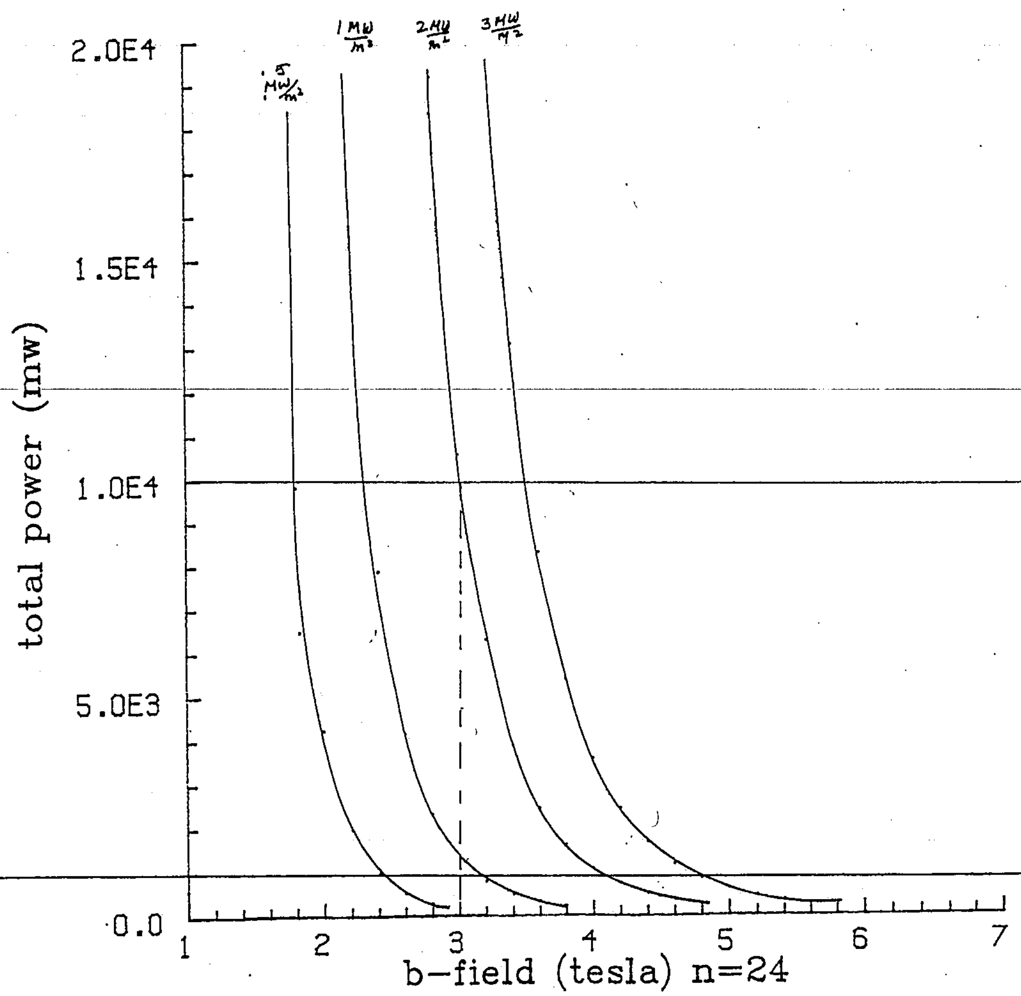
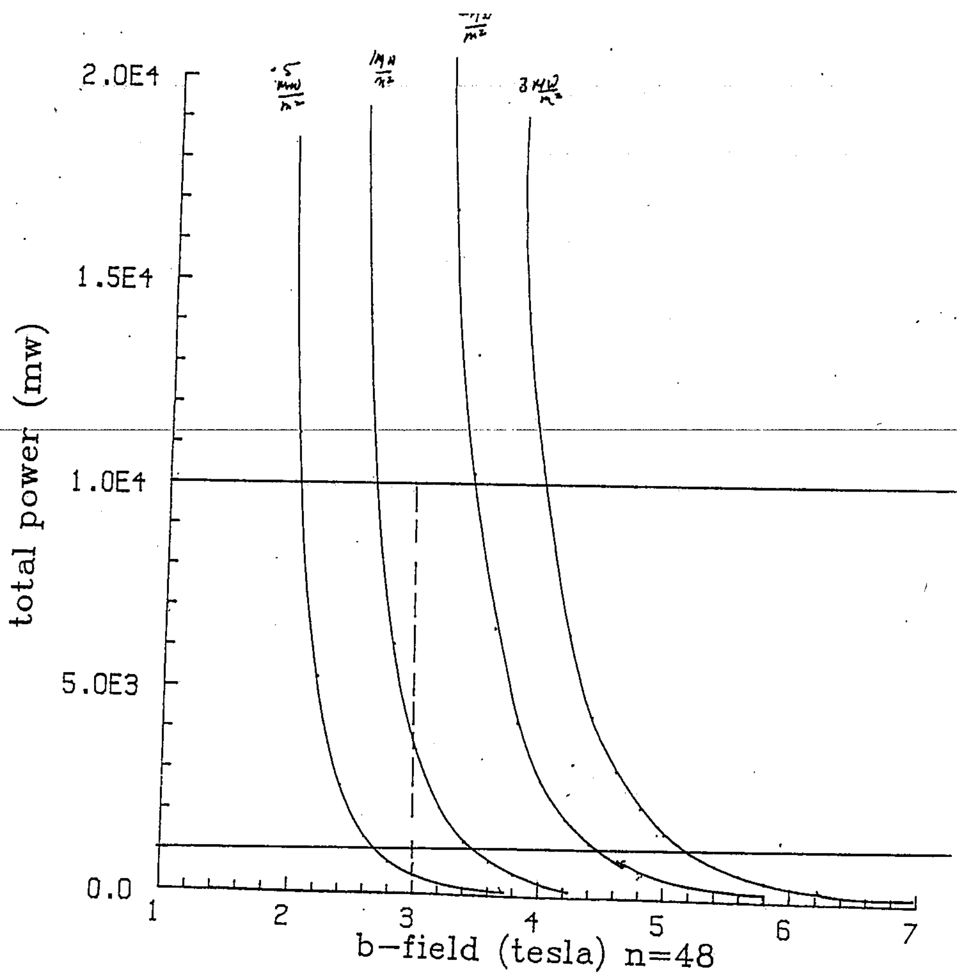
