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Saturation of a Single Mode Driven by an Energetic Injected Beam III. Alfvén Wave Problem

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Abstract

The saturation amplitude of Alfvén waves, excited by alpha particles produced in an ignited tokamak, is estimated. The formalism that has been developed to describe the saturation of a single mode, is generalized to toroidal geometry. The saturation level is estimated for the toroidal Alfvén gap mode. The alpha particle radial flux due to the finite wave amplitude is found to produce relatively weak energy losses compared to the usual energy drag losses.

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I. Introduction

Alpha particles created in a thermonuclear plasma can have a diamagnetic drift frequency greater than the Alfvén frequency and thereby can excite shear Alfvén wave instabilities. These waves are driven by the density gradient of the alpha particle distribution function. Alfvén waves are generally difficult to excite as they usually occur as modes in a continuous spectra that thereby have a finite damping rate.[1] However, when cylindrical and toroidal geometry is considered, shear Alfvén waves can have a discrete spectrum under special conditions.[2,3,4] The competition of the destabilizing nature of these modes compared to background dissipation mechanisms has been studied in Ref. 5. It was concluded that the toroidal Alfvén gap mode[4] is the most likely of the discrete Alfvén waves to be unstable in an ignited system. At long wavelengths this mode has a well-defined quantum number with a real frequency substantially greater than the growth rate. In this paper we attempt to describe the nonlinear saturation mechanism of this mode. We view the Alfvén wave as a well-defined standing wave that is only weakly perturbed by the alpha particles. As the wave grows we assume that the alpha particles can continue to be considered as a weak perturbation of the wave. However, the alpha particles themselves are strongly perturbed by the wave, especially in the region of particle-wave resonance. This perturbation alters the particle distribution and ultimately weakens the source of instability.

In our problem the alpha particles are assumed generated at fixed rate and isotropically at a fixed energy ($\sim 4\,\mathrm{MeV}$.) A steady-state distribution is formed by the slowing down of the particles from background plasma drag. Pitch-angle scattering of the alpha particles needs to be considered, especially for describing particles resonant with a finite amplitude standing wave. The competition of the particle source, classical transport processes and the particle response to a finite amplitude wave, determines the resonant particle interaction.

The description of how to treat this problem has been developed in previous papers.[6,7] The presentation in Ref. 7 is particularly appropriate to the application of the Alfvén wave problem. In this paper we need only extend the formalism so that the equations for Alfvén waves in toroidal geometry can be converted to the equations studied in Ref. 7. We shall use the results of Ref. 7 in this text and equations referring to Ref. 7 will be preceded by the symbol "II."

In the relaxation of an alpha particle distribution due to collisions, the drag rate, ν, is generally much larger than the diffusion rate ν_d. However, ν_d is of sufficient importance as it dominates the relaxation of alpha particles in describing the nonlinear resonance interaction with the wave. There are three regimes of collisionality for describing the nonlinear resonant particle-wave interaction.[7] In one regime (regime (a)) pitch-angle scattering is unimportant. In that case particles trapped in the wave convect across the entire eigenmode structure in a slowing down time. However, as the amplitude of the wave is small, even weak pitch-angle diffusion can scatter particles out of the trapping region. Two such scattering regimes were determined. In the low, but not negligible pitch-angle scattering regime (regime (b)), a trapped particle can convect a distance large compared to its oscillation amplitude (but less than a mode width) before being detrapped by pitch-angle scattering, and in a moderate pitch-angle diffusion regime (regime (c)), trapped particles scatter out of the trapping region before they convect an oscillation amplitude. It is the last regime that appears to be the most important for reactor parameters. Our detailed calculations will be confined to this regime.

The structure of this paper is as follows. In Sec. II we convert the equations for the Alfvén wave problem in toroidal geometry, to a standard form derived in Ref. 7. In Sec. III we calculate the specific power transfer for regime (c), of alpha particles to a finite amplitude Alfvén wave. We also estimate the wave saturation level by equating the resonant nonlinear alpha particle-wave power transfer rate to the power linearly dissipated by the wave. In

Sec. IV conclusions are presented.

II. Reduction of Alfvén Wave-Particle Interaction in Tokamak to Standard Form

To treat Alfvén waves we assume that the perturbed parallel electric field is zero, and the perturbed magnetic field is transverse to the equilibrium magnetic field. At low plasma beta such a perturbation can be represented by a potential function ξ with the properties

$$\begin{aligned} \mathbf{E} &= -\nabla \frac{\partial \xi}{\partial t} + \mathbf{b} \frac{\partial}{\partial s} \frac{\partial \xi}{\partial t} \\ \mathbf{B}_1 &= c\mathbf{b} \times \nabla \frac{\partial \xi}{\partial s} \end{aligned} \tag{1}$$

where we assume $\xi \nabla \times \mathbf{b} \ll \nabla \xi$.

