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Inversion of the Ballooning Transformation

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Abstract

The ballooning formalism can be viewed, not just as an eikonal representation, but as an integral transform, analogous to the Fourier or Laplace transforms, with a uniquely defined inverse. Here, the inversion theorem is proved, and an error in the previous literature is corrected.

A. Introduction

A field perturbation in an axisymmetric, toroidal confinement system is conveniently expressed in terms of its toroidal harmonics:

$$\Phi(q, \theta, \zeta) = \sum_n \exp(in\zeta)\varphi_n(q, \theta). \tag{1}$$

Here q is the safety factor, which is assumed to provide a suitable radial coordinate; the poloidal and toroidal angles are denoted as usual by θ and ζ. [Sums over all integers are indicated by omitting summation endpoints.] The *ballooning representation* is an assumed expression for φ_n,

$$\varphi_n = \sum_k \exp\left[-inq\left(\theta - \theta_0 + 2\pi k\right)\right] f_n\left(nq, \theta - \theta_0 + 2\pi k\right), \tag{2}$$

in terms of the ballooning mode amplitude, f_n, and the phase angle, θ_0. It allows the linear eigenmode problem for φ_n,

$$\mathcal{L}\varphi_n = 0,$$

to be analyzed by studying an associated linear problem for f_n,

$$\hat{\mathcal{L}} f_n = 0,$$

obtained by substitution.[1,2,3,4]

The operator $\hat{\mathcal{L}}$ is typically more tractable than $\mathcal{L}$, at least when n is large. The eigenvalue problem is especially simplified because the periodicity condition

$$\varphi_n(q, \theta + 2\pi m) = \varphi_n(q, \theta)$$

is enforced by (2) and not applied to f_n; indeed the θ-domain for f_n is the entire real axis. At the same time, (2) enforces the "flute-like ordering,"

$$\nabla_{||}\Phi \ll \nabla\Phi, \tag{3}$$

where $\nabla_{\|}$ is the gradient operator along the direction of the confining magnetic field:

$$\nabla_{\|} \propto \mathbf{B} \cdot \nabla \propto \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta}.$$

This ordering generally characterizes the most dangerous plasma instabilities, including the ballooning modes from which the representation earns its name.

The ballooning representation can be understood in the context of WKB theory, the amplitude f appearing as the coefficient of a conventional eikonal. (Infinite degeneracy of the lowest order WKB-problem allows formation of the periodic sum.) This point of view, adopted in some early studies,[3] was emphasized and developed by Dewar and Glasser,[5] who show that $\mathcal{L}$ and $\hat{\mathcal{L}}$ have the same spectrum. Characteristic of the eikonal method is that, while f evidently determines φ, the series cannot be inverted: a prescription for finding a unique ballooning mode amplitude corresponding to a given physical disturbance is not available, nor even deemed useful.

An alternative viewpoint also appears in the early literature[4]: one can view (2) as defining a *ballooning transformation*,

$$f_n \rightarrow \varphi_n,$$

in the same sense as the Fourier or Laplace transform. Just as the Fourier transformation, for example, is useful in the case of translation symmetry, so the ballooning transformation is appropriate when the flute-ordering, (3), is satisfied. The ballooning transformation in this second sense appears very naturally in analysis of the MHD energy principle.[6] Furthermore, it would seem potentially useful in *nonlinear* studies of flute-like disturbances: the study of dynamics, including, for example, correlations among different modes, involves more than spectral theory.

Of course, this point of view demands the existence of an inverse transformation,

$$\varphi_n \rightarrow f_n,$$

providing a unique f_n for any given φ_n. The inverse transform prescription provided in Ref. 4, however, was found to be incorrect,[5] although the scale-length separation requirements allowing a unique inverse were recognized.[1] Such scale-length orderings were later used to obtain the correct inverse transform,[7] but the uniqueness proof provided then has since been found[6] to be in error.

One purpose of the present note is to draw attention to the above distinction between ballooning representation and ballooning transformation, emphasizing the special usefulness of the latter. But our main purpose is to put the transformation on firm mathematical ground by providing a sound and explicit derivation of the inverse transform formula (Sec. B). In particular, the present derivation clarifies the constraint on f_n necessary for the inverse transform to exist. We finally explore issues of intrinsic mathematical interest, raised by the previous analysis (Sec. C).

