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Saturation of a Single Mode Driven by an Energetic Injected Beam II. Electrostatic "Universal" Destabilization Mechanism

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Abstract

The formalism to describe the saturation of a discrete mode that is destabilized by hot particles fed by neutral beam injection is extended. The destabilization mechanism described in this work arises from the density gradient in the distribution function formed from a spatially inhomogeneous source. Energetic particles are injected at a fixed speed and collisionally relax through drag and pitch-angle scattering with the background plasma. The distribution formed is solved self-consistently in the presence of a finite amplitude wave in a sheared magnetic field. Three regimes of collisionality are found and the expressions for the nonlinear wave particle power transfer is determined in each regime. With the dissipation processes of the background plasma given, the wave saturation level is then determined. When pitch-angle scattering is sufficiently weak, particles trapped in a wave convect across the magnetic field as they slow down, a phenomenon similar to the Ware pinch.

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I. Introduction

This is the second of a series of papers[1,2] where we investigate the saturation level of a plasma destabilized by a beam injected at high energy. In the first paper a formalism was generated and applied to a one-dimensional plasma wave problem when a single discrete plasma wave is excited. The classical transport processes effecting the high energy particles that were included were drag and particle annihilation. A formalism, similar to a neoclassical transport calculation,[3] was then developed to describe the particle-wave resonance interaction with a finite amplitude wave. In this work we extend the formalism to describe the excitation of a plasma wave that is driven by the density gradient free energy source of hot particles. The additional transport process of pitch-angle diffusion is included in the dynamics of the injected beam particles. The annihilation mechanism is discarded. It was previously needed to establish a destabilizing particle distribution in the plasma. Now we use a neutral beam source that varies across the sheared magnetic field. This gives rise to a steady-state distribution function with a spatial gradient perpendicular to the magnetic field which produces the "universal" instability drive[4,5] We shall investigate the electrostatic wave-particle interaction for a finite amplitude discrete wave when the "universal" instability drive is tapped.

Only a single well-defined discrete wave is assumed to be excited. This assumption is artificial for drift wave problems for which the "universal" instability mechanism most commonly applies. It is motivated so that we may ultimately describe the excitation of Alfvén waves by alpha particles in a tokamak.[2] Most of the formalism needed to describe the Alfvén problem can be developed for the simpler electrostatic wave problem in slab geometry. The Alfvén wave problem is further complicated by the toroidal and spatially inhomogeneous poloidal aspects of the geometry. These aspects are treated in Ref. 2. For the electrostatic problem in a sheared magnetic field we can treat the generic formalism of resonant particles with a density gradient being continuously driven and simultaneously relaxing by classical transport in the presence of a finite amplitude wave. The power transfer between resonant particles and the electrostatic field is calculated. At sufficiently high field amplitude the wave particle resonance interaction reduces the power transfer to a level comparable to the ambient dissipative processes present. At this stage a saturated steady-state wave can be established.

We find that the particle-wave interaction does not produce a simple flattening of the spatial gradient of the resonant particles. Instead there is a strong nonlinear effect at the transition between passing and trapped particles because trapped particles convect across magnetic field lines while the passing particles remain on a magnetic surface. Thus at the resonant boundary the source of passing particles comes from an injected particle source located at the original spatial point of injection. The passing particles slow down from high energy and pass through the resonance region. On the other hand, the source of trapped particles comes from points remote to the observation point as they are convected from the point of trapping across the magnetic field to the point of observation. This type of phenomena has been noted by Tennyson[6] to explain particle spreading observed in accelerator storage rings. There particles interacting with stray fields are trapped in island structures and are then able to convect a long distance in phase space because of their adiabaticity conservation. The mechanism of convection in our problem also resembles the Ware pinch.[7] For our problem there are three regions of collisionality to describe these processes and they are given in detail in the text.

The structure of this paper is as follows. In Sec. II we develop the formalism and derive the basic equations to describe the particle-wave interaction with finite amplitude wave. In Sec. III these equations are solved in the different regimes of collisionality. Section IV is devoted to conclusions.

II. Formalism

We shall solve the kinetic equation for the distribution f, of high energy particles of mass m and charge $q_\alpha = Ze$ in the presence of a standing electrostatic wave in a sheared magnetic field. Particles are supplied by a source varying spatially in x and injected at a speed v_0. Collisional effects are included that describe slowing down and pitch angle scattering from the background plasma. We shall assume $\omega^* \gg \omega$ where ω^* is the drift wave frequency of the hot species. The equation is

$$\frac{\partial f}{\partial t} + v_\| \frac{\partial f}{\partial s} - \frac{q_\alpha}{m}\frac{\partial \varphi}{\partial s}\frac{\partial f}{\partial v_\|} + c\frac{\mathbf{b}\times\nabla\varphi}{|\mathbf{B}|}\cdot\frac{\partial f}{\partial \mathbf{r}} = \nu(x)\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{\mathbf{v}}{|\mathbf{v}|^3}(v^3+v_I^3)f\right)$$

$$+\frac{m_i}{m}\frac{\nu(x)}{2}v_I^3\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{\mathbf{I}v^2-\mathbf{vv}}{v^3}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} + \frac{Q(x)}{4\pi v_0^2}\,\delta(v-v_0) \tag{1}$$

where the slowing down rate $\nu(x) = 4\pi n_e e^2 q_\alpha^2 \ln\Lambda / m_i v_I^3$, $v = |\mathbf{v}|$ and

$$v_I^3 = \frac{3\pi^{1/2}}{4}(m_e/m_i)(2T_e/m_e)^{3/2}$$

(for simplicity we treat v_I as a constant and assume the background plasma is a single ion species with atomic number unity), n_e is the background density, m_e the electron mass, m_i the ion mass of the background plasma and T_e the electron temperature. $Q(x)$ is the particle input rate where the particles are injected isotropically in velocity space with a speed v_0, s is the distance along a field line. The magnetic field $\mathbf{B}$, is sheared with $\mathbf{B} = B_0\left(\hat{\mathbf{z}} + \frac{x}{L_s}\hat{\mathbf{y}}\right)$. For the wave $\varphi(\mathbf{r}, t)$ we choose

$$\varphi(\mathbf{r},t) = \varphi_0(x)\cos\psi$$

with $\psi = (\omega t - k_y y) \doteq \omega t - k_y\eta - k_\|(x)s$ where $\eta = y - \frac{xz}{L_s}$, $s = z + \frac{xy}{L_s}$, $k_\| = \frac{k_y x}{L_s}$. By assuming $\omega \approx k_\| v \ll \omega^* \left(\omega^* = \frac{v_\| k_y}{\omega_c}\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial v_\|}\right)$, we can neglect the $q_\alpha \frac{\partial \varphi}{\partial s}$ in Eq. (1) compared to the $\mathbf{b}\times\nabla\varphi$ term. The distribution function f is then a function of x and ψ, and Eq. (1) can be written as

$$(\omega - k_\|(x)v_\|)\frac{\partial f}{\partial \psi} - \frac{ck_y}{B}\varphi_0(x)\sin\psi\frac{\partial f}{\partial x} - \cos\psi\frac{ck_y}{B}\frac{\partial \varphi_0}{\partial x}\frac{\partial f}{\partial \psi}$$

$$= \frac{\nu(x)}{v^2}\frac{\partial}{\partial v}\left[(v^3+v_I^3)f\right] + \nu_d(x,v)\frac{\partial}{\partial\lambda}(1-\lambda^2)\frac{\partial f}{\partial\lambda} + \frac{Q(x)\delta(v-v_0)}{4\pi v_0^2} \;. \quad (2)$$

where $v = |\mathbf{v}|$, $\lambda = \mathbf{v}\cdot\mathbf{B}/vB \equiv$ pitch angle parameter, and $\nu_d(x,v) = m_i\nu(x)v_I^3/(2v^3m)$.