We model a tokamak in the high aspect ratio limit. The magnetic field is $\mathbf{B} = B_0 \left(1 - \frac{r}{R_0} \cos\theta\right) \hat{\phi} + B_\theta(r) \hat{\theta}$ where r is the flux tube radius, θ the poloidal angle and R_0 the major radius. We denote φ as the toroidal angle, and choose an r, θ, φ right-handed coordinate system with $\mathbf{B} \cdot \hat{\phi} = B_\varphi \approx B$. The grad-$B$ drift is given by

$$\mathbf{v}_D = \frac{\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)}{\omega_c B} \mathbf{b} \times \nabla B = -v_{D0} \left(\sin\theta\, \hat{\mathbf{r}} + \cos\theta\, \hat{\boldsymbol{\theta}}\right) \tag{2}$$

with $v_{D0} = (v_\perp^2/2 + v_\parallel^2)/\omega_c R_0$, ω_c the cyclotron frequency and $v_\perp$ and $v_\parallel$ the perpendicular and parallel speeds respectively. Using the assumption that $\omega \ll \omega^*$, (with ω the mode frequency and ω^* the hot particle drift frequency) or more precisely

$$f \frac{d^2 \xi}{dt^2} \ll \frac{c}{qB} \mathbf{b} \times \nabla f \cdot \nabla \frac{\partial \xi}{\partial t} m v^2 \,,$$

the zero Larmor radius drift kinetic equation becomes

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s} + \mathbf{v}_D \cdot \nabla\right) f + \frac{c}{B} \frac{\partial f}{\partial \mathbf{r}} \cdot \left[\mathbf{b} \times \nabla \left(\frac{\partial \xi}{\partial t} + v_\parallel \frac{\partial \xi}{\partial s}\right)\right] \\ &- \nu_d(r,v) \frac{\partial}{\partial \lambda}(1 - \lambda^2) \frac{\partial f}{\partial \lambda} - \frac{\nu(r)}{v^2} \frac{\partial}{\partial v}\left((v^3 + v_I^3) f\right) - \frac{Q(r)}{4\pi v_0^2} \delta(v - v_0) = 0 \end{aligned} \tag{3}$$

where we have neglected the higher order curvature drift motion in the perturbed fields,

$$\nu_d(r,v) = \frac{2\pi e^2\, q_\alpha^2 \ln \Lambda n_e}{m^2\, v^3} \;, \nu(r) = \sum_i \frac{4\pi n_i e^2\, q_\alpha^2 \ln \Lambda}{m m_i\, v_I^3} \;, v_I^3 = \sum_i \frac{3\pi}{4} \left(\frac{m_e}{m_i}\right) \left(\frac{2T_e}{m_e}\right)^{3/2} \frac{n_i}{n_e} \;,$$

q_α the alpha particle charge, m the alpha particle mass, m_e the electron mass, m_i the background mass, note that $\nu(r)$ is independent of background ion mass, for simplicity we assume v_I independent of r, the ion species are singly charged and $n_e - n_i \ll n_e$.

We also note that $v_\parallel \dfrac{\partial}{\partial s} \doteq v_\parallel \left(\dfrac{1}{R}\dfrac{\partial}{\partial \varphi} + \dfrac{B_\theta}{rB_\varphi}\dfrac{\partial}{\partial \theta}\right)$ and we choose $\xi(\mathbf{r})$ to be of the form $\xi(\mathrm{r},t) = \xi_0(r)\sin(\omega t - n\varphi - m\theta)$.

With the distribution calculated from Eq. (3) we need to calculate the energy transfer function, which for waves with $E_\parallel = 0$, is given by

$$\begin{aligned}
\mathcal{P} &= q_\alpha \int d\varphi d\theta dr\, rR \int d^3 v \mathbf{v}_D \cdot \mathbf{E} f \\
&= -\frac{\omega q_\alpha}{2} \int d\varphi d\theta dr\, rR \int d^3 v v_{D0} \Bigg\{ \sin[\omega t - n\varphi - (m+1)\theta] \left(\frac{\partial \xi}{\partial r} - \frac{m\xi}{r}\right) \\
&\quad - \sin[\omega t - n\varphi - (m-1)\theta] \cdot \left(\frac{\partial \xi}{\partial r} + \frac{m}{r}\xi\right) \Bigg\} f
\end{aligned} \tag{4}$$

with q_α the α-particle charge.

In linear theory we put $f = f_0 + f_1$ where f_0 is determined from the kinetic equation with $\xi = 0$. Assuming $v_\parallel \dfrac{\partial}{\partial s} \gg \mathrm{v}_D \dfrac{\partial}{\partial \mathrm{r}}$ and $v_\parallel \dfrac{\partial}{\partial s} \gg \nu$, one readily finds

$$f_0 = F_0 - \frac{v_{D0}}{v_\parallel}\frac{rB}{B_\theta} \cos\theta \frac{\partial F_0}{\partial r} \;,$$

where

$$F = \frac{Q(r)}{4\pi \nu(r)\,(v^3 + v_I^3)} \;.$$

We note that the correction, $f_0 - F$, is of importance for low-m modes and has not been included in previous studies.[2,5] The linearized equation for f_1 becomes

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s} + \mathrm{v}_D \cdot \boldsymbol{\nabla}\right) f_1 = \frac{c}{rB}\frac{\partial f_0}{\partial r}\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s}\right)\frac{\partial \xi}{\partial \theta} - \frac{c}{rB}\frac{\partial f_0}{\partial r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s}\right)\xi \tag{5}$$

where as in previous treatments[6,7] we neglect the drag and diffusion term acting on f_1. In addition, $\mathcal{O}\left(\frac{r}{R} B_\theta / B\right)$ terms are neglected and we treat B on the right-hand side of Eq. (5) as a constant. By keeping the r/R corrections, we would only generate higher order corrections and significantly complicate the algebra.