B. Derivation of Inverse Transform

To simplify notation, we now suppress the subscript n and introduce the variables

$$x \equiv nq, \qquad y \equiv \theta - \theta_0.$$

Then (2) becomes

$$\varphi(x,y) \;=\; \sum_k \exp\left[-ix(y+2\pi k)\right] f(x, y+2\pi k), \tag{4}$$

while the inverse transform is[7]

$$f(x,y) \;=\; \int ds \frac{\sin \pi s}{\pi s} e^{iy(x+s)} \phi(x+s, y). \tag{5}$$

Here and below, integrals without endpoints are to be taken from $-\infty$ to $+\infty$.

The ballooning mode amplitude, f, is essentially a filtered version of φ. That is, if F is a displaced Fourier transform of φ,

$$F(t,y) \;\equiv\; \int dx \exp\left[ix(y+t)\right] \varphi(x,y), \tag{6}$$

then f is the inverse Fourier transform of a truncated version of F:

$$f(x,y) = \frac{1}{2\pi}\int dt \exp(-itx)w(t)F(t,y). \qquad (7)$$

Here w is a window function, defined by

$$\begin{aligned} w(t) &= 1, \quad -\pi < t < \pi, \\ &= 0, \quad |t| > \pi. \end{aligned}$$

The equivalence of (5) and (7) is easily deduced from the Fourier representation of the window function,

$$w(t) = \int ds\, e^{ist}\frac{\sin \pi s}{\pi s}. \qquad (8)$$

Convergence of the series (4) is guaranteed under rather mild conditions on the function φ. Suppose, for example, that φ satisfies

$$\int \left|\frac{\partial^2\phi(x,y)}{\partial x^2}\right| dx \leq M < \infty,$$

where M is independent of y. Then, after an integration by parts in (6), we obtain

$$|F(t,y)| \leq \frac{M}{(y+t)^2}$$

and therefore

$$|f(x,y)| \leq \frac{M}{\left(|y|-\pi\right)^2} = O\left(|y|^{-2}\right) \quad \text{as} \quad y \to \pm\infty.$$

The series (4) then converges uniformly in both variables.

The filtered nature of f corresponds to scale-length orderings that were recognized in the early ballooning literature, sometimes with emphasis.[8] It was the basis of the previous derivation of the inverse transform,[7] hereafter referred to as I, and its pivotal role was explicitly demonstrated in an analysis of the MHD energy principle,[6] hereafter referred to as II. The physical statement is that f must have nearly the same value on the nearest-neighbor rational flux surfaces corresponding to any given toroidal mode number, n; variation of φ from one rational surface to the next is accounted for explicitly, by the exponential factor. The mathematical point is that to specify f uniquely, given φ, we must restrict the bandwidth of f. Regarded as a function of x, for fixed y, it is required to have radian bandwidth no larger than π.

The relations (4) and (5) differ only in notation from those in I; an equivalent version is given in II. However the following proof is new.

To show that (4) and (5) are equivalent, we first assume (4) to hold. Then (5) follows if and only if the function

$$\begin{aligned} f'(x,y) &\equiv \int ds \frac{\sin \pi s}{\pi s} e^{iy(x+s)} \sum_k e^{-i(x+s)(y+2\pi k)} f(x+s, y+2\pi k) \\ &= \int ds \frac{\sin \pi s}{\pi s} \sum_k e^{-2\pi i k(x+s)} f(x+s, y+2\pi k) \qquad (9) \end{aligned}$$

coincides with f. At this point we apply the bandwidth requirement, expressing f in the form

$$f(x,y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt \exp(-itx) G(t,y),$$

or, in other words, assuming the Fourier transform G is identically zero for $|t| > \pi$. Then

$$f'(x,y) = \frac{1}{2\pi} \sum_k \int_{-\pi}^{\pi} dt\, G(t, y+2\pi k) e^{-ix(t+2\pi k)} \int ds \frac{\sin \pi s}{\pi s} e^{-is(t+2\pi k)}. \qquad (10)$$

The s-integral reproduces the window function, (8); then, since $|t| < \pi$, the $k = 0$ term is selected and we have

$$f'(x,y) = \frac{1}{2\pi} \int ds \exp(-isx) w(s) G(s,y) = f(x,y),$$

as desired. This replaces the incorrect part of the proof given in I.

To complete the proof we assume (5) and deduce (4). Thus, following I, we show that $\varphi' = \varphi$, where

$$\varphi'(x,y) \equiv \sum_k \int ds\, e^{is(y+2\pi k)} \varphi(x+s, y+2\pi k) \frac{\sin \pi s}{\pi s}.$$

Now invoking the periodicity property of φ, we rewrite $\varphi(x+s, y+2\pi k)$ in the integrand as $\varphi(x+s, y)$ and apply the sum rule

$$\sum_k \exp(2\pi iks) = \sum_n \delta(s-n).$$

This introduces a factor

$$\sum_n \frac{\sin \pi n}{\pi n} = \sum_n \delta_{n0} = 1$$

into the integrand, yielding finally

$$\varphi' = \varphi,$$

as claimed.

