DOE/ET-53088-398 IFSR #398

Current Sheets and Nonlinear Growth of the $m = 1$ Kink-Tearing Mode

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September 1989

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Abstract

A calculation is presented which accounts for the rapid nonlinear growth of the m=1 kink-tearing instability. The equilibrium analysis contained in the Rutherford theory of nonlinear tearing-mode growth is generalized to islands for which the constant-ψ approximation is not valid. Applying the helicity-conservation assumption introduced by Kadomtsev, we show the presence of a current-sheet singularity that gives rise to a narrow tearing layer and rapid reconnection. This rapid reconnection, in turn, justifies the use of the helicity-conservation assumption. The existence of a family of self-similar m=1 equilibrium islands is demonstrated. The formalism introduced here is shown to apply both to the case of the m=1 kink-tearing mode and to the case of forced reconnection. These two cases are compared and contrasted.

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I. INTRODUCTION

The nonlinear theory of tearing modes developed by Rutherford predicts algebraic growth of magnetic islands on a slow, diffusive time scale.[1] The Rutherford theory does not apply, however, when the mode under consideration is near marginal stability in the ideal limit. This is the case, in large aspect-ratio tokamaks, for the kink-tearing mode with poloidal mode number m=1. [2,3]

A qualitative picture of the nonlinear evolution of the kink-tearing mode has been proposed by Kadomtsev.[4,5] The Kadomtsev model predicts that the kink-tearing mode will grow on an intermediate time scale between the slow, diffusive time characterizing Rutherford growth and the much faster poloidal-Alfvèn time typical of ideal plasma motions.

Kadomtsev also calculated the final magnetic configuration after the complete reconnection of the flux within the q=1 surface, where q is the tokamak safety factor.[4] He based this calculation on the assumption that the helicity within two flux-tubes is conserved during their reconnection.

The Kadomtsev model has been confirmed by many numerical simulations of m=1 island growth.[6-9] However, the mechanism responsible for this instability is not well understood. It is sometimes argued that the reconnection is driven by the nonlinear evolution of the ideal kink instability. This explanation must be rejected, however, as it does not account for the rapid reconnection in ideally stable systems such as low-beta tokamaks[9,10].

An analytic description of rapid reconnection in helically symmetric configurations has been presented by Hazeltine *et. al.*[11]; however, in the present paper, a different approach is taken. Here, we seek equilibrium magnetic islands which satisfy Kadomtsev's helicity-conservation assumption. The focus on equilibrium configurations is motivated by the observation that the mode evolution is slow compared to the Alfvèn time. The helicity conservation assumption, by contrast, is justified by the rapidity of the reconnection compared to the resistive diffusion time.

The existence of a family of self-similar, helicity-conserving equilibrium islands is demonstrated below. The most important result of the analysis is that helicity-invariance requires the presence of a current-sheet singularity on the island separatrix. This result can be anticipated from the observation that the safety factor diverges on the separatrix of a regular island. The divergence of the safety factor is inconsistent with helicity invariance, and cannot be avoided without allowing for the presence of a current sheet on the separatrix.

The demonstration of the existence of a current-sheet singularity is significant in that it implies the self-consistency of the helicity conservation assumption, since the current sheet will give rise to a tearing layer in which rapid reconnection will occur. The conservation of magnetic helicity during rapid reconnection in a current sheet can be shown to follow from Ohm's law.

While the existence of the current sheet thus insures self-consistency of the helicity-conservation assumption, it might appear, upon first glance, to put into question the self-consistency of the equilibrium approach, since the current sheet causes a tearing layer in which inertial effects will be important. This is not the case, however, as the non-linear regime is characterized by the inequality $w_L << w$, where w_L represents the width of the tearing layer, and w represents the island width. The equilibrium solutions found in this paper can thus be interpreted as "exterior" solutions of a boundary-layer analysis.

A more serious criticism of the equilibrium approach is that the reconnection, although itself slow, may convert part of the magnetic energy into kinetic energy, thereby causing global oscillations of the island at the alfvén frequency[7]. However, the existence of current sheets is a consequence of the helicity-conservation assumption only, and is independent of whether the island evolves near equilibrium. Moreover, on a more fundamental level, it is the existence of nearby equilibria that determines what will occur during reconnection; thus the oscillations will play only an ancillary role.

The "interior" problem of reconnection within the tearing layer will not be considered here. A solution of this problem, assuming laminar flow conditions, has been given by Sweet and Parker[12]. The Sweet-Parker model has been generalized to include the effects of viscosity by Park *et. al.*[7] The application of these models to the current layers discussed in this paper will be considered in the conclusion.

The presentation is organized as follows. In section II the problem is formulated. The basic reduced nonlinear equilibrium equation for thin islands is derived in section III; the formulation used is an extension of the one introduced by Rosenbluth, Dagazian and Rutherford[2] in their study of the nonlinear saturation of the internal kink instability. In section IV, the helicity conservation assumption is applied to the thin island equilibrium equation. The reduced equilibrium equation is then shown in section V to be equivalent to a variational principle. This variational principle is used to demonstrate the existence of solutions to the flux-conserving equilibrium problem. The results are discussed in section VI.

II. FORMULATION

A. Equilibrium

The standard reduced magnetohydrodynamic (RMHD) equations of Strauss[13] describing large aspect-ratio, low-beta tokamaks will be taken as the starting point of the analysis. The equilibrium equation is

$$(\mathbf{B}\cdot\nabla)\, J = 0, \tag{1}$$

where J is the parallel current. The perturbed equilibrium will be assumed to be helically symmetric. It is then convenient to express the magnetic field in terms of the "auxilliary field" $\mathbf{B}_*$ as

$$\mathbf{B} = \hat{z} + \frac{\varepsilon}{q_s} r\, \hat{\theta} + \mathbf{B}_*,$$

where ε is the inverse aspect ratio and q_s is the safety factor on the resonant flux surface; that is, the rational flux surface where the winding ratio of the field lines matches the helicity of the equilibrium. The magnetic field is normalized to the value of the toroidal field B_z. The auxilliary field is related to the helical flux ψ by

$$\mathbf{B}_* = -\,\varepsilon(\hat{z} \times \nabla\psi),$$

while the parallel current J is given in terms of the helical flux by Ampere's law,

$$J = \nabla^2\psi - 2.$$

As a result of helical symmetry, the equilibrium equation, eq. (1), takes the simple form

$$\nabla\psi \times \nabla J = 0. \tag{2}$$

The most important feature of eq. (2) is its singular nature on resonant field lines. For such field lines, one has

$$\nabla\psi = 0.$$

The general solution of the equilibrium equation, away from resonant field lines, is

$$J = J(\psi). \tag{3}$$

On resonant field lines, however, it is apparent that current singularities violating eq. (3) are allowed by the equilibrium equation, eq. (2).