Now writing

$$f_1 = h + \frac{c}{rB}\left(\frac{\partial f_0}{\partial r}\frac{\partial \xi}{\partial \theta} - \frac{\partial f_0}{\partial \theta}\frac{\partial \xi}{\partial r}\right),$$

we find

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s} + \mathbf{v}_D \cdot \boldsymbol{\nabla}\right) h = -(\mathbf{v}_D \cdot \boldsymbol{\nabla})\left(\frac{c}{rB}\frac{\partial f_0}{\partial r}\frac{\partial \xi}{\partial \theta} - \frac{c}{rB}\frac{\partial f_0}{\partial \theta}\frac{\partial \xi}{\partial r}\right)$$

$$-\frac{\partial \xi}{\partial \theta}\left(\frac{v_\parallel B_\theta}{rB}\right)\frac{\partial}{\partial \theta}\left(\frac{c}{rB}\frac{\partial f_0}{\partial r}\right) + \frac{\partial \xi}{\partial r}\left(\frac{v_\parallel B_\theta}{rB}\right)\frac{\partial}{\partial \theta}\left(\frac{c}{rB}\frac{\partial f_0}{\partial \theta}\right) - \frac{c}{rB}\frac{\partial f_0}{\partial \theta}\frac{\partial \xi}{\partial \theta}\frac{\partial}{\partial r}\left(\frac{v_\parallel B_\theta}{rB}\right). \quad (6)$$

When we substitute f_0 into the right-hand side of Eq. (6), keeping only the lowest order (linear in v_{D0}) terms in Eq. (6), we obtain

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s} + \mathbf{v}_D \cdot \boldsymbol{\nabla}\right) h = \frac{c\, v_{D0}}{rB}\frac{\partial F}{\partial r}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \xi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \xi}{\partial \theta}\right).$$

Then neglecting $\mathbf{v}_D$ and aspect ratio correction on the left-hand side and using the explicit expression for ξ we obtain

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s}\right) h = \frac{c v_{D0}}{rB}\frac{\partial F}{\partial r}\frac{\partial}{\partial \theta}\left[\frac{\partial \xi_0}{\partial r}\sin\theta\,\sin(\omega t - n\varphi - m\theta) + \frac{m\xi_0}{r}\cos\theta\,\cos(\omega t - n\varphi - m\theta)\right]$$

$$= \frac{c\, v_{D0}}{2rB}\frac{\partial F}{\partial r}\left[(m+1)\left(\frac{\partial \xi_0}{\partial r} - \frac{m\xi_0}{r}\right)\sin\psi^+ - (m-1)\left(\frac{\partial \xi_0}{\partial r} + \frac{m\xi_0}{r}\right)\sin\psi^-\right] \quad (7)$$

where $\psi^\pm = \omega t - n\varphi - (m \pm 1)\theta$.

The solution, $h = h^+ + h^-$, with each component proportional to the corresponding phase $\psi^\pm$, is

$$h^\pm = \mp \frac{c(m \pm 1)}{2Br} v_{D0}\frac{\partial F}{\partial r}\left\{\left[\cos\psi^\pm P \frac{1}{\omega - v_\parallel\left(\frac{n}{R} + \frac{(m\pm 1)}{r}\frac{B_\theta}{B}\right)}\right]\right.$$

$$- \pi \sin \psi^{\pm} \delta \left[\omega - v_{\|} \left(\frac{n}{R} + \frac{m \pm 1}{r} \frac{B_\theta}{B} \right) \right] \Bigg\} \cdot \left(\frac{\partial \xi_0}{\partial r} \mp \frac{m \xi_0}{r} \right) . \tag{8}$$

Now substituting Eq. (8) into Eq. (4), yields for the linear power transfer

$$\mathcal{P}_{\rm L} = -\sum_m c \frac{\pi \omega q_\alpha}{8} \int d^3r\, d^3v\, \frac{1}{Br} \frac{\partial F}{\partial r} v_{D0}^2 \left[(m{+}1) \left(\frac{\partial \xi_{0m}}{\partial r} - \frac{m \xi_{0m}}{r} \right)^2 \delta \left(\omega - v_{\|} \left(\frac{n}{R} + \frac{(m+1)}{r} \frac{B_\theta}{B} \right) \right) \right.$$

$$\left. + \, (m-1) \left(\frac{\partial \xi_{0m}}{\partial r} + \frac{m \xi_{0m}}{r} \right)^2 \delta \left(\omega - v_{\|} \left(\frac{n}{R} + \frac{(m-1)}{r} \frac{B_\theta}{B} \right) \right) \right] , \tag{9}$$

where we have added a summation on m, as the eigenmode can contain multiple m-values