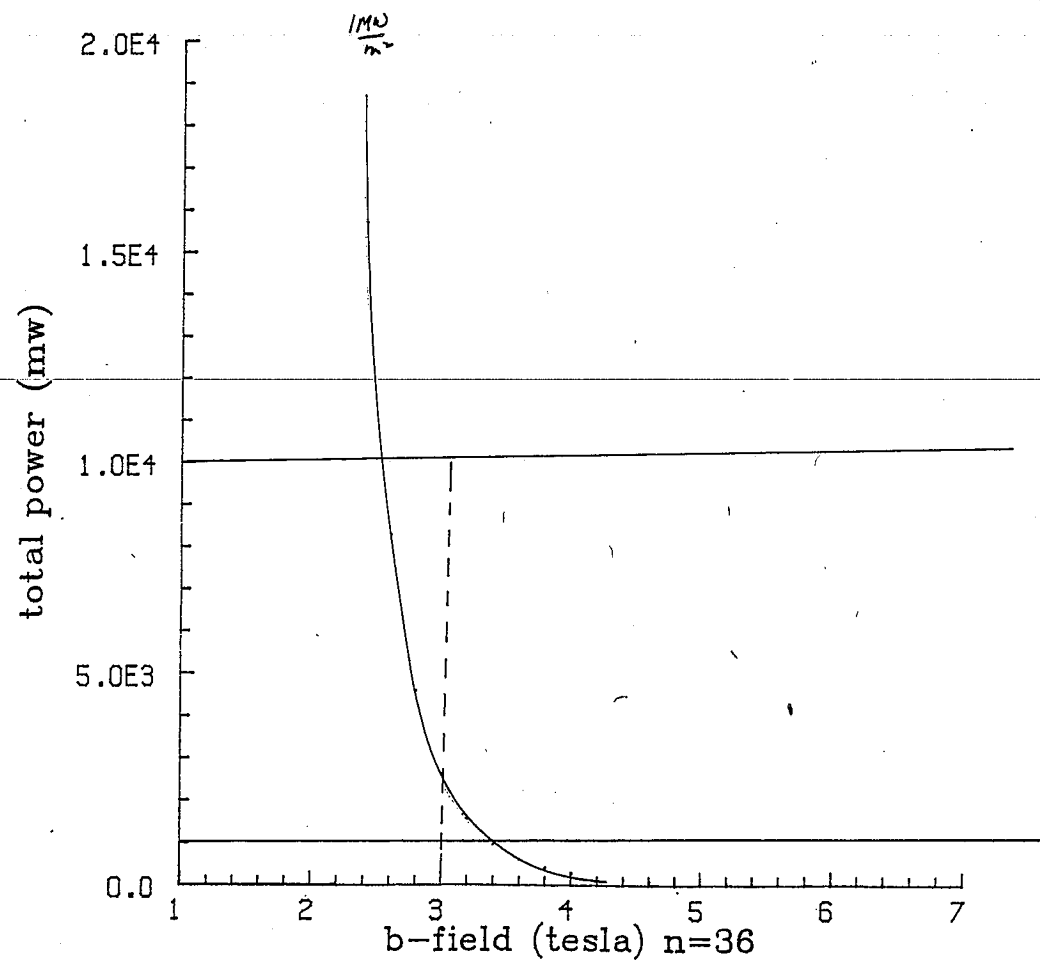