The simplest such configuration consists of a current filament lying on an isolated closed field line. It is interesting to consider the consequence of superposing such a current filament on an X-type neutral point. In the neighborhood of the current filament, the magnectic topology will be changed to that of an 0-type neutral point.[14] In the presence of resistivity, however, this structure will diffuse without further altering the topology.

More significant configurations arise if rational surfaces composed of closed field lines are present. Equilibria can then be found with ψ-independent current sheets lying on the rational surface. The bifurcated kinked equilibria of Rosenbluth *et. al.*[2], for example, fall under this category. Note that the auxilliary field will undergo an antisymmetric jump across such current sheets:

$$(\hat{n}\cdot\nabla\psi)_+ = -(\hat{n}\cdot\nabla\psi)_-, \tag{4}$$

where $\hat{n}$ is the normal to the flux surface.

Current-sheet singularities are also allowed on irrational magnetic surfaces, as long as they satisfy eq. (3). A jump condition can be derived by integrating eq. (2) around a closed loop lying along the current sheet.[15] One finds

$$(\hat{z}\times\hat{n})\cdot\nabla\left((\nabla\psi)_+^2 - (\nabla\psi)_-^2\right) = 0\,. \tag{5}$$

That is, the jump in the square of the auxilliary field is constant along the current-sheet.

The results of this subsection can be summarized as follows. Two types of current sheets can exist at equilibrium: on nonresonant flux surfaces, $J = J(\psi)$ must be satisfied, while on resonant surfaces the amplitude of a current-sheet may be field-line dependent.

B. Conservation of Magnetic Helicity

The precise statement of the helicity conservation assumption is now given.[4,5] The general definition of the helicity K in a flux tube is[16]

$$K = \int \mathbf{A} \cdot \mathbf{J} \, dV, \tag{6}$$

where J is the current and A is the magnetic vector potential. The integral in eq. (6) is taken over the volume of the flux tube being considered. For the large aspect-ratio, helically symmetric system considered here, it can be shown that conservation of K is equivalent to conservation of the integral C given by

$$C = \int \psi \, dV.$$

In an ideal plasma, the helicity integral C must be conserved for every flux tube individually. Kadomtsev's hypothesis[4,5], which is adopted here, is that the sum of the helicity integral for two reconnecting tubes of flux is conserved during their rapid reconnection. In addition, it is assumed that C is conserved separately for all the flux tubes which do not undergo reconnection during the period of interest. Note that the Kadomtsev assumption is related to but much more restrictive than that of global helicity conservation used by Taylor[16] and Bhattacharjee *et. al.*[17] to calculate the final state of a plasma after turbulent relaxation.

We now derive a more readily usable form of the helicity-conservation assumption. Consider two flux-tubes about to undergo reconnection. Since these flux-tubes share the separatrix, they correspond to the same value of ψ (fig.1) The helicity-conservation condition then takes the form

$$dV_f = dV_1 + dV_2, \tag{7}$$

where dV_f is the volume of the reconnected flux-tube and dV_1 and dV_2 are the volumes of the initial flux-tubes. Since $|d\psi_f| = |d\psi_1| = |d\psi_2|$, eq. (7) can be written

$$\left|\frac{dV}{d\psi}\right|_f = \left|\frac{dV}{d\psi}\right|_1 + \left|\frac{dV}{d\psi}\right|_2 . \tag{8}$$

Eq. (8) is the local form of the helicity-conservation assumption. It will be used in Section IV to calculate the current distribution within the island. Note that the "local helicity" $dV/d\psi$ is related to the safety factor q by

$$\frac{dV}{d\psi} = 2\pi\left(1 - \frac{1}{q}\right)^{-1}.$$

It can be evaluated with the general formula

$$\frac{dV}{d\psi} = \oint_{\psi} \frac{d\theta}{|\mathbf{B}_{*}\cdot\nabla\,\theta|} , \tag{9}$$

where the integration is carried out along a toroidal section of the flux-surface. The denominator in the integrand of eq. (9) is given by

$$\mathbf{B}_{*}\cdot\nabla\theta = \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)_{\theta.} .$$

III. EQUILIBRIUM IN THIN ISLANDS

A. The Thin-Island Approximation

The equilibrium problem can be greatly simplified for thin islands by using a second boundary-layer decomposition. In the present case, the decomposition is based on the smallness of the island's width, w, compared to the characteristic scale length of the unperturbed cylindrical equilibrium. Near the singular surface, defined as the surface where $\nabla\psi = 0$ in the cylindrical equilibrium, nonlinear effects will be important. In the bulk of the plasma, however, the perturbation occasioned by the growth of a thin island will be small, and a linearized description will be adequate.

In the thin nonlinear layer the effects of curvature may be neglected. In addition, gradients along the layer may be neglected compared to transverse gradients.

The simplifications described above were utilized by Rosenbluth *et. al.*,[2] to reduce the equilibrium problem for the bifurcation of the m=1 kink to a one-dimensional nonlinear equation. The analysis in this section follows closely that of Rosenbluth *et. al.*. A significant difference, however, is that the expansion parameter in their theory is the inverse aspect ratio ε, while the small parameter in the present analysis is the ratio w/r_s of the island width to the radius of the singular surface. In this respect, the thin island asymptotic analysis used here is more similar to that of Rutherford.[1]

B. Solution in the Nonlinear Layer

With the approximations discussed above, the equilibrium equation takes the form:

$$\nabla^2\psi \cong \left(\frac{\partial^2\psi}{\partial r^2}\right)_\theta = J(\psi) + 2 \, .$$

This equation can be integrated along a radius. One finds

$$\frac{1}{2}\left(\frac{\partial\psi}{\partial r}\right)_\theta^2 = F(\psi) - G(\theta), \tag{10}$$

where

$$F(\psi) = \int_{\psi_{os}}^{\psi} [J(\psi) + 2] \, d\psi$$

and $G(\theta)$ is a constant of integration.

A formal similarity between the island equilibrium problem and the problem of nonlinear electrostatic plasma oscillations[18] is apparent in eq. (10). The sign convention for $G(\theta)$, which is opposite to that chosen in ref. 2, has been chosen to make this similarity explicit. Solving eq. (10) for $(\partial\psi/\partial r)_\theta$ yields

$$\left(\frac{\partial\psi}{\partial r}\right)_\theta = \pm \left[2\left(F(\psi) - G(\theta)\right)\right]^{1/2}. \tag{11}$$

One thus finds that flux surfaces such that $F(\psi) < \mathrm{Max}\left(G(\theta)\right)$ will be "trapped" between "mirror points" θ_m given by $G(\theta_m) = F(\psi)$. These trapped surfaces correspond, of course, to reconnected flux surfaces lying within the island. The continuity of the flux can be assured by inverting the sign of the square root in eq. (11) when passing through mirror points.