In the nonlinear problem we shall assume ξ is small, but keep nonlinear terms associated with the resonant region where $\dot{\psi}^{\pm}$ is small. We assume $\mathbf{b} \times \nabla \ln v_{\|} \ll \mathbf{b} \times \nabla \ln \xi$, define $\tilde{f} = f + \frac{c}{B}(\mathbf{b} \times \nabla \xi) \cdot \frac{\partial f}{\partial \mathbf{r}}$, and we find

$$\widehat{D} \tilde{f} - \frac{c}{B} (\mathbf{b} \times \nabla \xi) \cdot \left(\nabla (\widehat{D} f) \right) - \frac{c}{B} (\nabla f) \cdot (\mathbf{b} \times \nabla (\mathbf{v}_D \cdot \nabla \xi))$$

$$- \, \nu_d(r,v) \frac{\partial}{\partial \lambda} (1 - \lambda^2) \frac{\partial f}{\partial \lambda} - \frac{\nu(r)}{v^2} \frac{\partial}{\partial v} [(v^2 + v_I^3) f] - \frac{Q(r)}{4 \pi v_0^2} \delta(v - v_0) = 0 \ . \tag{10}$$

with

$$\widehat{D} \equiv \left(\frac{\partial}{\partial t} + v_{\|} \frac{\partial}{\partial s} + \mathbf{v}_D \nabla \right) .$$

We note that the $\mathbf{v}_D \cdot \nabla \tilde{f}$ can now be dropped compared with $\left(\frac{\partial}{\partial t} + v_{\|} \frac{\partial}{\partial s} \right) \tilde{f}$.

To proceed further we assume that the nonlinear solution will be due to a large term independent of phase and small resonance terms depending on the phases $\psi^{\pm}$. The resonance regions are assumed sufficiently separated so that when considering, say a nonlinear resonance interaction involving ψ^{+} terms, one can neglect the effect of ψ^{-} terms as well as any other side band terms. With this approximation $\tilde{f}$ in Eq. (10) can be taken as f (as the difference $\tilde{f} - f$ is an oscillatory function in θ independent of ψ^{+} or ψ^{-}) and the term $\frac{c}{B} (\mathbf{b} \times \nabla \xi) \cdot \left(\nabla (\widehat{D} f) \right)$ which is a perturbation of the first term in Eq. (10) can be dropped. Then the kinetic equation reduces mathematically to the problem treated in Ref. 7. Specifically, taking $f = f(r, \psi^{\pm})$, we find that Eq. (10) can be rewritten as

$$\left[\omega - \left(\frac{n}{R} + \frac{B_\theta}{B}\,\frac{(m\pm 1)}{r}\right) v_{\|}\right] \frac{\partial f}{\partial \psi^{\pm}} \mp \frac{c(m\pm 1)v_{D0}}{2Br}\left(\frac{\partial \xi_0}{\partial r} \mp \frac{m}{r}\,\xi_0\right)\sin\psi^{\pm}\,\frac{\partial f}{\partial r}$$

$$-\nu_d(r,v)\,\frac{\partial}{\partial\lambda}\left(1-\lambda^2\right)\frac{\partial f}{\partial \lambda} - \frac{\nu(r)}{v^2}\,\frac{\partial}{\partial v}\left((v^3+v_I^3)f\right) - \frac{Q(r)}{4\pi v_0^2}\,\delta(v-v_0) = 0\ . \qquad (11)$$

If we assume $\dfrac{\delta r}{r} \approx \dfrac{\delta r}{\xi}\,\dfrac{\partial \xi}{\partial r} \ll 1$, where δr is the nonlinear radial excursion of a particle, Eq. (10) is quite similar to Eq. (II.2). One different aspect for the Alfvén wave problem is that some particles can pass through two separate resonance regions as they slow down.

We now follow the procedure we developed in Ref. 7. The characteristic equations for the nonlinear trajectories are

$$\dot{\psi}^{\pm} = \omega - \left(\frac{n}{R} + \frac{(m\pm 1)}{R}\,\frac{1}{q(r)}\right) v_{\|}$$

$$\dot{r} = \mp\frac{(m\pm 1)v_{D0}c}{2Br}\left(\frac{\partial \xi_0}{\partial r} \mp \frac{m}{r}\,\xi_0\right)\sin\psi^{\pm} \qquad (12)$$

where $q(r) = \dfrac{r}{R}\,\dfrac{B}{B_\theta(r)}$. By expanding Eq. (12) about $r = r^*_{\pm}$, we obtain a first integral

$$E = \frac{(r - r^*_{\pm})^2}{2} \pm A^{\pm}\cos\psi^{\pm} \qquad (13)$$

where $r^*_{\pm}$ satisfies the condition that

$$\omega - \frac{n v_{\|}}{R} - \frac{(m\pm 1)v_{\|}}{Rq(r^*_{\pm})} = 0 \qquad (14)$$

and

$$A^{\pm} = \frac{v_{D0}c}{2B}\,\frac{L_s}{v_{\|}}\left(\frac{\partial \xi_0(r^*)}{\partial r^*} \mp \frac{m\xi_0(r^*)}{r^*}\right) \qquad (15)$$

with $L_s^{-1} = \dfrac{r^*}{B}\,\dfrac{d}{dr^*}\left(\dfrac{B_\theta(r^*)}{r^*}\right) = \dfrac{r^*}{R}\,\dfrac{d}{dr^*}\left(1/q(r^*)\right)$. The region

$$-\,|A^{\pm}| < E < |A^{\pm}| \qquad (16)$$

is the trapping region. The period of oscillation of a deeply trapped particle is found to be

$$T_b^{\pm} \equiv \frac{2\pi}{\omega_b} = 2\pi \left[\frac{(m \pm 1)^2}{L_s} \frac{c v_{D0} v_{\|}}{2 r_{\pm}^{*2} B} \left(\frac{\partial \xi_0}{\partial r} (r_{\pm}^{*}) \mp \frac{m \xi_0 (r_{\pm}^{*})}{r_{\pm}^{*}} \right) \right]^{-1/2} . \tag{17}$$