Equation (2) determines the distribution function needed for calculating the average power transfer of the particles' energy into the wave. The power transfer of an electrostatic wave from the wave field to the particles in a strong magnetic field is given by

$$\mathcal{P} = \frac{k_y}{2\pi}\int dx \int_0^{2\pi/k_y} dy\; E_{\|}j_{\|} = -\frac{k_y q_\alpha}{2\pi}\int dx \int_0^{2\pi/k_y} dy\; \frac{\partial\varphi}{\partial s}\int d^3v\; v_{\|}\,f$$

$$= -q_\alpha\omega\int dx\,\varphi_0(x)\int_0^{2\pi}\frac{d\psi}{2\pi}\sin\psi\int d^3v\left(1+\frac{k_{\|}\delta v_{\|}}{\omega}\right)f \;, \quad (3)$$

where $\delta v_{\|} = v_{\|} - \omega/k_{\|}$. We anticipate that only a sharply peaked region of f around $v_{\|} = \omega/k_{\|}$ contributes to the integral so that we can neglect the $k_{\|}\delta v_{\|}/\omega$ term.

First we calculate the power transfer, $\mathcal{P}_{\mathrm{L}}$, from linear theory. We then need to solve Eq. (2) when φ_0 is arbitrarily small (the standard linear case). We choose $f = f_0 + f_1$, where f_1 is first order, so that f_0 and f_1 satisfy

$$\nu_d(x,v)\frac{\partial}{\partial\lambda}(1-\lambda^2)\frac{\partial f_0}{\partial\lambda} + \frac{\nu(x)}{v^2}\frac{\partial}{\partial v}\left((v^3+v_I^3)f_0\right)\Big|_{x,\lambda} + \frac{Q(x)\delta(v-v_0)}{4\pi v_0^2} = 0 \quad (4a)$$

$$\left(\omega - \frac{k_y x}{L_s}v_{\|}\right)\frac{\partial f_1}{\partial\psi} - \frac{ck_y\varphi_0\sin\psi}{B}\frac{\partial f_0}{\partial x} \quad (4b)$$

$$-\frac{\nu(x)}{v^2}\frac{\partial[(v^3+v_I^3)f_1]}{\partial v} - \nu_d(x,v)\frac{\partial}{\partial\lambda}(1-\lambda^2)\frac{\partial f_1}{\partial\lambda} = 0 \;.$$

The solution to the equilibrium distribution is

$$f_0 = \frac{S(x)\theta(v_0-v)}{v^3+v_I^3} \quad (5)$$

where $S(x) = \frac{Q(x)}{4\pi}\Big/\nu(x)$. It can be shown that the collisional terms in Eq. (4b) can be ignored if

$$\frac{\nu}{\omega}\left(\frac{v}{\Delta v}\right)^2 \ll 1 \quad \text{and} \quad \frac{\nu_d}{\omega}\left(\frac{v}{\Delta v}\right)^3 \ll 1 \quad (6)$$

where Δv is the width of the equilibrium distribution function at resonance. In order to fulfill the conditions near the injected energy, we may suppose the injection source has a sufficiently broad spread to satisfy (6), but is otherwise narrow.

The solution to f_1 is then found straightforwardly to be

$$f_1 = \mathrm{Re}\left\{-e^{-i\psi}\left[P\,\frac{1}{\omega - k_y\,xv_\parallel/L_s} - i\pi\,\delta\left(\omega - \frac{k_y\,xv_\parallel}{L_s}\right)\right]\right\}\frac{ck_y\varphi_0}{B(v^3+v_I^3)}\,\frac{\partial S}{\partial x}\,\theta(v_0 - v)$$

$$= -\frac{ck_y\varphi_0}{B}\left[\cos\psi\,P\,\frac{1}{\omega - k_yxv_\parallel/L_s} - \pi\delta\left(\omega - \frac{k_yxv_\parallel}{L_s}\right)\sin\psi\right]\;\frac{\frac{\partial S}{\partial x}\,\theta(v_0-v)}{v^3+v_I^3}\;. \qquad (7)$$

If Eq. (7) is substituted in Eq. (3), the $\cos\psi$ vanishes after the ψ integration, while the delta function term gives

$$\mathcal{P}_\mathrm{L} = -\frac{ck_y}{2B}\,q_\alpha\pi\omega\int dx\,d^3v\;\delta\left(\omega - \frac{k_yxv_\parallel}{L_s}\right)\varphi_0^2(x)\,\frac{\theta(v_0-v)}{v^3+v_I^3}\,\frac{\partial S}{\partial x}$$

$$= -\frac{cq_\alpha\pi\omega L_s}{2B}\int\frac{d^3v}{|v_\parallel|}\,\frac{\theta(v_0-v)}{v^3+v_I^3}\;\mathrm{sgn}\left(\frac{L_s}{k_y}\right)\varphi_0^2(x^*)\,\frac{\partial S}{\partial x}(x^*) \qquad (8)$$

where $x^* = \omega L_s/k_yv_\parallel$ and $\mathrm{sgn}(x)$ is the sign of the argument. If $k_yq_\alpha\omega\,\frac{\partial S}{\partial x} > 0$, it follows that $\mathcal{P}_\mathrm{L} < 0$ and thus destabilizing.

Now let us consider the nonlinear problem. We note, if collisional terms are neglected, that the characteristics of Eq. (2) are

$$\dot{\psi} = \omega - \frac{k_yx}{L_s}\,v_\parallel - \frac{ck_y}{B}\,\frac{\partial\varphi_0(x)}{\partial x}\cos\psi$$

$$\dot{x} = -\frac{ck_y\varphi_0(x)}{B}\sin\psi\;. \qquad (9)$$

Equation (9) has an approximate first integral of motion that is found by considering

$$\dot{\psi} = \dot{x}\,\frac{d\psi}{dx} = -\frac{ck_y\varphi_0(x)\sin\psi}{B}\,\frac{d\psi}{dx} = \omega - \frac{k_yxv_\parallel}{L_s} - \frac{ck_y}{B}\,\frac{\partial\varphi_0\cos\psi}{\partial x}\;.$$

This equation is solved by expanding $\varphi_0(x)$ about the resonance point x^* which satisfies

$$\omega - \frac{k_yx^*v_\parallel}{L_s} = 0\;, \qquad (10)$$

and then neglecting all higher order derivative terms in φ_0. The conditions for this procedure is determined *a posteriori*. We can then integrate in x and find a constant of motion E, given by

$$E = \frac{(x - x^*)^2}{2} + A \cos\psi \tag{11}$$

where $A = c\varphi_0(x^*)L_s/Bv_{||}$ (to simplify notation A is assumed positive). Using that $(x - x^*) \approx A^{1/2}$, we find that the condition for neglecting the $\dfrac{\partial \varphi_0}{\partial x}$ terms in Eq. (10) is $A^{1/2} < \Delta x$ where $\Delta x = \left(\dfrac{\partial}{\partial x} \ln \varphi_0\right)^{-1}$, i.e., the nonlinear oscillation amplitude is less than the mode width.

Note that Eq. (11) allows for two groups of particles, a passing group for which $E > A$, where ψ (modulo 2π) can vary from 0 to 2π, and a trapped group for which $-A < E < A$ and ψ is constrained to vary from $\cos^{-1}(E/A) < \psi < 2\pi - \cos^{-1}(E/A)$. It readily follows from Eq. (9) that the period, T_b, for oscillatory motion in the x-direction of a deeply trapped particle ($E = -A + \epsilon$) is

$$T_b = 2\pi/\omega_b = 2\pi/[k_y(\varphi_0 \, v_{||}/L_s B)^{1/2}] \,. \tag{12}$$

T_b is the characteristic period for the particles in which $E \approx A$. Of course near the separatrix where $E \doteq A$, the period approaches infinity logarithmically.