Flux surfaces such that $F(\psi) > \mathrm{Max}\left(G(\theta)\right)$, by contrast, are "untrapped" and continue to encircle the plasma core.

The displacement of the flux surfaces from their position in the cylindrical configuration can now be derived. Radial distances are henceforth measured from the radius r_s of the singular surface in the unperturbed cylindrical configuration. This is accomplished by the substitution $r - r_s \rightarrow r$.

The flux in the unperturbed configuration can be expanded close to the singular surface:

$$\psi = \psi_{0s} + \frac{1}{2}\psi_0'' r^2 + O(r^3).$$

Here $\psi_{0s} = \psi_0(0)$.

Note that the helical-flux ψ is not a good flux-surface label for unreconnected flux-surfaces, since there are two distinct flux surfaces with the same value of ψ. It is therefore preferable to label the flux-surfaces with their radial position in the unperturbed configuration. As in ref. 2, this quantity is denoted by x. For reconnected flux surfaces, it is convenient to label the "left" branch of the flux-surface between the two mirror points with the position of the corresponding "left-hand-side" flux-surface in the initial configuration, and respectively, the "right" branch with the "right-hand-side" position. With this convention the sign of the square root in eq. (11) is identical to the sign of x. The radial displacement of a flux-surface with respect to its position in the initial equilibrium is given by

$$\xi\,(x,\theta) = r\,(x,\theta) - x. \tag{12}$$

An expression for ξ is now derived. Taking the reciprocal of eq. (11) and substituting $F(\psi) = \frac{1}{2}\left(\psi_0''\right)^2 f(x)$ and $G(\theta) = \frac{1}{2}\left(\psi_0''\right)^2 g(\theta)$, one finds

$$\frac{\partial}{\partial x}\Big(r(x,\theta)\Big) = \frac{|x|}{\big(f(x) - g(\theta)\big)^{1/2}}. \tag{13}$$

Integrating eq. (13) from the mirror point to a flux-surface x and combining the result with eq. (12) yields the sought-after equation for ξ:

$$\xi\,(x,\theta) = \xi\Big(x_m\,(\theta),\theta\Big) + \int_{x_m(\theta)}^{x} dx\,|x|\left[\frac{1}{\big(f(x) - g(\theta)\big)^{1/2}} - \frac{1}{|x|}\right]. \tag{14}$$

The quantity $x_m\,(\theta)$ in eq. (14) is the value of x for the flux-surface whose mirror-point lies on the radius of integration. At the mirror point, the flux surface originating from $-\left|x_m(\theta)\right|$ must be connected to that coming from $+\left|x_m(\theta)\right|$ (fig.2), so that

$$\xi\Big(-\left|x_m(\theta)\right|,\,\theta\Big) - \xi\Big(+\left|x_m\,(\theta)\right|,\,\theta\Big) = 2\left|x_m\,(\theta)\right|. \tag{15}$$

It should be emphasized that the variable ξ does not represent the fluid displacement, but only the radial displacement of a flux-surface. Eq. (15) is thus nothing more than a continuity condition on the flux.

In order to match the linear solution to the solution inside the nonlinear layer, an asymptotic expansion for the latter must be found. An expression for f(x) at large x can be derived by assuming that the helicity-conservation equation is asymptotically valid for large x. Note that this will be the case not only for rapid reconnection, but also for the slow tearing mode evolution of Rutherford. There results from eqs. (8) and (9)

$$f(x) \sim x^2 + g_0\,, \tag{16}$$

where g_0 is the zeroth harmonic of g:

$$g_0 = \oint d\theta\; g(\theta)\;.$$

The asymptotic expansion of the displacement at large values of x follows from eqs.(14) and (16). One finds

$$\xi\,(x,\,\theta)\;\sim\;\xi_{\pm}(\theta) + \frac{\xi_0(\theta)}{x}\,, \tag{17}$$

where

$$\xi_{\pm}(\theta) = \xi\big(\pm\left|x_m(\theta)\right|,\,\theta\big) \pm \int_{\left|x_m(\theta)\right|}^{\infty} dx\;\; x\left[\frac{1}{\big(f(x) - g(\theta)\big)^{1/2}} - \frac{1}{x}\right] \tag{18}$$

and

$$\xi_0(\theta) = -\frac{1}{2}\big(g(\theta) - g_0\big). \tag{19}$$

The relevant quantity as regards the asymptotic matching is the value of the jump in ξ across the layer, which we denote $\xi_\infty\,(\theta)$:

$$\xi_\infty\,(\theta) = \xi_+\,(\theta) - \xi_-\,(\theta). \tag{20}$$

Combining eqs. (15), (18) and (20) yields

$$\frac{1}{2}\,\xi_\infty(\theta) = \int_{x_m(\theta)}^{\infty} dx\; x\left[\frac{1}{\left(f(x) - g(\theta)\right)^{1/2}} - \frac{1}{x}\right] - x_m(\theta). \tag{21}$$

C. Linear Solution in the exterior region.

The linear solution is found by integrating the Euler-Lagrange equations derived from the well-known energy principle.[19-21] The case of free reconnection should be distinguished here from that of forced reconnection. The former refers to the spontaneous growth of a tearing mode driven only by the energy contained in the equilibrium configuration, whereas the latter occurs as a result of imposing a helical perturbation on an equilibrium. These two situations require distinct boundary conditions. We consider each case in turn.

In the case of free reconnection, the exterior problem is identical to that for the linear tearing and kink-tearing modes. The familiar result is that the asymptotic behavior of the exterior solution in the neighborhood of the singular surface can be described by the coefficient Δ'. Neglecting the effects of curvature, the asymptotic form of the exterior solution near the singular layer is[20,21]

$$\xi \sim A_\pm + \frac{A_0}{x}, \tag{22}$$

where the A_+ and A_- should be used for $x>0$ and $x<0$ respectively. The Δ' coefficient is then defined by $\Delta' = A_+ - A_-/A_0$

Two important results of the linear analysis in low aspect-ratio tokamaks should be recalled here. The first is that modes with poloidal mode number m=1 are distinguished by rather large values of Δ'.[2,3] Specifically, $\Delta' \sim \varepsilon^{-2}$ where ε is the inverse aspect-ratio. The second result is that the average curvature vanishes, to lowest order in the inverse aspect-ratio, for all the modes which are resonant on the flux surface where q=1.[22] The neglect of curvature effects thus appears particularly appropriate for these modes.