We transform from r to E, and find the equation for f can be written as (to accommodate notation we now explicitly write the equation for the positive side band and suppress the negative side band term; the $\pm$ notation will now be used to denote whether $r - r^*$ is greater than zero (+) or less than zero (−));

$$\begin{aligned}
&\mp \frac{(m+1) v_{\|}}{r^* L_s} \frac{\partial f^{\pm}}{\partial \psi} |r(E,\psi) - r^*| = \frac{\nu(r^*)}{v^2} \frac{\partial}{\partial v} \left[(v^3 + v_I^3) f^{\pm} \right] + \frac{Q(r^*)}{4\pi v_0^2} \delta(v - v_0) \\
&\pm |r(E,\psi| - r^*| \left\{ \frac{\nu'(r^*)}{v^2} \frac{\partial}{\partial v} \left[(v^3 + v_I^3) f^{\pm} \right] + \frac{Q'(r^*)}{4\pi v_0^2} \delta(v - v_0) \right\} \\
&\qquad - \frac{\nu'(r^*)}{v^2} \frac{\partial r^*}{\partial v} (v^3 + v_I^3) |r(E,\psi) - r^*|^2 \frac{\partial f^{\pm}}{\partial E} \\
&\mp \frac{\nu(r^*)}{v^2} (v^3 + v_I^3) |r(E,\psi) - r^*| \frac{\partial r^*}{\partial v} \frac{\partial f^{\pm}}{\partial E} + \nu_d(r^*, v) \left(\frac{\partial}{\partial \lambda} \mp |r(E,\psi) - r^*| \frac{\partial r^*}{\partial \lambda} \frac{\partial}{\partial E} \right) (1 - \lambda^2) \\
&\quad \cdot \left(\frac{\partial}{\partial \lambda} \mp |r(E,\psi) - r^*| \frac{\partial r^*}{\partial \lambda} \frac{\partial}{\partial E} \right) f^{\pm} \pm \frac{\nu'(r^*)}{v^2} |r(E,\psi) - r^*| \frac{\partial A}{\partial v} (v^3 + v_I^3) \frac{\partial f^{\pm}}{\partial E} \cos\psi \\
&+ \frac{\nu(r^*)}{v^2} \frac{\partial A}{\partial v} (v^3 + v_I^3) \frac{\partial f^{\pm}}{\partial E} \cos\psi + \cdots
\end{aligned} \tag{18}$$

where $|r(E,\psi) - r^*| = \sqrt{2}\,(E - A\cos\psi)^{1/2}$. This is essentially the same equation (Eq. (II.13)) as obtained for the electrostatic problem of Ref. 7.

The power transfer function, $\mathcal{P}$, is given by Eq. (4). We transform from r to E, use that

$$2 \frac{v_{\|} B}{L_s c} \frac{\partial}{\partial \psi} |r(E,\psi) - r^*| = v_{D0} \frac{\left(\frac{\partial \xi_0}{\partial r} (r^*) - m \frac{\xi_0(r^*)}{r^*} \right) \sin\psi}{|r(E,\psi) - r^*|} , \tag{19}$$

and perform an integration by parts, to obtain

$$\mathcal{P} = \frac{2\pi R q_\alpha B \omega}{c} \int d^3v \int_{-|A|}^{\infty} dE \int_{\psi_{\min}}^{\psi_{\max}} d\psi \, \frac{|r(E,\psi) - r^*| r^* v_{\|}}{L_s} \frac{\partial (f^+ + f^-)}{\partial \psi} , \tag{20}$$

where $\psi_{\min}$ and $\psi_{\max}$ have the obvious definition. Except for trivial factors, this is exactly the same as Eq. (II.14). Thus, the electromagnetic Alfvén wave problem has been reduced exactly to the electrostatic wave problem in a sheared magnetic field, and can be solved in an identical manner. There are three separate regimes of solution, $\nu_d\omega^2/\nu\omega_b^2 < 1$, $\nu_d\omega^2/\omega_b^2 > \nu > \nu_d\omega/\omega_b$, and $\nu_d\omega/\omega_b > \nu$. The last region is the one that is physically the most relevant, and is the one we shall discuss in detail.