Thus (4) and (5) are indeed a transform pair; the bandwidth constraint gives the ballooning transformation a unique inverse. In other words, we have established a one-to-one mapping between the space of functions $\varphi(x, y)$, 2π-periodic in y, and the space of functions $f(x, y)$, bandlimited in x, of radian bandwidth $\leq \pi$.

C. Alternative Proof

The uniqueness of the inverse ballooning transformation is established in the previous section. The alternative proof given here is presented for intrinsic interest, and because otherwise the error of I is left mysterious. Indeed, it is not at all obvious that I *is* in error; moreover, once the error is appreciated it is rather surprising that a rigorous version of the argument can after all be constructed.

We recapitulate the (incorrect part of the) argument of I. It uses a version of (9) in which the finite bandwidth of f, expressed by Eq. (7), is replaced by a "smoothness requirement": that f should be entire, in the sense that its Taylor series in x has infinite radius of convergence. Suppose we substitute the Taylor series into (9) and integrate term by term.

Then (10) is replaced by

$$f' = \sum_k \sum_m \int ds \frac{\sin \pi s}{\pi s} \exp\left[-i2\pi k(x+s)\right] \frac{s^m}{m!} \frac{\partial^m f(x, y+2\pi k)}{\partial x^m}. \tag{11}$$

Here the obvious problem with convergence is addressed in a conventional way,

$$\exp(-i2\pi ks) \rightarrow \exp\left(-i2\pi ks - \varepsilon |s|\right).$$

Then the exponential decay of the integrand for large $|s|$ is supposed to justify the term-by-term integration, with the result

$$f'(x,y) = \sum_k \sum_m I_{mk} \exp[-i2\pi kx] \frac{1}{m!} \frac{\partial^m f(x, y+2\pi k)}{\partial x^m},$$

where

$$I_{mk} = \lim_{\varepsilon \to 0} \int ds \frac{\sin \pi s}{\pi s} \exp\left(-i2\pi ks - \varepsilon |s|\right) s^m.$$

From this form it is not hard to show that

$$I_{mk} = \delta_{m0} \delta_{k0} \tag{12}$$

so we have

$$f'(x,y) = f(x,y). \tag{13}$$

To see that this argument has to be wrong, we note that, since the left-hand side of (9), like $(\sin \pi x)/\pi x$, has bandwidth π, (13) would imply that every entire function f has bandwidth $\leq \pi$. Evidently the error lies in the improper interchange of integration and summation orders—a difficulty that can also occur in more elementary contexts.[9]

The argument leading to (13), while fallacious regarding its original intention, correctly verifies a useful fact. It shows that, if the sum over m can be interchanged with the integral, then f has bandwidth $\leq \pi$. We repair the proof of I, while preserving its essential method, by also showing the converse: that bandwidth $\leq \pi$ permits the interchange.

It is convenient to consider a slightly generalized problem. Let

$$\begin{aligned} f(t) &= \int ds\, \varphi(s) \frac{\sin \pi(s-t)}{\pi(s-t)} \\ &= \int ds \frac{\sin \pi s}{\pi s} \varphi(s+t) \end{aligned}$$

be the filtered version of some absolutely integrable, entire function φ; let

$$f_\varepsilon(t) = \int ds \frac{\sin \pi s}{\pi s} \exp\left(-\varepsilon |s|\right) \varphi(s+t); \tag{14}$$

and let

$$g_\varepsilon(t) = \sum_k \frac{1}{k!} \frac{d^k \varphi(t)}{dt^k} \int ds\, s^k \frac{\sin \pi s}{\pi s} e^{-\varepsilon |s|}. \tag{15}$$

It is clear that

$$\lim_{\varepsilon \to 0} f_\varepsilon(t) = f(t),$$

and, since φ is entire, that

$$f_\varepsilon(t) = \int ds \frac{\sin \pi s}{\pi s} e^{-\varepsilon |s|} \sum_k \frac{s^k}{k!} \frac{d^k \varphi(t)}{dt^k}. \tag{16}$$

Then two issues remain: first, whether term-by-term integration is permissible, so that

$$f_\varepsilon(t) = g_\varepsilon(t), \tag{17}$$

and second, whether the $\varepsilon \longrightarrow 0$ limit can be taken term-wise, so that

$$\lim_{\varepsilon \to 0} g_\varepsilon(t) = \sum_k \frac{1}{k!} \frac{d^k \varphi(t)}{dt^k} \delta_{k0} = \varphi(t) \tag{18}$$

[see (12)]. Note that (17) and (18) together would imply that $\varphi = f$, or in other words that φ was unaltered by the filtering operation. It must have been of bandwidth $\leq \pi$ to begin with.