We now use Eq. (11) to transform from x to E, approximate $\nu(x)$ and $Q(x)$ as a power series around $x = x^*$ (keeping up to two terms). Then we find that Eq. (2) can be written as

$$\mp \frac{k_y v_{||}}{L_s} |x(E,\psi) - x^*| \frac{\partial f^\pm}{\partial \psi} = \frac{\nu(x^*)}{v^2} \frac{\partial}{\partial v} [(v^3 + v_I^3) f^\pm] + \frac{Q(x^*)}{4\pi v_0^2} \delta(v - v_0)$$

$$+ \frac{\nu(x^*)}{v^2} \frac{\partial A}{\partial v} (v^3 + v_I^3) \frac{\partial f^\pm}{\partial E} \cos\psi \pm \frac{\nu'(x^*)}{v^2} |x(E,\psi) - x^*| \frac{\partial A}{\partial v} (v^3 + v_I^3) \frac{\partial f^\pm}{\partial E} \cos\psi$$

$$- \frac{\nu'(x^*)}{v^2} (v^3 + v_I^3) \frac{\partial x^*}{\partial v} |x(E,\psi) - x^*|^2 \frac{\partial f^\pm}{\partial E}$$

$$\pm\, |x(E,\psi) - x^*| \left\{ \frac{\nu'(x^*)}{v^2} \frac{\partial}{\partial v} \left[(v^3 + v_I^3) f^\pm \right] + \frac{Q'(x^*)}{4\pi v_0^2}\, \delta(v - v_0) \right\}$$

$$\mp\, \frac{\nu(x^*)}{v^2} (v^3 + v_I^3) |x(E,\psi) - x^*| \frac{\partial x^*}{\partial v} \frac{\partial f^\pm}{\partial E}$$

$$+\, \nu_d(x^*) \left(\frac{\partial}{\partial \lambda} \mp |x(E,\psi) - x^*| \frac{\partial x^*}{\partial \lambda} \frac{\partial}{\partial E} \right) (1 - \lambda^2) \left(\frac{\partial}{\partial \lambda} \mp |x(E,\psi) - x^*| \frac{\partial x^*}{\partial \lambda} \frac{\partial}{\partial E} \right) f^\pm$$

$$+ \cdots \tag{13}$$

where $|x(E,\psi) - x^*| = \sqrt{2}(E - A\cos\psi)^{1/2}$ and $+$ and $-$ refer to $x - x^* > 0$ and $x - x^* < 0$ respectively.

Observe that the left-hand side of Eq. (13) is proportional to $\omega_b f$. Roughly, away from resonance ($E \approx x^{*^2} \gg A$) the right-hand side scales as νf. However, if we assume that near resonance $\dfrac{\partial f}{\partial E} \sim f/A$ ($E \approx A$), then $\partial/\partial E$ terms acting on f allow the last two terms on the right-hand side to scale near resonance as $\nu f x^*/A^{1/2}$ and $\nu_d f x^{*^2}/A$. Thus near resonance we can develop a perturbation theory for the nonlinear problem that assumes $\omega_b \gg \nu_{\text{eff}} \equiv \max\left[\dfrac{\nu x^*}{A^{1/2}},\ \nu_d x^{*2}/A\right]$. ν_{eff} is the rate a resonant particle would slip out of resonance due to either drag (wherein $\nu_{\text{eff}} = \nu x^*/A^{1/2} \approx \nu\omega/\omega_b$, where we have used Eqs. (10) and (12)) or due to diffusion (wherein $\nu_{\text{eff}} = \nu_d x^{*2}/A \approx \nu_d \omega^2/\omega_b^2$). We shall assume throughout that $\nu_d \ll \nu$. However, if $A/x^{*^2} \approx \omega_b^2/\omega^2 < (\nu_d/\nu)^2$, the diffusion term dominates the dynamics in the resonance region. We also note that our calculational method is very similar to the treatment of the banana regime in neoclassical transport theory, viz. in an effective collision time, ν_{eff}^{-1}, a particle's oscillatory orbit is well defined and ν_{eff}, effective collision rate, is enhanced above the bare inverse drag rate due to the small phase space of the trapping region.

One other important point should be noted. For E approaching the separatrix value A, $\omega_b \to 0$. Therefore for this region there is a local breakdown of the above perturbative procedure. In this region a special boundary matching procedure must be invoked, which will be discussed below.

Ultimately the power transfer needs to be calculated. If we use E as an integration variable, the power transfer function, given by Eq. (3), can be written as

$$\mathcal{P} = -q_\alpha \omega \int d^3v \int_{-A}^{\infty} dE \int_{\psi_{\min}}^{\psi_{\max}} \frac{d\psi}{2\pi} \frac{\sin\psi (f^+ + f^-)[\varphi_0(x^*) + \cdots]}{\sqrt{2}(E - A\cos\psi)^{1/2}}$$

$$= q_\alpha \frac{B\omega}{c} \int d^3v \int_{-A}^{\infty} dE \int_{\psi_{\min}}^{\psi_{\max}} \frac{d\psi}{2\pi} \frac{v_\parallel |x(E,\psi) - x^*|}{L_s} \frac{\partial(f^+ + f^-)}{\partial\psi} \tag{14}$$

where we have integrated by parts to obtain the final form. Thus the power transfer function depends only on the left-hand side of Eq. (13). When ω_b is large, we can then use Eq. (13) to obtain the power transfer in terms of lower order quantities.

To solve Eq. (13) we seek solutions in an expansion of $\epsilon = \nu_{\text{eff}}/\omega_b$. Then writing $f = F + f_1$, to lowest order we have

$$\mp \frac{k_y v_\parallel |x(E,\psi) - x^*|}{L_s} \frac{\partial F^\pm}{\partial \psi} = 0 \tag{15}$$

and therefore $F^\pm$ is independent of ψ. We also note that for trapped particles $F^+ = F^- \equiv F$, since at turning points F must be continuous. To next order we obtain

$$\mp \frac{k_y v_\parallel |x(E,\psi) - x^*|}{L_s} \frac{\partial f_1^\pm}{\partial \psi} = Q^\pm(\psi, F^\pm) \tag{16}$$

where $Q^\pm(\psi, F^\pm)$ is the right-hand side of Eq. (13) with $f^\pm$ replaced by $F^\pm$. As we assume that $\nu_d < \nu$, we anticipate that the only important term in the diffusion term (the last term in Eq. (13)) is

$$\nu_d(x^*)\Big(x(E,\psi) - x^*\Big) \left(\frac{\partial x^*}{\partial \lambda}\right)^2 (1 - \lambda^2) \frac{\partial}{\partial E} \Big(x(E,\psi) - x^*\Big) \frac{\partial F^\pm}{\partial E} .$$

This is the only part of the diffusion term that we will explicitly retain.

To solve Eq. (16) we demand that $f_1^\pm$ be periodic in ψ. Therefore for passing particles ($E > A$) we divide Eq. (16) by $|x(E,\psi) - x^*|$ and integrate over a period in ψ, to find as a solubility condition

$$\left[\frac{\nu(x^*)}{v^2}\frac{\partial}{\partial v}\widetilde{F}^{\pm}+\frac{Q(x^*)}{4\pi v_0^2}\,\delta(v-v_0)\right]\left\langle\frac{1}{|x(E,\psi)-x^*|}\right\rangle$$

$$+\frac{\nu(x^*)}{v^2}\frac{\partial A}{\partial v}\frac{\partial\widetilde{F}^{\pm}}{\partial E}\left\langle\frac{\cos\psi}{|x(E,\psi)-x^*|}\right\rangle\pm\frac{\nu'(x^*)}{v^2}\frac{\partial\widetilde{F}^{\pm}}{\partial v}-\frac{\nu'(x^*)}{v^2}\frac{\partial x^*}{\partial v}\left\langle|x(E,\psi)-x^*|\right\rangle\frac{\partial\widetilde{F}^{\pm}}{\partial E}$$

$$\pm\frac{Q'(x^*)}{4\pi v_0^2}\,\delta(v-v_0)\mp\frac{\nu(x^*)}{v^2}\frac{\partial\widetilde{F}^{\pm}}{\partial E}\frac{\partial x^*}{\partial v}$$

$$+\nu_d(x^*)\left(\frac{\partial x^*}{\partial\lambda}\right)^2(1-\lambda^2)\frac{\partial}{\partial E}\left\langle|x(E,\psi)-x^*|\right\rangle\frac{\partial F^{\pm}}{\partial E}=0 \qquad (17)$$

where $\langle G(E,\psi)\rangle=\int_0^{2\pi}\frac{d\psi}{2\pi}G(E,\psi)$ and $\widetilde{F}^{\pm}=(v^3+v_I^3)F^{\pm}$.