III. Calculation of Power Transfer and Wave Saturation

As in the electrostatic wave problem, if $\nu_{\text{eff}} \equiv \nu_d\omega^2/\omega_b^2 < \omega_b$, the most important contribution to the nonlinear power transfer, $\mathcal{P}_{\text{NL}}$ comes from the diffusion term including the interface region between passing and trapped particles. Assuming $\nu_d\omega/\omega_b > \nu$, leads to the equation (compare with corresponding terms of Eq. (II.24))

$$\mathcal{P}_{\text{NL}} = -\frac{(2\pi)^2 B\omega R q_\alpha}{c(m+1)} \int d^3v \nu_d(r^*, v) \left(\frac{\partial r^*}{\partial \lambda}\right)^2 (1-\lambda^2) r^{*2} \left\{ \left\langle \left(r(A,\psi) - r^*\right)^2 \right\rangle \frac{\partial \Delta F\,(E = A(1+\delta))}{\partial E} \right.$$

$$+ \int_A^\infty dE \left[\left\langle \left(r(E,\psi) - r^*\right) \frac{\partial}{\partial E} \left(\left(r(E,\psi) - r^*\right) \frac{\partial \Delta F}{\partial E} \right) \right\rangle \right.$$

$$\left.\left. - \left\langle r(E,\psi) - r^* \right\rangle \frac{\partial}{\partial E} \left(\left\langle r(E,\psi) - r^* \right\rangle \frac{\partial \Delta F}{\partial E} \right) \right] \right\} \tag{21}$$

where

$$\langle G(\psi) \rangle = \int_0^{2\pi} \frac{d\psi}{2\pi} G(\psi) \qquad \text{and} \qquad \Delta F = F^+ - F^-$$

with ΔF satisfying the equation

$$\frac{\partial}{\partial E}\left\langle\sqrt{2}\,(E-A\cos\psi)^{1/2}\right\rangle\frac{\partial\Delta F}{\partial E}=0\;. \tag{22}$$

The solution is the same as in Ref. 7 (see Eq. (II.57))

$$\frac{\partial\Delta F}{\partial E}=\frac{1}{2\pi(v^3+v_I^3)}\frac{\partial}{\partial r^*}\left(\frac{Q(r^*)}{\nu(r^*)}\right)\Big/\left\langle\sqrt{2}\,(E-A\cos\psi)^{1/2}\right\rangle\;, \tag{23}$$

where r^* is the resonant position. Substituting Eq. (23) into Eq. (21), assuming that the eigenmode is localized radially at $r=r_0\approx r^*$, defining $\nu_{d0}=\nu_d\,v^3/v_0^3$, and including the two side band terms (we again use the $\pm$ notation to refer to upper and lower side bands), we find

$$\mathcal{P}_{\rm NL}=\frac{-2\pi q_\alpha\omega B}{c}\,Rr_0^2\,\nu_{d0}\,\frac{\partial}{\partial r_0}\left(\frac{Q(r_0)}{\nu(r_0)}\right)I_2\,v_0^3$$

$$\sum_m\int\frac{d^3v_I(1-\lambda^2)}{v^3(v^3+v_I^3)}\left[\left(\frac{\partial r_+^*}{\partial\lambda}\right)_m^2\frac{\sqrt{|A_m^+|}}{(m+1)}+\left(\frac{\partial r_-^*}{\partial\lambda}\right)_m^2\frac{\sqrt{|A_m^-|}}{(m-1)}\right] \tag{24}$$

with $I_2=1.38$ (see Eqs. (II.61) and (II.62) and a summation on mode amplitudes has been added.

As previously, we can calculate the ratio of $\mathcal{P}_{\rm NL}/\mathcal{P}_{\rm L}$ and find

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}}\approx\frac{\nu_{d0}\,\omega^2}{\omega_b^3}\;.$$

Let us evaluate this ratio somewhat more precisely for the Alfvén Toroidal Gap mode. This mode has the properties that it is defined by a radius r_0 where

$$k_{||}(n,m)\equiv\frac{n}{R}+\frac{mB_\theta(r_0)}{Br_0}=-k_{||}(n,m-1)\;. \tag{25}$$

These conditions determine the point r_0 and mode frequency such that

$$\begin{aligned} q(r_0)&=\frac{Br_0}{B_\theta(r_0)R}=\frac{-(2m-1)}{2n}\equiv q_0\\ \omega&=k_{||}v_A=v_A\left(\frac{n}{R}+\frac{mB_\theta(r_0)}{r_0B}\right)=\frac{v_A}{2q_0R}\;. \end{aligned} \tag{26}$$

(A typical mode is $m = -1$, $n = 1$, giving $q_0 = 3/2$).

For simplicity we assume a mode $\xi(r)$ localized close to r_0 with $\dfrac{\partial \xi}{\partial r} \equiv \xi/\Delta$ with $\Delta < r_0/m$ so that $\dfrac{\partial \xi}{\partial r} + \dfrac{m}{r}\xi \approx \dfrac{\partial \xi}{\partial r}$. We take as a test function

$$\xi(r,\theta) = \hat{\xi} \exp\left(-(r-r_0)^2/\Delta^2\right) \{\exp(im\theta) + \exp\left[i(m-1)\theta\right]\}$$

and choose the negative side band for the m amplitude and the positive side band for the $m-1$ amplitude, and we assume that the other side bands do not resonate with alpha particles. (It is readily shown that the other side bands need $|v_\parallel| \approx 3v_A$ and thus do not contribute if $v_0 < 3v_A$). The particle resonance condition for the m amplitude is

$$\omega = \frac{v_A}{2q_0 R} = v_\parallel \left[\frac{n}{R} + (m-1)\frac{B_\theta(r^*)}{Br^*}\right] .$$