.5 MW/m²
1 MW/m²
2 MW/m²
3 MW/m²
2.0E4
1.5E4
1.0E4
5.0E3
0.0
total power (mw)
1
2
3
4
5
6
7
b-field (tesla) n=24

FIGURE #9



2.0E4
1.5E4
1.0E4
5.0E3
0.0
total power (mw)
1
2
3
4
5
6
7
b-field (tesla) n=48

FIGURE #10


1MW/m²
2.0E4
1.5E4
1.0E4
5.0E3
0.0
total power (mw)
1
2
3
4
5
6
7
b−field (tesla) n=36

CONCLUSION

From the preceding analysis, it is clear that the interacting ring model of stability in ring stabilized plasmas has replaced rigid ring theory in describing where the plasma will be stable with respect to the betas of core and ring plasma. Hence one is forced to use its more pessimistic predictions in deciding upon the feasibility of a future reactor. Now the problems with the Q-value feasibility criterion that could have existed due to the necessity of operating at a lower core beta seem to be solved by adopting a tandem mirror modification to the EBT concept. The long straight sections provide an enhancement factor of close to an order of magnitude in the volume ratio $(\frac{V_P}{V_A})$ which more than adequately compensates for the lower core beta and enables one to predict that a $Q_E > 15$ is probable.

The problem arises most severely in the wall-loading criterion for feasibility. Here no volume ratio appears in the calculations and the tandem mirror configuration has no advantages over the present EBT design. What has been seen is that a small window of feasibility exists. Figure 1 will illustrate this point. Here it is assumed that a 1 MW/M^2 wall-loading is necessary for feasibility and that a 3-Tesla field is the maximum allowable given present technology. Clearly this figure indicates feasibility, albeit the margin of error is small.