We now consider the case of forced reconnection. In this case, the linear perturbation in the bulk plasma is the sum of a "homogeneous" solution identical to that found above for free reconnection and of a solution of the same equation satisfying the inhomogeneous boundary condition imposed by the external perturbation on the plasma surface. The inhomogeneous solution may be chosen to satisfy A_0=0 on the singular surface.

The asymptotic behavior of the complete solution near the singular surface again takes the form given by eq (22), but the coefficients are now related by

$$A_0 \Delta' = A_+ - A_- - A_{+inh}. \tag{23}$$

Note that A_{-inh}=0. In the following section eq. (23) will be used to derive the reduced equilibrium equation.

D. Reduced Equilibrium Equation.

The nonlinear layer solution, given by eqs. (17)-(21), and the exterior linear solution, given by eqs. (22) and (23), are now matched together. The matching condition yields the reduced equilibrium equation,

$$\int_{x_m}^{\infty} dx\, x \left(\frac{1}{\left(f(x)-g(\theta)\right)^{1/2}} - \frac{1}{x} \right) - x_m(\theta) = -\frac{1}{4} \sum_{n=1}^{\infty} (\Delta'_n g_n + \delta_n) \cos(n\theta), \tag{24}$$

where δ=-2A_{+inh}, and

$$g_n = 2 \oint d\theta \; g(\theta) \cos(n\theta), \qquad n \neq 0.$$

Eq (24) is the basic nonlinear equilibrium equation. It allows, at least in principle, the function $g(\theta)$ to be determined for a given current distribution $J(\psi)$. It should be emphasized that only the thin-island approximation was used in the derivation of eq. (24), and that it therefore has very general validity. It can be used to describe both free and forced reconnection, Rutherford reconnection, and the bifurcated kinked equilibria of Rosenbluth et al., provided the appropriate current distribution is used.

Eq (24) can be simplified further in either of the following two cases depending on the size of the parameter $\Delta' r_s$. For $\Delta' r_s \sim 1$, the constant-ψ approximation may be used. This case was treated by Rutherford, and is only included here for the purpose of comparison. For $\Delta' r_s >> 1$, however, the constant-ψ approximation fails, and a different approach must be taken.

The simplification of eq (24) in the "small-Δ' " case is the subject of the following subsection. The "large-Δ' " case will be treated in sec. IV.

E. Constant-Psi Islands

The constant-ψ approximation consists in neglecting the perturbation $\tilde{\psi}$ in the poloidal component of the auxilliary field $\mathbf{B}*$. This is justified, when $\Delta' \sim r_s^{-1}$, since $\partial\tilde{\psi}/\partial r \sim \Delta'\tilde{\psi} \sim \Delta' w\,(\partial\psi_0/\partial r)$ implies that $\partial\tilde{\psi}/\partial r \sim (w/r_s)\,\partial\psi_0/\partial r$. The neglect of $\partial\tilde{\psi}/\partial r$ is thus consistent with the thin-island approximation $w/r_s << 1$.

The poloidal component of the auxilliary field is then given by

$$\left(\frac{\partial\psi}{\partial r}\right)_\theta = \psi_0'' r \,. \qquad (25)$$

The constant-ψ approximation also allows the flux to be written

$$\psi = \psi_0'' \frac{r^2}{2} + \tilde{\psi} \qquad (26)$$

where

$$\tilde{\psi} = \sum_{n=1}^{\infty} \tilde{\psi}_n(t)\cos(2\pi n\theta).$$

An equation equivalent to eq (11) for constant-ψ islands can now be obtained from eqs (25) and (26) :

$$\left(\frac{\partial\psi}{\partial r}\right)_\theta = \left(2\psi_0''(\psi-\tilde{\psi})\right)^{1/2} . \qquad (27)$$

Comparing eq (27) to eq (11), one concludes that, for constant-ψ islands, $f \cong f_0 = x^2$ and $g_n = 2\tilde{\psi}_n/\psi_0''$. That is, the current perturbation $\tilde{f}$ is of order w/r_s and may be neglected in eq (27). It must be retained in eq (24), however. To lowest order, this equation yields

$$\int_{x_m(\theta)}^{\infty} dx \cdot \frac{df}{dx} \cdot \frac{1}{\sqrt{x^2 - g(\theta)}} = \frac{1}{2}\sum_{n=1}^{\infty} \Delta'_n \, g_n \cos(2\pi n\theta) .$$

This is clearly equivalent to eq (14) in Rutherford's paper.[1]

The significance of the constant-ψ approximation, however, is that it allows a direct calculation of the current from the flux-surface-averaged Ohm's law. This is a consequence of the fact that $\partial\psi/\partial t$ and $\partial\psi/\partial r$ are independent of the current, to lowest order in w/r_s.

In large-Δ' islands, this simplification cannot be used. As a result, the current can no longer be calculated explicitly from Ohm's law. In the following section, the helicity-conservation assumption is used to calculate the current-profile in large-Δ' islands. We repeat for emphasis that the resulting current profile is consistent with Ohm's law, thus justifying the use of the helicity conservation assumption.

IV HELICITY-CONSERVING ISLANDS

A. Helicity Conservation and Current Distribution

The thin-island formalism introduced in sec. III.A is now applied to the helicity conservation condition, eq (8). Note that for unreconnected flux surfaces, the integral in this equation extends over the two flux tubes labeled by ψ. For reconnected flux-surfaces, the same integral must range over the two branches of the flux-surface corresponding to the $\pm$ sign in eq (11). The helicity-conservation constraint therefore takes the following compact form :

$$\frac{1}{|x|} = \oint d\theta \cdot \frac{H\big(f(x)-g(\theta)\big)}{\big(f(x)-g(\theta)\big)^{1/2}}, \qquad (28)$$

where we have inserted the Heaviside function H to define the range of integration. It is defined by

$$H(y) = \begin{cases} 0, & \text{for } y<0 \\ 1, & \text{for } y\geq 0 \end{cases}.$$

Eq (28) is an implicit equation for the total current f(x) contained within a flux-surface, as a function of the flux-label x. Note that the helicity-conservation constraint can be applied individually to flux-surfaces which did not undergo reconnection, resulting in $f(x) = f(-x)$.

It is the solution of eq. (28) for f(x) that demonstrates the existence of the current sheet. The inversion of this equation yields three solution-branches for f(x). In order to obtain an island topology, it is necessary to jump from one branch to another. The resulting jump in f(x) corresponds to a current-sheet lying on the separatrix.