Then, expanding about $r^* = r_0$, using that $\dfrac{n}{R} + \dfrac{m}{q_0 R} = \dfrac{1}{2q_0 R}$, the resonance condition is

$$\begin{aligned} v_\parallel \equiv \lambda v &= \frac{-v_A}{1 - 2(r^* - r_0)\dfrac{(m-1)r_0}{B_\theta(r_0)}\left(\dfrac{B_\theta(r_0)}{r_0}\right)'} \\ &\doteq -v_A \left[1 + \frac{2(r^* - r_0)(m-1)r_0}{B_\theta(r_0)}\left(\frac{B_\theta(r_0)}{r_0}\right)'\right] . \end{aligned} \tag{27}$$

If we assume $2\Delta(m-1)\dfrac{r_0}{B_\theta}\left(\dfrac{B_\theta(r_0)}{r_0}\right)' \ll 1$, it follows that all the resonant particles have a parallel velocity near the negative of the Alfvén speed. Similarly, resonant particles for the $m-1$ amplitude are near the Alfvén speed and the resonance condition is

$$\lambda v = v_A \left[1 - \frac{2(r^* - r_0)m}{B_\theta(r_0)}\left(\frac{B_\theta(r_0)}{r_0}\right)'\right] .$$

Using Eqs. (9) and (24) we can write the expressions for $\mathcal{P}_{\rm L}$ and $\mathcal{P}_{\rm NL}$ as

$$\mathcal{P}_{\rm L} = \frac{-\pi^3 c\,\omega\, q_\alpha R}{4B}\left(\frac{Q}{\nu}\right)' \int_{v_A}^{v_0} \frac{dv\, v^2}{v^3 + v_I^3} \int_{v_A/v}^{1} d\lambda\, v_{D0}^2$$

$$\left[\frac{(m-1)\left|\dfrac{\partial \xi_m(r^*)}{\partial r^*}\right|^2}{\left|\lambda v(m-1)B^{-1}\left(\dfrac{B_\theta(r^*)}{r^*}\right)'\right|} + \frac{m\left|\dfrac{\partial \xi_{m-1}(r^*)}{\partial r^*}\right|^2}{\left|\lambda v m B^{-1}\left(\dfrac{B_\theta(r^*)}{r^*}\right)'\right|} \right] \tag{28}$$

$$\mathcal{P}_{\rm NL} = -\frac{(2\pi)^2 \omega\, B R r_0^2\, q_\alpha \left(\dfrac{Q}{\nu}\right)' \nu_{d0}\, I_2\, v_0^3}{c} \cdot \int_{v_A}^{v_0} \frac{dv\, v^2}{v^3(v^3+v_I^3)} \int_{v_A/v}^{1} d\lambda (1-\lambda^2)$$
$$\cdot \left[\left(\frac{\partial r^*}{\partial \lambda}\right)_m^2 \frac{\sqrt{|A_m^-(r^*)|}}{m-1} + \left(\frac{\partial r^*}{\partial \lambda}\right)_{m-1}^2 \frac{\sqrt{|A_{m-1}^*(r^*)|}}{m} \right] . \tag{29}$$

We use

$$\left(\frac{\partial r^*}{\partial \lambda}\right)_m = \frac{-r_0 L_s\, v}{2(m-1) q_0\, R\, v_A} ,$$
$$\left(\frac{\partial r^*}{\partial \lambda}\right)_{m-1} = \frac{-r_0 L_s v}{2m\, q_0 R v_A}$$

and we define $\overline{A} = c v_A\, L_s \widehat{\xi}/(\Delta \omega_c R B)$ and then find with the subscripts and superscripts on A and r^* suppressed,

$$A = \overline{A}\, \frac{(r^* - r_0)}{\Delta} \exp\left(-\frac{(r^*-r_0)^2}{\Delta^2}\right) \frac{v_A}{2\lambda v} \left[\left(\frac{\lambda v}{v_A}\right)^2 + \frac{v^2}{v_A^2} \right] .$$

Now we transform the λ integration to r^*, note that we can approximately set $\lambda v = v_A$, set $v^3 + v_I^3 \doteq v^3$, and we find that $\mathcal{P}_{\rm NL}/\mathcal{P}_{\rm L}$ can be written as

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} = \frac{4\, r_0^3 L_s^3\, \nu_{d0}\, v_0^3\, I_2}{\pi v_A^4 q_0^2 R^2\, \overline{A}^{3/2}}$$

$$\cdot \frac{\displaystyle\int_1^{v_0/v_A} \frac{dy}{y^5}(y^2-1)\left(\frac{1+y^2}{2}\right)^{1/2} \int_0^\infty dx\, x^{1/2} \exp\left(-\frac{x^2}{2}\right)\left[\frac{1}{(m-1)\,|m-1|} + \frac{1}{|m|\,m}\right]}{\displaystyle\int_1^{v_0/v_A} \frac{dy}{y^2}\left(\frac{1+y^2}{2}\right)^2 \int_0^\infty dx\, x^2 \exp(-2x^2)(2m-1)} . \tag{30}$$

We note that

$$\frac{\displaystyle\int_0^\infty dx\, x^{1/2} \exp\left(-\frac{x^2}{2}\right)}{\displaystyle\int_0^\infty dx\, x^2 \exp(-2x^2)} \approx 6.6$$

and

$$I_3 \equiv \frac{\displaystyle\int_1^{v_0/v_A} \frac{dy}{y^5}(y^2-1)\left(\frac{1+y^2}{2}\right)^{1/2}}{\displaystyle\int_1^{v_0/v_A} \frac{dy}{y^2}\left(\frac{1+y^2}{2}\right)^2} \longrightarrow \begin{cases} \left(\dfrac{v_0}{v_A}-1\right), \text{if} & \dfrac{v_0}{v_A}-1 \ll 1 \\ 6.2\,\dfrac{v_A^3}{v_0^3}, & \text{if} \quad \dfrac{v_0}{v_A} \gg 1 \end{cases} \quad . \tag{31}$$

The maximum value of I_3 is ≈ 0.17.