For trapped particles, boundary conditions demand

$$\begin{aligned} f_1^+(\psi_{\min}) &= f_1^-(\psi_{\min}) \\ f_1^-(\psi_{\max}) &= f_1^+(\psi_{\max})\ . \end{aligned} \qquad (18)$$

If we divide Eq. (16) by $|x(E,\psi)-x^*|$ and integrate the $f^{\pm}$ equations from $\psi_{\min}$ to $\psi_{\max}$, we have

$$\begin{aligned} -f_1^+(\psi_{\max})+f_1^+(\psi_{\min}) &= \frac{2\pi L_s}{k_y v_\parallel}\left\langle\frac{Q^+(\psi,F)}{|x(E,\psi)-x^*|}\right\rangle \\ f_1^-(\psi_{\max})-f_1^-(\psi_{\min}) &= \frac{2\pi L_s}{k_y v_\parallel}\left\langle\frac{Q^-(\psi,F)}{|x(E,\psi)-x^*|}\right\rangle \end{aligned} \qquad (19)$$

where $\langle G(\psi)\rangle=\frac{1}{2\pi}\int_{\psi_{\min}}^{\psi_{\max}}d\psi\,G$. Note that it is consistent to use the same symbol for $\langle G(\psi)\rangle$ for passing and trapped particles as for passing particles $\psi_{\min}=0$ and $\psi_{\max}=2\pi$.

Now adding the two equations in Eq. (19), and noting from Eq. (18) that the left-hand side vanishes, leads to the solubility condition for trapped particles

$$\left[\frac{\nu(x^*)}{v^2}\frac{\partial\widetilde{F}}{\partial v}+\frac{Q(x^*)}{4\pi v_0^2}\,\delta(v-v_0)\right]\left\langle\frac{1}{|x(E,\psi)-x^*|}\right\rangle$$

$$+\frac{\nu(x^*)}{v^2}\frac{\partial A}{\partial v}\left\langle\frac{\cos\psi}{|x(E,\psi)-x^*|}\right\rangle\frac{\partial\widetilde{F}}{\partial E}-\frac{\nu'(x^*)}{v^2}\frac{\partial x^*}{\partial v}\left\langle|x(E,\psi)-x^*|\right\rangle\frac{\partial\widetilde{F}}{\partial E}$$

$$+ \nu_d(x^*)(1-\lambda^2)\left(\frac{\partial x^*}{\partial \lambda}\right)^2 \frac{\partial}{\partial E}\left\langle |x(E,\psi)-x^*|\right\rangle \frac{\partial F}{\partial E} = 0 \ . \tag{20}$$

Note that Eqs. (17) and (20) differ and this leads to considerably different behavior for passing and trapped particles.

If we use Eqs. (17) and (20), we find that Eq. (16) can be written as

$$\mp \frac{k_y v_\parallel}{L_s}\frac{\partial f_1^\pm}{\partial \psi} = \left[\frac{\nu(x^*)}{v^2}\frac{\partial \widetilde{F}^\pm(E,v)}{\partial v} + \frac{Q(x^*)}{4\pi v_0^2}\,\delta(v-v_0)\right]$$

$$\cdot\left[\frac{1}{|x(E,\psi)-x^*|} - \left\langle \frac{1}{|x(E,\psi)-x^*|}\right\rangle\right]$$

$$+\nu_d(x^*)(1-\lambda^2)\left(\frac{\partial x^*}{\partial \lambda}\right)^2 \frac{\partial}{\partial E}\left[|x(E,\psi)-x^*| - \left\langle |x(E,\psi)-x^*|\right\rangle\right]\frac{\partial F^\pm}{\partial E}$$

$$+\frac{\nu(x^*)}{v^2}\frac{\partial A}{\partial v}\frac{\partial \widetilde{F}^\pm}{\partial E}\left[\frac{\cos\psi}{|x(E,\psi)-x^*|} - \left\langle\frac{\cos\psi}{|x(E,\psi)-x^*|}\right\rangle\right] \pm \frac{\nu(x^*)}{v^2}\frac{\partial A}{\partial v}\frac{\partial \widetilde{F}^\pm}{\partial E}\cos\psi$$

$$-\frac{\nu(x^*)}{v^2}\frac{\partial x^*}{\partial v}\frac{\partial \widetilde{F}^\pm}{\partial E}\left[|x(E,\psi)-x^*| - \left\langle |x(E,\psi)-x^*|\right\rangle\right] \ . \tag{21}$$

To obtain the nonlinear power transfer, $\mathcal{P}_{\rm NL}$, we apparently need only substitute Eq. (21) into Eq. (14). However, to do only this would overlook an essential contribution from the vicinity of the separatrix, where the large ω_b ordering fails. To include this region in the power transfer, we note that we can integrate the E variation of Eq. (14) in the vicinity of the separatrix between $A(1-\delta) < E < A(1+\delta)$, where $\delta \ll 1$ but $\omega_b\Big(A(1\pm\delta)\Big) \gg \nu_{\rm eff}$. From Eq. (13) we find

$$\int_{A(1-\delta)}^{A(1+\delta)} dE\,\frac{v_\parallel}{L_s}\,|x(E,\psi)-x^*|\,\frac{\partial(f^+ + f^-)}{\partial\psi}$$

$$= \frac{\nu(x^*)}{k_y v^2}(v^3+v_I^3)|x(A,\psi)-x^*|\frac{\partial x^*}{\partial v}\left[F^+\Big(A(1+\delta)\Big) + F^-\Big(A(1+\delta)\Big) - 2F\Big(A(1-\delta)\Big)\right]$$

$$-\frac{\nu'(x^*)}{k_y v^2}\frac{\partial A}{\partial v}(v^3+v_I^3)|x(A,\psi)-x^*|\cos\psi\left[F^+(A(1+\delta)) + F^-(A(1+\delta)) - 2F(A(1-\delta))\right]$$

$$
-\frac{\nu_d(x^*)}{k_y}\left(\frac{\partial x^*}{\partial \lambda}\right)^2\left(x(A,\psi)-x^*\right)^2\left[\frac{\partial F^+\Big((A(1+\delta)\Big)}{\partial E}-\frac{\partial F^-\Big(A(1+\delta)\Big)}{\partial E}\right]\cdot(1-\lambda^2)
$$

$$
+\mathcal{O}(A\delta)\ . \tag{22}
$$

In Eq. (22) we have neglected all the terms containing $F^+(A(1+\delta)) - F^-(A(1+\delta))$ since at the separatrix we have $F^+ = F^-$. At the moment, we do not assume F^+ and F^- to be equal to $F(A(1-\delta))$. They actually become equal if ν_d is not too small (see Sec. III.B) while as $\nu_d \Rightarrow 0$ the distribution function is not in general continuous at the separatrix (see Sec. III.A.)

Now we substitute Eqs. (21) and (22) into Eq. (14) to obtain

$$
\mathcal{P}_{\rm NL}=\frac{q_\alpha B\omega}{ck_y}\int d^3v\left[\!\!\left[\int_{A(1+\delta)}^{\infty} dE\left\{-\frac{\nu(x^*)}{v^2}\frac{\partial[(v^3+v_I^3)\Delta F]}{\partial v}\left[1-\left\langle x(E,\psi)-x^*\right\rangle\left\langle\frac{1}{x(E,\psi)-x^*}\right\rangle\right]\right.\right.\right.
$$

$$
+\left\langle|x(E,\psi)-x^*|\right\rangle\left\langle\frac{\cos\psi}{|x(E,\psi)-x^*|}\right\rangle\frac{\nu(x^*)}{v^2}\frac{\partial A}{\partial v}(v^3+v_I^3)\frac{\partial\Delta F}{\partial E}
$$

$$
+\left[\left\langle|x(E,\psi)-x^*|^2\right\rangle-\left\langle|x(E,\psi)-x^*|\right\rangle^2\right]\frac{\nu'(x^*)}{v^2}\frac{\partial x^*}{\partial v}(v^3+v_I^3)\frac{\partial\Delta F}{\partial E}
$$

$$
-\left\langle|x(E,\psi)-x^*|\right\rangle\frac{\nu'(x^*)}{v^2}\frac{\partial A}{\partial v}(v^3+v_I^3)\frac{\partial}{\partial E}(F^++F^-)
$$