To demonstrate the above assertions, consider the function $\chi(f)$ defined by

$$\chi(f) = \left[\oint d\theta \ \frac{H\big(f - g(\theta)\big)}{\big(f - g(\theta)\big)^{1/2}}\right]^{-1}.$$

In terms of this function, eq (28) takes the simple form $|x| = \chi(f)$. Two features of $\chi(f)$ are immediately apparent. First, $\chi(0) = 0$ is required by eq (28). This will be satisfied when $g(\theta)=o(\theta^2)$ in the neighborhood of the island's magnetic axis, which we take to be $\theta=0$. Second, the integral in $\chi(f)$ contains a non-integrable singularity for $f=g_{max}$, where $g_{max}=\mathrm{Max}\big(g(\theta)\big)$, so that $\chi(g_{max}) = 0$. This singularity reflects of course the well-known divergence of the safety factor on the separatrix of regular islands, as discussed in the introduction.

These two properties, combined with the positiveness of $\chi(f)$, imply the existence of a maximum χ_{max} of $\chi(f)$ between f=0 and f=g_{max}. The analysis of χ is completed by the asymptotic expansion for large f :

$$\chi(f) \sim f^{1/2}\left(1 - \frac{g_0}{2f}\right) + 0\left(f^{-3/2}\right).$$

The function $\chi(f)$ is represented schematically in fig.4 (a); its inverse is shown in fig. 4 (b).

Recall that $f<g_{max}$ corresponds to "trapped" surfaces, while $f>g_{max}$ corresponds to untrapped surfaces. Thus, if the reconnected flux is given, or equivalently if the value of x on the separatrix, x_s is given, the inverse function f(x) will be determined unambiguously. That is, the lower branch of $\chi^{-1}(x)$ must be chosen for $x<x_s$ in order for those flux-surfaces to be "trapped", while the upper branch must be chosen for $x>x_s$.

The current-sheet corresponding to the jump in f must therefore lie on the separatrix. Note that the "inner" value of f on the separatrix f_i, is less than g_{max}. This has the following important consequence for the island structure : the outermost flux surface in the island, corresponding to f_i, will not extend all the way around the magnetic axis of the tokamak, but will remain "trapped" within the arc defined by $g(\theta) \leq f_i$.

In the complementary arc, $g(\theta) > f_i$, the innermost unreconnected flux-surfaces, corresponding to f_e, are contiguous. Across this segment of the separatrix, the magnetic field undergoes an antisymmetric, θ-dependent jump from $+\left(f_e - g(\theta)\right)^{1/2}$ to $-\left(f_e - g(\theta)\right)^{1/2}$. Such a current sheet is similar to the one found by Rosenbluth *et al.*, and is consistent with the equilibrium equation (2), as previously discussed.

To summarize, one finds that the X-point structure of regular islands is replaced in helicity-conserving islands by two Y-points, separated by a ribbon generated by closed field-lines. This ribbon supports a field-line dependent current-sheet which satisfies the equilibrium condition given in eq. (4), while the body of the island is surrounded by a flux-dependent current-sheet satisfying eq. (5).

B. Helicity-Conserving Equilibrium Islands

The magnetic configuration of rapidly reconnecting m=1 islands is described by eq. (24), with f(x) determined by eq. (28) as discussed in the previous subsection. We begin by considering the simplifications of eq. (24) which can be justified within the ordering assumptions $w \ll r_s$ and $\Delta'^{-1} \ll r_s$, independently of the relative size of w and Δ'^{-1}. Eq. (24) can be written symbolically as

$$-\xi_{\infty n} = \frac{1}{2}(\Delta'_n g_n + \delta_n), \quad n=1, 2, \ldots \qquad (29)$$

Note that the zeroth harmonic of eq. (24) is identically satisfied.

The relative order of the terms in eq. (29) can be seen to be $\xi \sim g^{1/2} \sim w$. One concludes that the r.-h.s. of this equation should be neglected, for $n \neq 1$. For the n = 1 harmonic, however, the r.-h.s. must clearly be retained. The general form of eq. (29) in either the large-Δ' case or for driven reconnection is therefore

$$-\xi_{\infty 1} = \frac{1}{2}(\Delta'_1 g_1 + \delta_1), \qquad (30)$$

$$\xi_{\infty n} = 0, \qquad n=2,3,\ldots \qquad (31)$$

This system of equation can be simplified further, for the case of free reconnection, in the regime where $\Delta'^{-1}_1 \ll w \ll r_s$. Eq. (30) may then be replaced with

$$g_1 \cong 0. \qquad (32)$$

The most significant consequence of this approximation is that it implies that freely reconnecting m=1 islands become self similar for $w \gg \Delta'^{-1}$; that is, they are invariant under the scale transformation $x_s \rightarrow \alpha x_s$, $f \rightarrow \alpha^2 f$, $g_n \rightarrow \alpha^2 g_n$. Note that under this scale transformation, the island width w and the displacement of the plasma core $\xi_{\infty 1}$ become αw

and $\alpha\xi_{\infty 1}$, respectively. One consequence is that the scaling law for constant-ψ islands, $w \sim \psi_s^{1/2}$, also applies to m=1 islands. Another important consequence of eq. (32) is that g will have a minimum at $\theta=\pm\pi$, in addition to the one at $\theta=0$, where $\theta=0$ is the position of the island's magnetic axis. This means that the current will peak at the center of the ribbon of closed field lines joining the tips of the island.

We list here two other properties of the solution that can be deduced by inspection of the integrals in eqs. (24) and (28) in the neighborhood of the points $\theta=0$ and $\theta=\pm\theta_y$, where the θ_y, which denote the positions of the tips of the island, are given by $g(\theta_y)=f_i$.

First, near $\theta=0$, one finds that eq. (28) requires that $g(\theta)=O(\theta^8)$ Second, near $\theta=\pm\theta_y$, the singularity in the denominator of eq. (24) must be compensated by a zero in the numerator in order for $\xi_{\infty 1}$ to remain analytic. Thus, it is necessary that dx/df=0 be satisfied for $f=f_i$; that is, $g(\theta)$ must be such that $\chi_{max}=x_s$ is satisfied. These qualitative features are sketched in figs. 3 and 4.

We conclude this section by considering the case of forced reconnection. For thin islands or large perturbations such that $\Delta' w^2 << \delta$, the Δ' term may be neglected in eq. (30). For $x_s=0$, the resulting problem is identical to that considered by Rosenbluth *et al.*[2], except that the amplitude of the kinked equilibrium is now determined by δ. It should be recalled that the sign convention used here is opposite to that of ref.2. As the island grows, a small localized well will appear at the point θ_0 for which $g(\theta)=O((\theta-\theta_0)^8)$ in the unreconnected kinked equilibrium. This point will become the island's magnetic axis. Note that our conclusion in this respect differs from that of ref.2,.where it was suggested that the point θ_0 would become an X-type neutral point during reconnection.