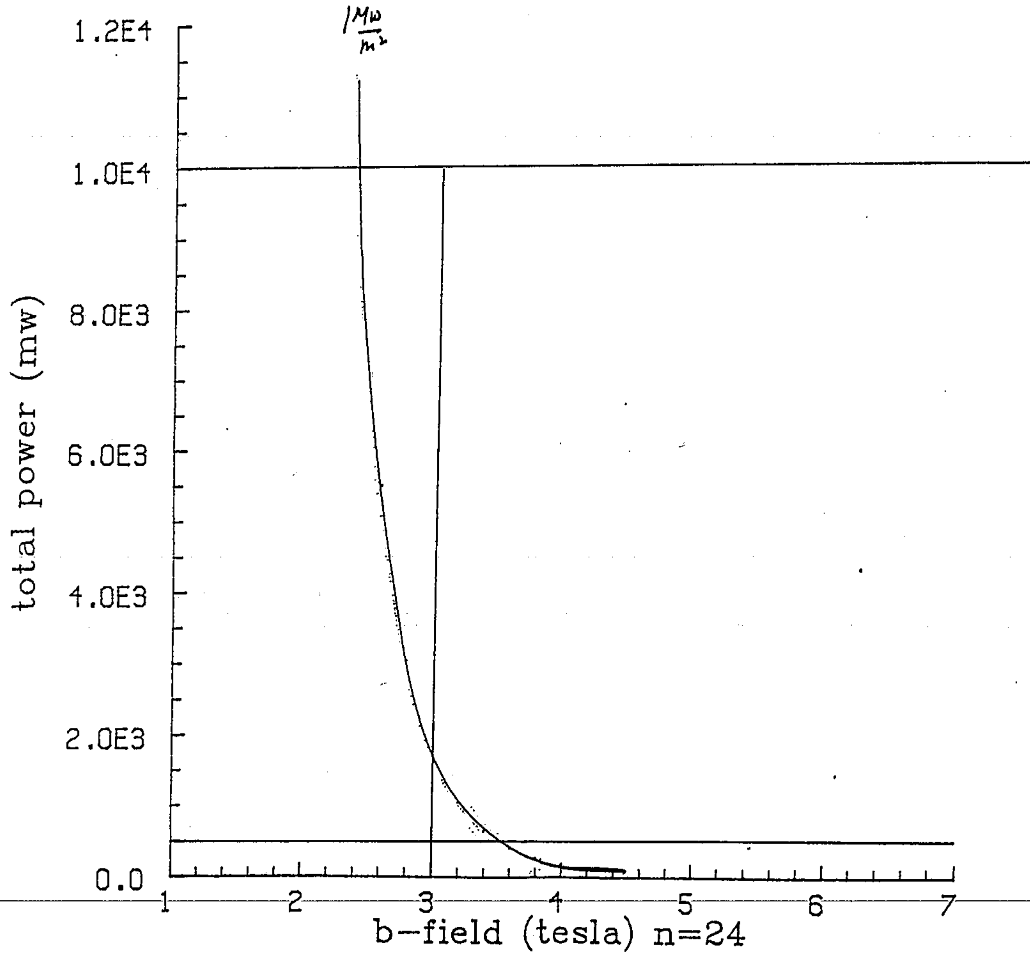

1MW/m²
1.2E4
1.0E4
8.0E3
6.0E3
4.0E3
2.0E3
0.0
total power (mw)
1
2
3
4
5
6
7
b-field (tesla) n=24

FIGURE #1

One final point concerns the geometry of the configuration. At 3-Tesla, the machine would have a 10 meter major radius and a minor radius of over a meter (See figure 2). Anything less than 3-Tesla would require a geometry that precludes the small aspect ratio calculations of the past. The minor radius would increase and the major radius would decrease proportional to B^4. Given these geometrical considerations it would seem rather tenuous to go with a field of under 3-Tesla as the margin of error would nearly disappear.

The success of any solution to a problem rests with the accuracy of its underlying assumptions. For this reason, interacting ring theory provides further insight into fusion research. Given the smallness of the window of feasibility, the success of the program of magnetic confinement rests squarely on the shoulders of interacting ring theory predictions. These predictions must be accurate to within 20% for the reactor to succeed. If the core beta is required to dip below .08 for stability, the wall-loading criterion will doom the reactor to failure or force it to rely on a larger than 3-Tesla field. The fusion program is confronted with these facts on all fronts and this forces a conclusion that fits well within contemporary fusion research. There is a chance that a reactor program will succeed, but it is close.