$$
\left.-\nu_d(x^*)(1-\lambda^2)\left(\frac{\partial x^*}{\partial\lambda}\right)^2\left[\begin{array}{l}\left\langle\left(x(E,\psi)-x^*\right)\frac{\partial}{\partial E}\left(x(E,\psi)-x^*\right)\frac{\partial\Delta F}{\partial E}\right\rangle\\ -\left\langle x(E,\psi)-x^*\right\rangle\frac{\partial}{\partial E}\left\langle x(E,\psi)-x^*\right\rangle\frac{\partial\Delta F}{\partial E}\end{array}\right]\right\}
$$

$$
+2\left\{\int_{-A}^{A(1-\delta)} dE\left\langle|x(E,\psi)-x^*|\right\rangle\left[-\left(\frac{\nu'(x^*)}{v^2}\frac{\partial[(v^3+v_I^3)F]}{\partial v}+\frac{Q'(x^*)\delta(v-v_0)}{4\pi v_0^2}\right)\right.\right.
$$

$$+\frac{\nu(x^*)}{v^2}(v^3+v_I^3)\frac{\partial x^*}{\partial v}\frac{\partial F}{\partial E}\Bigg]\Bigg]\Bigg\}$$

$$-2\left\{\int_{-A}^{A(1-\delta)} dE \left\langle |x(E,\psi)-x^*|\cos\psi\right\rangle \frac{\nu'(x^*)}{v^2}\frac{\partial A}{\partial v}(v^3+v_I^3)\frac{\partial F}{\partial E}\right\}$$

$$+\frac{\nu(x^*)}{v^2}\frac{\partial x^*}{\partial v}(v^3+v_I^3)\left\langle |x(E,\psi)-x^*|\right\rangle\left[F^+(A(1+\delta))+F^-(A(1+\delta))-2F(A(1-\delta))\right]$$

$$-\frac{\nu'(x^*)}{v^2}\frac{\partial A}{\partial v}(v^3+v_I^3)\left\langle |x(E,\psi)-x^*|\cos\psi\right\rangle\Big[F^+(A(1+\delta))$$

$$+F^-(A(1+\delta))-2F(A(1-\delta))\Big]$$

$$-\nu_d(x^*)(1-\lambda^2)\left(\frac{\partial x^*}{\partial\lambda}\right)^2\left\langle\left(x(A,\psi)-x^*\right)^2\right\rangle\frac{\partial\Delta F(E=A(1+\delta))}{\partial E}\Bigg]\Bigg] \qquad (23)$$

where $\Delta F = F^+ - F^-$.

The dominant terms of Eq. (23) can be determined by estimating $dE \approx A$, $\frac{\partial F}{\partial E}\approx\frac{\partial \Delta F}{\partial E}\approx F/A$ and $A\approx |x^*-x|^2$. Then keeping only the dominant terms we find that $\mathcal{P}_{\rm NL}$ can be simplified to

$$\mathcal{P}_{\rm NL}=\frac{-q_\alpha B\omega}{ck_y}\int d^3v\Bigg\{\nu_d(x^*)(1-\lambda^2)\left(\frac{\partial x^*}{\partial\lambda}\right)^2\left\langle\left(x(A,\psi)-x^*\right)^2\right\rangle\frac{\partial\Delta F(E=A(1+\delta))}{\partial E}$$

$$-\frac{\nu(x^*)}{v^2}\frac{\partial x^*}{\partial v}(v^3+v_I^3)\left\langle|x(A,\psi)-x^*|\right\rangle\left[F^+(A(1+\delta))+F^-(A(1+\delta))-2F(A(1-\delta))\right]$$

$$+\nu_d(x^*)(1-\lambda^2)\left(\frac{\partial x^*}{\partial\lambda}\right)^2\int_A^\infty dE\Bigg[\left\langle\left(x(E,\psi)-x^*\right)\frac{\partial}{\partial E}\left(x(E,\psi)-x^*\right)\frac{\partial\Delta F}{\partial E}\right\rangle$$

$$-\left\langle x(E,\psi)-x^*\right\rangle\frac{\partial}{\partial E}\left\langle x(E,\psi)-x^*\right\rangle\frac{\partial\Delta F}{\partial E}\Bigg] \quad .$$

$$- \frac{2\nu(x^*)}{v^2} \frac{\partial x^*}{\partial v} \int_{-A}^{A} dE \left\langle |x(E,\psi) - x^*| \right\rangle \frac{\partial \widetilde{F}}{\partial E} \Bigg\} . \tag{24}$$

We now evaluate the power transfer in three regimes (a) $\nu > \nu_d {x^*}^2/A \simeq \nu_d \omega^2/\omega_b^2$ (b) $\nu_d \omega^2/\omega_b^2 > \nu > \nu_d x^*/A^{1/2} \simeq \nu_d \omega/\omega_b$, and (c) $\nu_d \omega/\omega_b > \nu$.

III. Solutions in Various Collisionality Regimes

A. Non-diffusive Regime $\nu_d \omega^2/\omega_b^2 < \nu$

In this regime diffusion is not an important effect. Particles slow down from their initial injection speed v_0 in a time scale $\sim \nu^{-1}$, and trapped particles do not have a chance to diffuse away from the trapping region. Thus, in Eqs. (17) and (20) we can neglect ν_d terms. With the boundary condition $F(v_0 + \epsilon) = 0$, we see from Eq. (17) that

$$\widetilde{F}(v_0 - \epsilon) = S(x_0^*) \pm \left\langle \frac{1}{|x(E,\psi) - x_0^*|} \right\rangle^{-1} S'(x_0^*) \underset{E \gg A}{\longrightarrow} S(x_0^*) \pm \sqrt{2E}\, S'(x_0^*) \tag{25}$$

where $\omega - k_y v_{\|0} x_0^*/L_s = 0$, and the subscript "0" refers to the injection coordinate. For the remaining determination of $F^{\pm}$, we use only the dominant terms of Eq. (17), which upon using x^* as a variable instead of v, reduces to the equation

$$\frac{\partial \widetilde{F}^{\pm}}{\partial x^*} \mp \left\langle \frac{1}{\sqrt{2}(E - A\cos\psi)^{1/2}} \right\rangle^{-1} \frac{\partial \widetilde{F}^{\pm}}{\partial E} = 0 \, . \tag{26}$$

The characteristic of this equation is

$$\frac{dE}{dx^*} = \mp \left\langle 2[E - A\cos\psi]^{-1/2} \right\rangle^{-1} \tag{27}$$

which yields a constant of integration C given by

$$C(x^*, E) = x^* \pm \left\langle \sqrt{2}(E - A\cos\psi)^{1/2} \right\rangle \left[1 + \mathcal{O}(A^{1/2}/x^*)\right] \, .$$

For concreteness we assume that if $x(E,\psi) - x^* > 0$, a particle injected at $x > x^*$ will reach resonance as it slows down to a speed v_r (the subscript r denotes the resonance value). Then for $v < v_r$, we have $x(E,\psi) - x^* < 0$, and using that $C(x_r^* + \epsilon, A_r) = C(x_r^* - \epsilon, A_r)$, we have

$$C(x^*, E) \equiv x_0^* + \left\langle \sqrt{2}(E_0 - A_0 \cos\psi)^{1/2} \right\rangle$$
$$= \begin{cases} x^* + \left\langle \sqrt{2}(E - A\cos\psi)^{1/2} \right\rangle \ , & x - x^* > 0 \\ x^* - \left\langle \sqrt{2}(E - A\cos\psi)^{1/2} \right\rangle + 2\left\langle \sqrt{2}\, A^{1/2}(1-\cos\psi)^{1/2} \right\rangle \ , & x - x^* < 0 \end{cases} \tag{28}$$

We also note that $C(x^*, E) \doteq x_0^* + \sqrt{2E_0}$ if $E_0 - E \gg A$.