In the next section, we introduce a variational principle to show the existence of solutions to the free reconnection problem.

V. VARIATIONAL PRINCIPLE

The existence of a solution to eqs. (31) and (32), describing freely reconnecting large-Δ' islands in the regime where $\Delta'^{-1} << w << r_s$, can be demonstrated with the help of a variational principle. The variational functional is most readily constructed by the standard method of combining the energy integral with the integral invariants of motion and constraints related to the boundary conditions. We choose

$$I = E + C + \Lambda, \tag{33}$$

where E is twice the energy,

$$E = \int |\nabla\psi|^2 dV; \tag{34}$$

C is the general helicity invariant,

$$C = \int K(\psi)\, dV; \tag{35}$$

and where Λ is a constraint related to the boundary condition represented by eq. (32),

$$\Lambda = \lambda \int \frac{\partial\psi}{\partial r} r \cos\theta\, d\theta. \tag{36}$$

The integrals in eqs (34) and (35) extend over the volume of the plasma. The poloidal integral in eq. (36) should be taken over a flux surface in the asymptotic region $r >> w$.

The function K must be determined so as to satisfy the helicity-conservation constraint. Similarly, the Lagrange parameter λ is determined by requiring that eq. (32) be satisfied, or equivalently that $\Lambda=0$. The variational functional I can be written in terms of the functions f(x) and $g(\theta)$ by substituting the expression for $\partial\psi/\partial r$ from eq. (11). It is convenient to subtract from the functional I its value for the unperturbed cylindrical state, so as to allow the integrals over x to be formally extended to infinity. The arguments of f(x) and $g(\theta)$ will be omitted from the equations below for clarity. The various terms in $I(f,g)-I_0$ then take the following form:

$$E-E_0 = 2\int_0^\infty dx\, x\left(\oint d\theta\, H(f-g)\,(f-g)^{1/2} - x\right) \quad ;$$

$$C-C_0 = 2\int_0^\infty dx\, xK(x)\left(\oint d\theta\, H(f-g)\,(f-g)^{-1/2} - x^{-1}\right) \; ;$$

$$\Lambda-\Lambda_0 = \lambda \oint d\theta\, g(\theta)\cos\theta \, .$$

The variation of I with respect to f is given by

$$\frac{\delta I}{\delta f} = x\oint_0^\infty d\theta\, H(f-g)(f-g)^{-1/2} + 2xK(x)\frac{d}{df}\left(\oint_0^\infty d\theta\, H(f-g)(f-g)^{-1/2}\right).$$

The function K is now chosen so as to ensure that the flux-conservation constraint, eq. (25), is statisfied for the extremum of I, $f_e(x)$. This condition is fulfilled for

$$K(x)=2x\frac{df_e}{dx} \, . \tag{37}$$

The equilibrium equation now follows by requiring that I be extremal under variations of g. The variation of I with respect to g is given by

$$\frac{\delta I}{\delta g} = \int_0^\infty dx\, x\frac{H(f-g)}{(f-g)^{1/2}} - 2\int_0^\infty dx\, xK(x)\,\frac{d}{df}\left(\frac{H(f-g)}{(f-g)^{1/2}}\right) + \lambda\cos\theta \, . \tag{38}$$

Combining eqs (37) and (38) and integrating the second term on the r.-h.s. by parts, one concludes that $\delta I/\delta g=0$ is equivalent to eq. (31), and that the value of the lagrange multiplier λ for which the minimum satisfies the constraint given by eq. (32) can be identified with $\xi_\infty/2$.

Is has thus been established that the functional I, with K given by eq (37), is extremal for the $g(\theta)$ and $f(x)$ which satisfy the equilibrium equation. The existence of solutions to the equilibrium equation follows from the fact that E is positive, and must therefore reach a minimum on the set of functions $g(\theta)$, $f(x)$ satisfying the constraints, eqs. (28) and (32).

The variational functional constructed here is related to that used by Rosenbluth *et.al.*[2] in their analysis of the bifurcated-kink problem, but differs from it in two important ways. First, since x_s must be kept constant when minimizing the energy of an island configuration, the scaling law described in Sec. IV.B cannot be used to decouple the equilibrium equation from the asymptotic matching equation as in the bifurcated-kink problem. Second, the variational principle used here has been constructed so as to be a functional of the two independent functions $f(x)$ and $g(\theta)$. The variational functional $H(g)$ for the bifurcated kink problem, by contrast, depends only on the function $g(\theta)$. An analogous functional $\widehat{H}(g)$ can be constructed for the helicity-conserving island problem, but this functional cannot be used to show the existence of equilibrium solutions. The reason for this is as follows: as a result of the restriction to a fixed x_s, it is not possible to satisfy eq. (28) for the trial functions $g(\theta)$ such that $\chi_{max}<x_s$, since for such functions one cannot jump to an inner solution branch of $f(x)$ at x_s. Thus, the domain of $\widehat{H}(g)$ must be restricted to the set of functions $g(\theta)$ such that $\chi_{max}>x_s$. However, this implies that the solution of the equilibrium equation will lie on the boundary of the domain of $\widehat{H}(g)$. This makes the functional $\widehat{H}(g)$ inappropriate for showing the existence of a solution to eqs. (31) and (32), because it cannot be asserted, in general, that $\delta\widehat{H}=0$ at a minimum lying on the boundary of the domain of $\widehat{H}$.

VI. DISCUSSION

The equilibrium analysis contained in Rutherford's theory of nonlinear tearing-mode growth has been generalized to rapidly growing islands for which the constant-ψ

approximation fails, such as the islands resulting from the growth of the m=1 kink-tearing mode. These islands are found to have distinctive properties, which we briefly summarize here. First, they do not completely encircle the plasma in the poloidal direction, as is usually the case; instead, the extremities of the islands are connected together by a segment of the separatrix composed of closed field lines. A current sheet lies on the separatrix, and the parallel current is found to be field line dependent on the ribbon separating the tips of the island. This should be contrasted to the more usual equilibrium property that the parallel current is everywhere a flux-function.

The equilibrium analysis presented here can be applied both to the case of free reconnection and to that of forced reconnection. During forced reconnection, the current sheet is induced through an external perturbation. For small island widths, the amplitude of the current sheet is determined by the applied perturbation. During free reconnection, however, the amplitude of the current sheet is determined by the island width; for moderately large islands, satisfying $\varepsilon^2 \ll w/a \ll 1$, the islands become self-similar.