We next show that the finite bandwidth assumption indeed allows term-by-term treatment. The proof depends on the Lebesgue convergence theorem, which permits the interchange of summation and integration provided, roughly, that the integrand series defines an absolutely integrable function even after replacement of each term by its absolute value.[10]

Consider, more generally, a function φ of arbitrary bandwidth W:

$$\varphi(t) = \int_{-W}^{W} ds\, e^{-ist}\Phi(s). \tag{19}$$

The derivatives of φ can be bounded, as follows:

$$\left|\frac{d^k\varphi(t)}{dt^k}\right| \leq W^k I, \tag{20}$$

where

$$I = \int_{-W}^{W} ds\, |\Phi(s)|.$$

(We are assuming, of course, $I < \infty$.) The integrand in (16) is then dominated, term by term, by the series

$$Ie^{-\varepsilon|s|}\sum_k \frac{|Ws|^k}{k!} = Ie^{(W-\varepsilon)|s|}.$$

Thus the term-by-term integration is permitted, verifying (17), provided

$$\varepsilon > W.$$

Of course this lower bound is not acceptable: we must show that (17) holds for all $\varepsilon > 0$. This is accomplished by demonstrating that both members of (17) are analytic in the right half ε-plane, and then using analytic continuation. In other words, we note that term-by-term integration is permitted if the functions $f_\varepsilon(t)$ and $g_\varepsilon(t)$ are analytic in ε for $\mathrm{Re}(\varepsilon) \geq 0$.

Analyticity of $f_\varepsilon(t)$ in the right half plane is clear from (14) and the boundedness of φ. The analyticity property for $g_\varepsilon(t)$ is more interesting. Returning to the definition (15), we note that terms with odd k vanish trivially, and that the even, $k = 2n$, terms are given by

$$M_n \equiv \frac{1}{(2n)!}\frac{d^{2n}\varphi(t)}{dt^{2n}}\int ds\, s^{2n}\frac{\sin \pi s}{\pi s}e^{-\varepsilon|s|}.$$

But

$$\int ds\, s^{2n}\frac{\sin \pi s}{\pi s}e^{-\varepsilon|s|} \quad = \quad \left(\frac{2}{\pi}\right)(2n-1)!\left(\pi^2+\varepsilon^2\right)^{-n}\sin(2n\theta) \quad (\text{for } n \neq 0),$$

$$= \frac{2\theta}{\pi} \quad (\text{for } n = 0),$$

where

$$\theta = \tan^{-1}\left(\frac{\pi}{\varepsilon}\right).$$

Since $\theta \to \pi/2$ for $\varepsilon \to 0$, this result reproduces (12); it also yields the ε-independent estimate

$$|M_n| \leq \frac{1}{(2n)!} W^{2n} I \frac{2}{\pi} (2n-1)! \left(\pi^2\right)^{-n}$$
$$= \frac{I}{\pi n}\left(\frac{W}{\pi}\right)^{2n}.$$

Assume first that $W < \pi$. Since the series

$$\frac{1}{\pi}\sum_n \left(\frac{W}{\pi}\right)^{2n} \frac{1}{n}$$

then converges, we may conclude that the series in (15) converges uniformly with respect to ε. Moreover, since its individual terms are analytic, we have shown that g_ε is indeed analytic in the right half ε-plane. Thus Eq. (17) is confirmed for any real positive ε.

Furthermore, uniform convergence of the series for g_ε suffices also to justify term-wise evaluation of the $\varepsilon \to 0$ limit: Eq. (18) is established.

The foregoing argument fails if $W = \pi$, since the comparison series $\sum n^{-1}$ diverges. Let us assume, however, that the function Φ is not only absolutely integrable, as assumed above, but also bounded. We can then strengthen the estimate (20) (for $W = \pi$) to

$$\left|\frac{d^k\varphi(t)}{dt^k}\right| \leq \frac{2\pi^{k+1}}{k+1}\max(\Phi)$$

and thus obtain the convergent comparison series $\sum n^{-1}(2n+1)^{-1}$ in the place of $\sum n^{-1}$. The same conclusions will follow as before.

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