The general solution to Eq. (26) is $\widetilde{F} = \widetilde{F}(C(E, x^*))$. If $E_0 \gg A_0$, this solution is readily matched to the boundary condition given by Eq. (25), and we find for particles that go through resonance (i.e., $x^* > x_0^*$),

$$\widetilde{F} = \begin{cases} S(x^*) + \left\langle \sqrt{2}(E - A\cos\psi)^{1/2} \right\rangle S'(x^*) \ , & x(E,\psi) - x^* > 0 \\ S(x^*) - \left\langle \sqrt{2}(E - A\cos\psi)^{1/2} \right\rangle S'(x^*) + C_2 \ , & x(E,\psi) - x^* < 0 \end{cases} \tag{29}$$

with

$$C_2 = 2\sqrt{2}\, A^{1/2} \left\langle (1-\cos\psi)^{1/2} \right\rangle S'(x^*) \ . \tag{30}$$

We now examine the solution for trapped particles where $-A < E < A$. If we change variables from v to $A\Big(A = cL_s\varphi_0\big(x^*(v)\big)\Big/ B\, v\lambda\Big)$, Eq. (20) can be written as

$$\frac{\partial \widetilde{F}(A,E)}{\partial A} \left\langle \frac{1}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle + \left\langle \frac{\cos\psi}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle \frac{\partial \widetilde{F}(A,E)}{\partial E}$$
$$- \frac{1}{\nu}\frac{\partial \nu}{\partial A} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \widetilde{F}(A,E)}{\partial E} = 0 \ . \tag{31}$$

where it is assumed $v < v_0$, so that the source terms at $v = v_0$ are ignored. If we now introduce the variable

$$J = \sqrt{2}\,\nu(x^*) \int_{\psi_{\min}}^{\psi_{\max}} d\psi (E - A\cos\psi)^{1/2} \tag{32}$$

and transform from E to J, Eq. (31) then takes the form

$$\frac{\partial \widetilde{F}(A,J)}{\partial A} = 0 \ . \tag{33}$$

Equation (33) indicates that J is an adiabatic invariant as the equation does not contain the term $\frac{\partial F}{\partial J}$. From Eq. (32) we need $E < A$ when a particle is trapped. As $|E| \approx \frac{1}{2}(x - x^*)^2$ and $x^*(v)$ changes as a particle slows down, we conclude that the x position of a trapped particle convects across field lines to track with x^* in order to maintain the conservation of J. The situation is illustrated in Fig. 1. As $v_{\|}$ (or equivalently x^*) changes, passing particles move along ψ to the resonant point where they are reflected and then move in the other direction of ψ. The x-motion is limited to a relatively small oscillatory amplitude. Trapped particles are confined to a limited phase of ψ and are then forced to convect across the field lines as they slow down (i.e., as x^* changes). Some of the passing particles can be trapped (or a trapped particle detrapped) as indicated in Fig. 2. Thus it follows from Eq. (33), that for a given J, $\widetilde{F}$ remains constant as the speed v decreases. To determine $\widetilde{F}$, we note that $\widetilde{F}$ must have the value it had when a particle first becomes trapped. This arises when $E = A$, and from Eq. (32) it follows that at the separatrix,

$$J = J_s \equiv 8\nu(x^*)\left(\frac{c\varphi_0(x^*)k_y x^*}{B\omega}\right)^{1/2} . \tag{34}$$

We first consider the injection point $x_0^* \equiv x^*(v_0)$. At the point x_0^* the distribution consists of particles injected directly into the resonant region. To lowest order, $\widetilde{F} = S(x_0^*)$, which applies to untrapped and trapped particles. The trapped particles in the region $0 < J < J_0 = 8\nu(x_0^*)\left(\frac{L_s\varphi_0\left(x_0^*\right)}{\lambda B v_0}\right)^{1/2}$ will then have an $\widetilde{F}$-value given by

$$\widetilde{F} = S\left(x_0^*\right) . \tag{35}$$

As $x^*(v)$ increases from x_0^*, the separatrix width either increases or decreases depending on whether $A \propto \varphi(x^*)/\lambda v$ is increasing or decreasing (note that for $\varphi(x^*)$ constant A increases). If the separatrix width increases, passing particles are entrapped by the rising separatrix. The distribution $\widetilde{F}$ of trapped particles adjacent to the separatrix then has the value $S(x^*)$, i.e., the passing particle distribution function adjacent to the separatrix.

Formally, we define a function $x_s^*(J)$ which is defined by Eq. (34); $x_s^*(J)$ is the x^*-value a particle has when it is first trapped. Then, at $x = x^*$ for $J_0 < J < J_s$, $\widetilde{F}(J)$ is given by

$$\widetilde{F}(J) = \widetilde{F}\left(x_s^*(J)\right) \ . \tag{36}$$

When the separatrix decreases with increasing x^* (this ultimately happens when $\varphi(x^*)$ decreases sufficiently rapidly) trapped particles cross the separatrix and become passing particles. There is then a discontinuity of F at the separatrix, with the passing particle distribution given by $S(x^*)$ and the trapped particle distribution given by

$$\begin{aligned} &S(x_0^*) \ ; && \text{if} \ \ J_0 < J_s(x^*) \\ &S\left(x^*(v_0)\right) \ ; && \text{if} \ \ J_s(x^*) < J_0 \ . \end{aligned} \tag{37}$$

We now calculate the power transfer $\mathcal{P}_{\rm NL}$. We account only for the most important terms in Eq. (24). These terms come from the trapping region (the last term in Eq. (24)) and the discontinuity of F at the separatrix. We find

$$\begin{aligned} \mathcal{P}_{\rm NL} = {} & \frac{2q_\alpha B\omega}{ck_y} \int d^3v \, \frac{1}{v^2} \frac{\partial x^*}{\partial v} \\ & \cdot \left\{ \nu(x^*) \left\langle |x(A,\psi) - x^*| \right\rangle \left[S(x^*) - S(x_s^*) \right] + \int_0^{J_s(x^*)} \frac{dJ}{2\pi} J \frac{\partial \widetilde{F}}{\partial J} \right\} \end{aligned} \tag{38}$$

with

$$\frac{\partial \widetilde{F}}{\partial J} = -\frac{\omega L_s}{k_y v_s^2 \lambda} \frac{\partial v}{\partial J} S' \, \theta(J - J_0) \ , \tag{39}$$

and we have used the convention that $x_s^* = x^*$ if the separatrix is outwardly increasing.

With the further approximation that $S(x^*) - S(x_s^*) = (x^* - x_s^*) S'(x_s^*)$ and some additional algebra we find,

$$\begin{aligned} \mathcal{P}_{\rm NL} = {} & \frac{-8q_\alpha B\omega^3 L_s^2}{\pi c k_y^3} \int d^3v \, \frac{S'(x^*)\nu(x^*)}{v^4 \lambda^2} \\ & \left[\left(\frac{1}{v} - \frac{1}{v_s(v)} \right) \left(\frac{cL_s\varphi_0\left(x^*(v)\right)}{\lambda B v} \right)^{1/2} + \int_{v_s(v)}^{v_0} \frac{dv'}{v'^2} \left(\frac{cL_s\varphi_0\left(x^*(v')\right)}{\lambda B v'} \right)^{1/2} \right] \end{aligned} \tag{40}$$

where $v_s(v)$ is determined from the relation,

$$\nu^2\left(x^*(v_s)\right)\,\frac{\varphi\left(x^*(v_s)\right)}{v_s}=\nu^2\left(x^*(v)\right)\,\frac{\varphi\left(x^*(v)\right)}{v}$$

and only one relative maximum of the separatrix width is assumed present. If a solution for $v_s > v$ cannot be found, we shall take $v_s(v) = v$.

In order to compare the scaling of $\mathcal{P}_{\mathrm{NL}}$ and $\mathcal{P}_{\mathrm{L}}$ we define the variable

$$\omega_b^2(v)=\frac{c\lambda k_y^2 v\varphi\left(x^*(v)\right)}{BL_s}\,.$$

Then the ratio of $\mathcal{P}_{\mathrm{NL}}$ to $\mathcal{P}_{\mathrm{L}}$ can be written as

$$\frac{\mathcal{P}_{\mathrm{NL}}}{\mathcal{P}_{\mathrm{L}}}=\frac{16\omega^2}{\pi}\,\frac{\displaystyle\int\frac{d^3v}{v^4\lambda^3}\,S'(x^*)\nu(x^*)\theta(v_0-v)\left[\frac{\omega_b(v)}{v}\left(\frac{1}{v}-\frac{1}{v_s(v)}\right)+\int_{v_s(v)}^{v_0}\frac{dv'}{v'^3}\,\omega_b(v')\right]}{\displaystyle\int d^3v\,\frac{S'(x^*)\theta(v_0-v)\omega_b^4(v)}{\lambda^3v^3(v^3+v_I^3)}}\,. \tag{41}$$

From this expression it is evident that

$$\frac{\mathcal{P}_{\mathrm{NL}}}{\mathcal{P}_{\mathrm{L}}}\simeq\frac{\alpha\omega^2\nu}{\omega_b^3(v_0)}$$

with α a numerical coefficient.