Now using Eq. (17), we define a mean bounce frequency as

$$\varpi_b^2 = \frac{m^2 \overline{A} v_A^2}{2r_0^2 L_s^2} \; .$$

We then find

$$\frac{\mathcal{P}_{\mathrm{NL}}}{\mathcal{P}_{\mathrm{L}}} = G\left(\frac{v_A}{2q_0R}\right)^2 \frac{\nu_{d0}\, v_0^3}{\varpi_b^3\, v_A^3} \frac{m^3\left[\dfrac{1}{(m-1)|m-1|}+\dfrac{1}{m|m|}\right]}{(2m-1)} \tag{32}$$

where $G \approx 4.1\, I_3$. As $\omega = v_A/2q_0R$, this result agrees to within a numerical factor, with our rough estimates of $\mathcal{P}_{\mathrm{NL}}/\mathcal{P}_{\mathrm{L}}$ if we use $\varpi_{b,m}^- \approx \omega_{b,m-1}^+$.

Saturation is predicted when

$$\mathcal{P}_{\mathrm{NL}} = \mathcal{P}_d \tag{33}$$

where $\mathcal{P}_d$ is the background dissipation. Now, the mean perturbed magnetic field, $\overline{\delta B}$, is given by

$$\frac{\overline{\delta B}}{B} \approx \frac{\left|c\mathbf{b} \times \nabla \dfrac{\partial \xi}{\partial s}\right|}{B} \approx \frac{c\hat{\xi}}{2Bq_0R\Delta} \; .$$

Then using the expression for ϖ_b from Eqs. (32) and (33), we have

$$\varpi_b^2 \approx \frac{m^2\, v_A^3 q_0}{L_s r_0^2 \omega_c} \frac{\overline{\delta B}}{B} \; .$$

Thus at saturation we have

$$\frac{\overline{\delta B}}{B} \approx \left(\frac{\nu_{d0} R}{4v_A}\right)^{2/3} \frac{L_s r_0^2 \omega_c}{v_A (q_0 R)^2} \frac{G^{2/3}}{q_0 (m-1)^2} \left(\frac{v_0}{v_A}\right)^2 \left(\frac{\mathcal{P}_{\mathrm{L}}}{\mathcal{P}_d}\right)^{2/3} \; . \tag{34}$$

As an example let us consider a reactor system with parameters $B \equiv 50\,\mathrm{kg}$, $n = 10^{14}\,\mathrm{cm}^{-3}$, $R = 4\,\mathrm{m}$, $r_0 = 1\,\mathrm{m}$, $T_e = 15\,\mathrm{keV}$ and an effective atomic number of 2.5. Then $v_A = 6.9 \times 10^8\,\mathrm{cm/sec}$, $v_0 = 1.4 \times 10^9\,\mathrm{cm/sec}$ and $\nu_{d0} = 1\,\mathrm{sec}^{-1}$. For simplicity we take $L_s = q_0 R$, and we find

$$\frac{\overline{\delta B}}{B} \approx \frac{1.3 \times 10^{-5}}{q_0^{4/3}\, m^2} \left(\frac{G\mathcal{P}_{\mathrm{L}}}{\mathcal{P}_d}\right)^{2/3} < \frac{1.0 \times 10^{-5}}{q_0^{4/3}\, m^2} \left(\frac{\mathcal{P}_{\mathrm{L}}}{\mathcal{P}_d}\right)^{2/3} . \tag{35}$$

In Ref. 5 $\mathcal{P}_{\mathrm{L}}/\mathcal{P}_d \sim 3$ was calculated for typical reactor parameters.

IV. Conclusion

A formalism has been developed to describe the saturation of Alfvén waves excited by the density gradient free energy drive from alpha particles created in an ignition system. In the most relevant region of physical interest the saturation level for the perturbed magnetic field is found to scale according to Eq. (34) and for typical reactor parameters a saturation level of $\delta B/B \approx 2 \times 10^{-5}$ is to be expected. This saturation level is low enough to prevent appreciable anomalous loss of alpha particle energy. Only particles trapped in the wave diffuse an appreciable distance across field lines. Their rate of flow is roughly $\dot{r} = \nu\, r$ and only a fraction f of the alpha particles ($f \approx A^{1/2}/\Delta$) are trapped. From Eqs. (35) and (36) we have f less than one percent. Thus, the anomalous energy loss of alpha particles due to outward flow is negligible compared to the energy deposited by drag directly to the background plasma. Thus, if our description of a discrete mode for the particle-Alfvén wave interaction is valid, it appears that even if Alfvén waves are excited the quality of plasma containment will not be adversely affected.

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