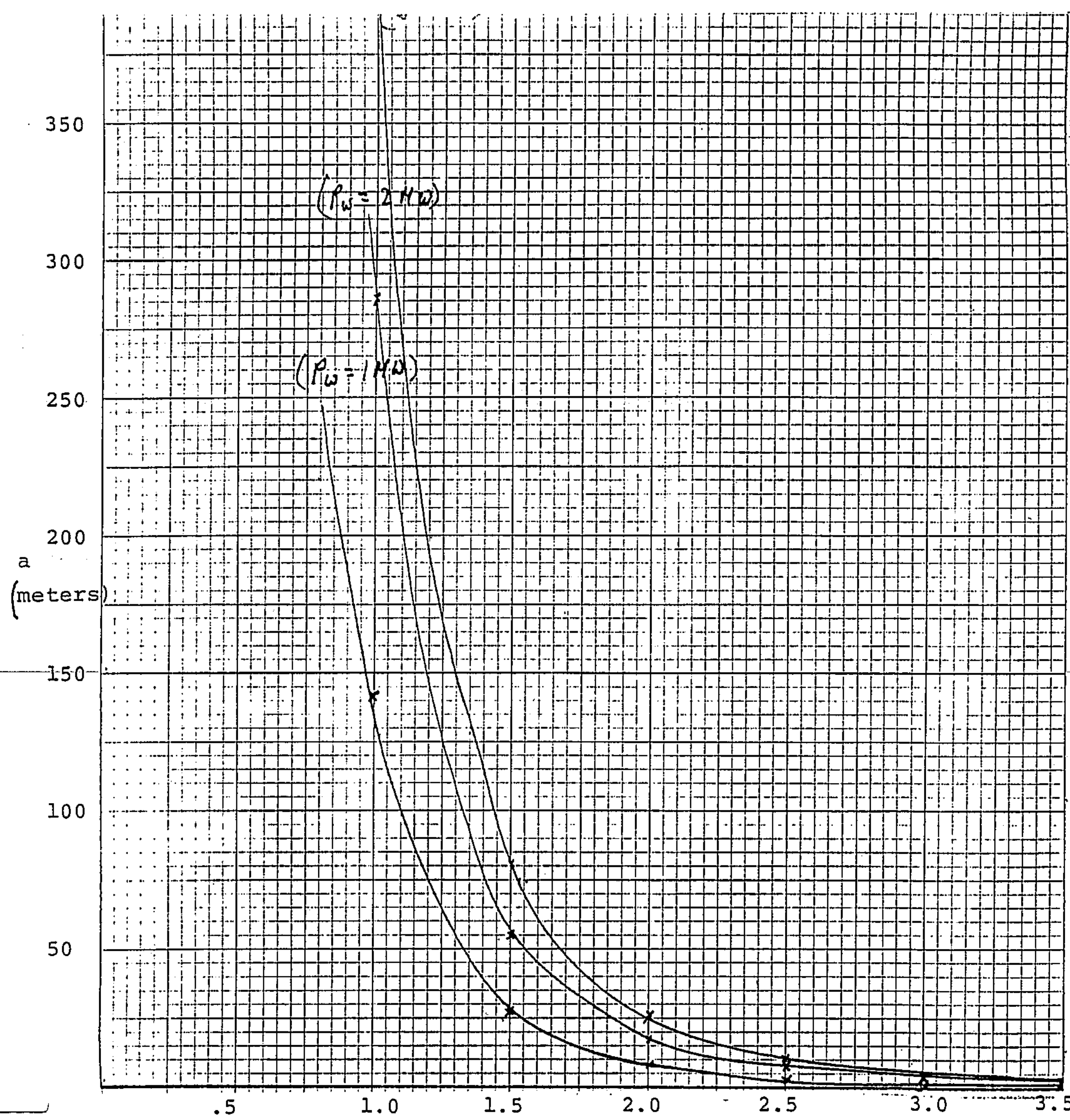


FIGURE #2

Minor radius as a function of magnetic field with power at the walls as a parameter.
a in terms of meters; B in terms of Tesla
(with β_c =.1 held fixed)

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