Marginal stability arises when

$$\mathcal{P}_{\mathrm{NL}}=\mathcal{P}_d$$

where $\mathcal{P}_d$ is the power damping rate to the background plasma. Roughly, the saturated amplitude is then determined from

$$\omega_b\approx(\nu\omega^2)^{1/3}\left(\frac{\mathcal{P}_{\mathrm{L}}}{\mathcal{P}_d}\right)^{1/3}\,.$$

The above ordering is valid if ω_b is greater than the rate in which particles pass through the resonant region, i.e., $\omega_b > \nu x^*/A^{1/2} \approx \nu\omega/\omega_b$. Note that $\mathcal{P}_{\mathrm{NL}} > \mathcal{P}_{\mathrm{L}}$ if

$$\nu\omega<\omega_b^2<\nu\omega^2/\omega_b$$

where the left-hand side of the inequality is necessary for the validity of the expression for $\mathcal{P}_{\mathrm{NL}}$. Thus, over a considerable band of ω_b, the linear growth rate is enhanced as $\mathcal{P}_{\mathrm{NL}} > \mathcal{P}_{\mathrm{L}}$. Only in the region $\nu\omega^2/\omega_b^3 < 1$ is the nonlinear growth rate less than the linear one.

B. Diffusive Regimes

If $\nu < \nu_d \omega^2/\omega_b^2$, diffusion is important in solving for $\widetilde{F}$. In regime (b), $\nu_d \dfrac{\omega}{\omega_b} < \nu < \nu_d \omega^2/\omega_b^2$, trapped particles scatter out of the trapping region so that they no longer remain trapped during a characteristic slowing down time. However, they can still drift across the magnetic field over a length that is many separatrix widths. In regime (c) $\nu < \nu_d \omega/\omega_b$, the trapped particles scatter out of the trapping region before they drift the distance of a single separatrix width.

To solve for $\widetilde{F}$, let us consider Eq. (17) for the passing particles with the diffusion term. To leading order it can be written as

$$\left\langle \frac{1}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle \frac{\partial \widetilde{F}^{\pm}}{\partial x^*} \mp \frac{\partial \widetilde{F}^{\pm}}{\partial E} + \beta A^{1/2} \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \widetilde{F}^{\pm}}{\partial E} = 0 \qquad (42)$$

with

$$\beta = (1-\lambda)^2 \frac{\nu_d}{\nu} \frac{\partial v}{\partial x^*} \frac{v^2}{v^3 + v_I^3} \left(\frac{\partial x^*}{\partial \lambda} \right)^2 \frac{1}{A^{1/2}} .$$

Similarly, Eq. (20) for the trapped particle distribution can be written to leading order as

$$\left\langle \frac{1}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle \frac{\partial \widetilde{F}}{\partial x^*} + \beta A^{1/2} \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \widetilde{F}}{\partial E} = 0 . \qquad (43)$$

The boundary conditions for these equations are: (1) the matching of the passing particle solution to the solution when $E \gg A$. Thus we require

$$\widetilde{F}^{\pm}(E, x^*) \underset{E \gg A}{\longrightarrow} S(x^*) \pm \sqrt{2E}\; S'(x^*)$$

(2) the continuity of $\widetilde{F}$ across the separatrix, viz. in the limit $\delta \Rightarrow 0$,

$$\widetilde{F}^{\pm}\,(A(1+\delta)) = \widetilde{F}^{-}\,(A(1+\delta)) = \widetilde{F}\,(A(1-\delta)) \ ;$$

(3) the continuity of particle flux across separatrix

$$\left[\frac{\partial \widetilde{F}^{+}(E, x^*)}{\partial E} + \frac{\partial \widetilde{F}^{-}(E, x^*)}{\partial E} \right]_{E = A(1+\delta)} - 2 \left[\frac{\partial \widetilde{F}(E, x^*)}{\partial E} \right]_{E = A(1-\delta)} = 0 .$$

Now for the passing particles, consider $\overline{F} \equiv \widetilde{F}^{+} + \widetilde{F}^{-}$ and $\Delta\widetilde{F} = \widetilde{F}^{+} - \widetilde{F}^{-}$. Equation (42) can then be written as

$$\left\langle \frac{1}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle \frac{\partial \overline{F}}{\partial x^*} - \frac{\partial \Delta\widetilde{F}}{\partial E} + \beta A^{1/2} \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \overline{F}}{\partial E} = 0$$

$$\left\langle \frac{1}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle \frac{\partial \Delta\widetilde{F}}{\partial x^*} - \frac{\partial \overline{F}}{\partial E} + \beta A^{1/2} \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \Delta\widetilde{F}}{\partial E} = 0 \,. \quad (44)$$

To lowest order in the asymptotic form $\overline{F} = 2S(x^*)$, which is independent of E. Thus, we can take $\dfrac{\partial \overline{F}}{\partial x^*} = 2S'(x^*)$ and $\dfrac{\partial \Delta\widetilde{F}}{\partial x^*} = 0$. By using

$$\frac{\partial \overline{F}}{\partial E} = \beta A^{1/2} \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \Delta\widetilde{F}}{\partial E} \quad (45)$$

we construct a second order equation for $g = \dfrac{\partial \Delta\widetilde{F}}{\partial E}$,

$$g - 2\beta^2 A \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial}{\partial E} \left(g \left\langle (E - A\cos\psi)^{1/2} \right\rangle \right) = \left\langle \frac{\sqrt{2}\,S'(x^*)}{(E - A\cos\psi)^{1/2}} \right\rangle . \quad (46)$$

We can solve Eq. (46) in two limits, $\beta \ll 1$ (regime b) and $\beta \gg 1$ (regime c).

Now we solve for g in the limit $\beta \ll 1$. One can readily ascertain that the contribution from the trapping region is the most important contribution to the power transfer. In the trapped particle region we have to solve Eq. (43). The first term in this equation is actually known from the boundary condition at the separatrix

$$\widetilde{F}(E, x^*)_{E=A} \doteq S(x^*) \,. \quad (47)$$

To show this, we estimate each of the two terms in Eq. (43) as

$$\left\langle \frac{1}{\sqrt{2}\,(E - A\cos\psi)^{1/2}} \right\rangle \frac{\partial \widetilde{F}}{\partial x^*} \approx \frac{\widetilde{F}}{A^{1/2} x^*} \;;\; \beta A^{1/2} \frac{\partial}{\partial E} \left\langle \sqrt{2}\,(E - A\cos\psi)^{1/2} \right\rangle \frac{\partial \widetilde{F}}{\partial x^*} \approx \beta \frac{\widetilde{F}}{A} \,.$$

It follows from these estimates that second term is much larger than the first one, while according to Eq. (43) these two terms have equal absolute values. To make them equal $\widetilde{F}$

needs to be almost constant in E in the whole range of trapped particles. Therefore, to a good approximation $\widetilde{F}$ in the trapping region is given by Eq. (47) so that we may substitute $S'(x^*)$ for $\partial\widetilde{F}/\partial x^*$ in Eq. (43). We then integrate Eq. (43) under the condition that the flux of particles at the bottom of the potential well equals zero. The solution for $\partial\widetilde{F}/\partial E$ is

$$\frac{\partial\widetilde{F}}{\partial E} = -4\,\frac{S'(x^*)}{\gamma\pi}\,A^{1/2}\,. \tag{48}$$

This equation is essentially all the information that is needed for the calculation of the power transfer in regime (b). However, for completeness we indicate the solution $\widetilde{F}$ for passing particles.