The kink-tearing instability is important because it is widely presumed[2-4,23] to be responsible for the minor disruptions or sawtooth crashes observed in tokamak experiments[24]. The growth rate of this instability can be determined by matching the equilibrium solutions to a model for the reconnection within the current-sheets. We will consider the result of applying the Sweet-Parker[12], Park *et. al.*[7] model for the current-sheet reconnection to the equilibria described in sec.IV.B. However, it should be pointed out that the Sweet-Parker model is probably not appropriate for the parameter regimes of interest for present and future tokamaks, as it does not take into account the kinetic effects which are expected to be important in high temperature plasmas. Nevertheless, this model provides a useful reference point for comparing the analytical results with the results of numerical simulations, as well as with the experimental observations.

The reconnection rate predited by the Sweet-Parker[12], Park *et. al.*[7] model is given by

$$\dot{\psi}_s \cong K\,\eta^{1/2}\left(1+\frac{\mu}{\eta}\right)^{-1/4}\frac{B_{*\theta}^{3/2}}{r_s^{1/2}}, \tag{39}$$

where ψ_s is the helical-flux at the separatrix, K is a geometrical constant, and $B_{*\theta}$ is the auxilliary magnetic field just outside the current layer. From the similarity law for freely reconnecting m = 1 islands, one finds that ψ_s scales as $\psi_0'' w^2$, while $B_{*\theta}$ scales as $\psi_0'' w$. Eq (39) can then be integrated to find the evolution of the island width w during the reconnection. After restoring the dimensions, one finds

$$\frac{w}{r_s} \cong K'\left(\frac{\lambda t}{\tau_K}\right)^2, \tag{40}$$

where $\lambda=(1+\mu/\eta)^{-1/4}$. The Kadomtsev reconnection time τ_K which appears in eq. (40) is the geometric mean between the resistive skin time and the shear-Alfvén time:

$$\tau_K = (\tau_S\,\tau_{A*})^{1/2}\,.$$

The resistive skin time is given by $\tau_S = 4\pi r_s^2/c^2\eta$, and the shear-alfvén time by $\tau_{A*} = R_0/v_A r_s q'$. The Alfvén velocity v_A is defined by $v_A^2 = B_0^2/4\pi\rho$, where ρ is the fluid density.

In the case of forced reconnection, by contrast, the amplitude of the current singularity is determined by the applied perturbation and is essentially constant for small islands. If the amplitude of the displacement imposed on the surface of the plasma is ξ_e, the island width will grow as

$$\frac{w}{r_s} \cong K''\left(\frac{\xi_e}{r_s}\right)^{3/4}\left(\frac{\lambda t}{\tau_K}\right)^{1/2}.$$

These growth rates are in good agreement with the numerical results. The qualitative predictions of the equilibrium theory, such as the separation of the X-point into two Y-points and the peaking of the current in the center of the ribbon joining the two Y-points are also consistent with the numerical simulations.[7,8]

It should be noted, with respect to this model, that steady growth requires that the viscosity be much larger than the resistivity in order for the kinetic energy produced during the reconnection to be dissipated within the layer. When this condition is not fulfilled, the simulations of Park *et. al.*[7] indicate that ideal oscillations are superposed upon the steady growth described by eq. (37). However, these superposed oscillations do not significantly alter the average rate of reconnection.

An important question with respect to the tearing-layer problem concerns the role played by turbulence. Numerical simulations of helically symmetric systems suggest the likelyhood that the tearing-layer undergoes a transition to turbulence as the resistivity is decreased.[8] In three dimensions, however, the nonlinear coupling between modes with different helicities is likely to lead to much lower thresholds for turbulence, and to significantly faster growth in the turbulent regime. The effect of magnetohydrodynamic turbulence on the growth of constant-ψ islands has been investigated by Diamond *et.al.*[25]

We conclude by remarking that the similarity between free reconnection and forced reconnection, as far as the presence of a current-layer is concerned, suggests that forced reconnection experiments may yield some insight into the current-layer reconnection processes.

Acknowledgements

The author is grateful to Marie-Noëlle Bussac, Richard Hazeltine, Ahmet Aydemir and Jean-François Luciani for encouragement and many helpful discussions. The author also wishes to thank Guy Laval and Marie-Noëlle Bussac for providing him with the opportunity to complete this paper at the Centre de Physique Théorique of the Ecole Polytechnique.

This work was supported in part by the U.S. Department of Energy under Contract No. DE-FG05-80ET-53088.

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Figure Captions

1. Flux-function in the initial cylindrical state, as a function of the volume contained within the flux-surface.

2. Magnetic island showing the mapping between the initial flux-surfaces (dashed line) and the final reconnected flux surface.

3. General features of the function $g(\theta)$. For $f<g_{max}$, the flux surfaces are "trapped" between the values of θ given by $f=g(\theta)$.

4. General features of the function $\chi(f)$ (a) and its inverse $f(x)$ (b) showing the three solution branches for $x<\chi_{max}$. In order for the flux-surfaces to be continuous it is necessary for g to be such that $\chi_{max}=x_s$.

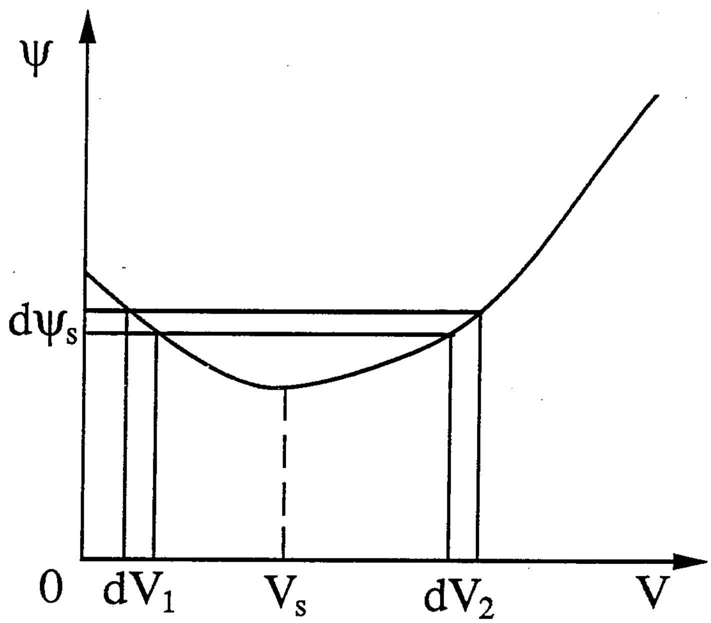

fig. 1

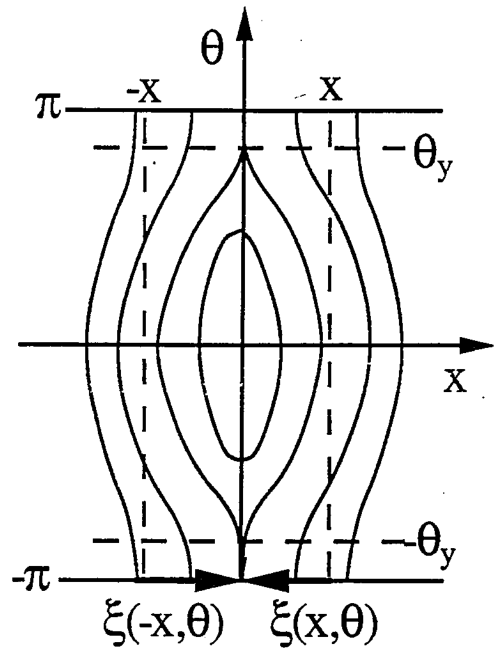

fig. 2

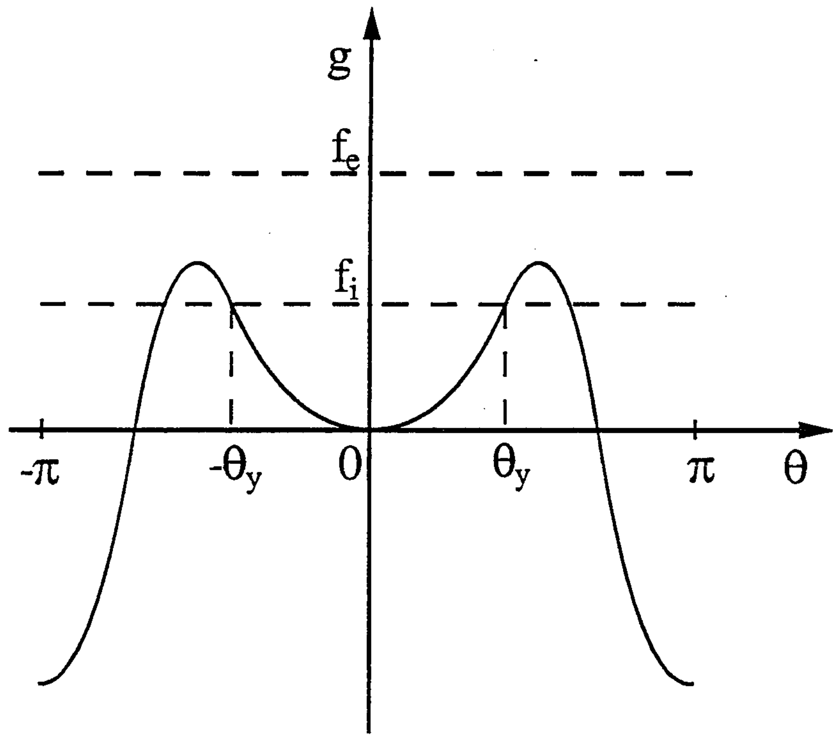

fig. 3

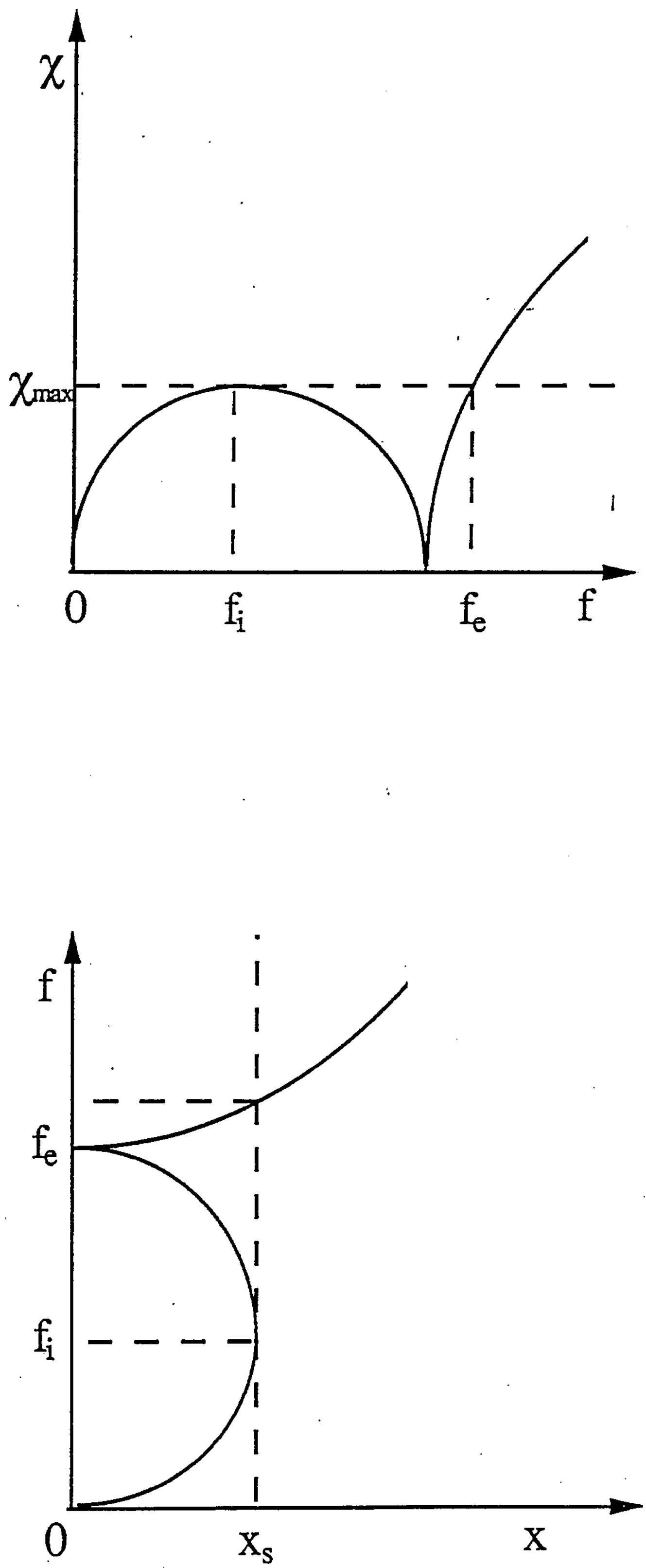

fig. 4