For passing particles the diffusion term is unimportant except near the separatrix. Therefore for $E > A$ we can use Eq. (29) and then find

$$\Delta\widetilde{F} = 2\sqrt{2}\,S'(x^*)\left\langle(E - A\cos\psi)^{1/2}\right\rangle$$

and

$$g \equiv \frac{\partial\Delta\widetilde{F}}{\partial E} = \left\langle\frac{\sqrt{2}\,S'(x^*)}{(E - A\cos\psi)^{1/2}}\right\rangle\,.$$

Near $E = A$ the diffusive character is important in order to match boundary conditions at $E = A$. In solving Eq. (46) we can set $E = A$ in the diffusion coefficients and Eq. (46) becomes

$$g - \gamma^2\,\frac{\partial^2 g}{\partial E^2} = \left\langle\frac{\sqrt{2}\,S'(x^*)}{\langle(E - A\cos\psi)^{1/2}\rangle}\right\rangle \tag{49}$$

with $\gamma = 4A\beta/\pi$. The solution for g that is bounded at infinity can then be written as

$$g = \left[C\,\exp\left(\frac{A-E}{\gamma}\right)\right] + \frac{1}{2\gamma}\int_A^\infty ds\left\langle\frac{\sqrt{2}\,S'(x^*)}{(s - A\cos\psi)^{1/2}}\right\rangle\exp\left(-\frac{|s-E|}{\gamma}\right) \tag{50}$$

where C is a constant.

We integrate Eq. (50) once more, using that $(\Delta\widetilde{F})_{E=A} = 0$, to find

$$\Delta\widetilde{F} + \gamma C\left[\exp\left(-\frac{E-A}{\gamma}\right) - 1\right] = \frac{1}{2\gamma}\int_A^E dx\int_A^\infty ds\left\langle\frac{\sqrt{2}\,S'(x^*)}{(s - A\cos\psi)^{1/2}}\right\rangle\exp\left(-\frac{|s-x|}{\gamma}\right)\,. \tag{51}$$

When $E - A \gg \gamma$ the integral on the right-hand side of this equation can be written as

$$2\sqrt{2}\, S'(x^*) \left\langle (E - A\cos\psi)^{1/2} - A^{1/2}(1-\cos\psi)^{1/2} \right\rangle \; .$$

Thus, in order to have

$$\Delta\tilde{F} \underset{E-A\gg 1}{\longrightarrow} 2\sqrt{2}\, S'(x^*) \left\langle (E - A\cos\psi)^{1/2} \right\rangle \; ,$$

we require

$$C = \frac{2\sqrt{2}\, S'(x^*)}{\gamma} \left\langle A^{1/2}(1-\cos\psi)^{1/2} \right\rangle = 8\, \frac{S'(x^*)}{\gamma\pi}\, A^{1/2} \; . \tag{52}$$

Using Eq. (50), we can now substitute $g \equiv \partial\Delta\tilde{F}/\partial E$ in Eq. (45) and find $\partial\overline{F}/\partial E$ near $E = A$.

$$\frac{\partial\overline{F}}{\partial E} = -8\, \frac{S'(x^*)}{\gamma\pi}\, A^{1/2} \exp\left(-\frac{E-A}{\gamma}\right) \; . \tag{53}$$

At the separatrix Eq. (53) agrees with Eq. (48) as $\tilde{F}^+ = \tilde{F}^- = \overline{F}/2$. Note that the main contribution to $\partial\overline{F}/\partial E$ comes from the first term of Eq. (50).

We are now in a position to calculate the power transfer in regime (b), $\nu_d\, \omega/\omega_b < \nu < \nu_d\, \omega^2/\omega_b^2$. The largest term in $\mathcal{P}_{\rm NL}$ in Eq. (24) comes from the trapping region and gives

$$\mathcal{P}_{\rm NL} = \frac{2q_\alpha\, B\omega}{c\, k_y} \int d^3v \frac{\partial x^*}{\partial v} \frac{\nu(x^*)}{v^2} \int_{-A}^{A} dE \left\langle \sqrt{2}(E - A\cos\psi) \right\rangle \frac{\partial\tilde{F}}{\partial E} \; .$$

The contribution in Eq. (24) from passing particles, using Eqs. (51) and (52) is smaller by a factor $\nu_d\, \omega/\nu\, \omega_b$. Using the calculated value of $\partial\tilde{F}/\partial E$ in the trapping regime (see Eq. (48)), gives

$$\mathcal{P}_{\rm NL} = -\frac{64 q_\alpha\, B\omega}{9\pi\, ck_y} \int d^3v\, (v^3 + v_I^3)\, \frac{\nu^2(x^*)\lambda^2\, S'(x^*) A^{3/2}}{(1-\lambda^2)v^6\, \nu_d(x^*)} \; . \tag{54}$$

Note that Eq. (54) includes logarithmically diverging integral over λ. To estimate the integral we have to recall the applicability condition, $\beta \gg (A^{1/2})/x^*$, used in deriving Eq. (54). As β is proportional to $(1-\lambda^2)$, this condition is violated as $\lambda \Rightarrow 1$. Therefore, we have to cut off the integral somewhere at $\beta \sim (A^{1/2})/x^*$. The ratio of $\mathcal{P}_{\rm NL}$ to $\mathcal{P}_{\rm L}$ can then be written as

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} \approx \frac{\nu^2\Delta}{\nu_d\, \omega_b} \tag{55}$$

with a large logarithmic factor Δ.

Thus, in regime (b), saturation occurs when

$$\omega_b \approx \frac{\nu^2 \mathcal{P}_{\rm L}}{\nu_d \, \mathcal{P}_d} \, . \tag{56}$$

The calculation for regime (b) is valid if $\omega_b^2 > \nu\omega$. Observe that if $\nu\omega < \omega_b^2 < \nu^2 \omega_b \Delta/\nu_d$, our calculation predicts that $\mathcal{P}_{\rm NL}$ exceeds $\mathcal{P}_{\rm L}$.

In regime (c) $\beta \gg 1$, and we need only keep the β^2 term in Eq. (46). A particular solution is

$$g \equiv \frac{\partial \Delta \widetilde{F}}{\partial E} = \frac{\sqrt{2}\, S'(x')}{\left\langle \sqrt{E} - A\cos\psi)^{1/2} \right\rangle} \tag{57}$$

and it is readily verified that it matches to the asymptotic limit for large E. From Eq. (45) we then have

$$\frac{\partial \overline{F}}{\partial E} = 0 \, . \tag{58}$$

In the trapping region, $\beta x^*/A^{1/2} \gg 1$, only the β term is important in Eq. (43). The solution is

$$\widetilde{F} = \alpha - \delta \int_E^A \frac{ds}{\langle (s - A\cos\psi)^{1/2} \rangle} \; ; \; \frac{\partial \widetilde{F}}{\partial E} = \frac{\delta}{\langle (E - A\cos\psi)^{1/2} \rangle} \tag{59}$$

with α and δ constants. These constants are determined from continuity of F and the derivative at the separatrix. We then find that

$$\alpha = S(x^*) \quad ; \quad \delta = 0 \, . \tag{60}$$

Equations (57)-(60) are the relevant solutions for (regime c).

In calculating the power transfer in regime (c) we note that the contribution of the discontinuity of the diffusion term at the separatrix in Eq. (24) is important. The drag terms from the trapping region are negligible. Now substituting Eqs. (57), (59) and (60) into the largest term of Eq. (24) yields

$$\mathcal{P}_{\rm NL} = -2 \frac{q_\alpha B \omega}{c k_y} \int d^3v \, \nu_d(x^*)(1-\lambda^2) \left(\frac{\partial x^*}{\partial \lambda} \right)^2 \frac{S'(x^*) A^{1/2} I_2}{(v^3 + v_I^3)} \, , \tag{61}$$

with

$$I_2 = \sqrt{2} - \frac{1}{\sqrt{2}} \int_1^\infty dy \left[\frac{1}{\langle (y - \cos\psi)^{1/2} \rangle} - \frac{1}{y^{1/2}} \right] \doteq 1.38 \ . \tag{62}$$

Very roughly, Eq. (61) yields

$$\mathcal{P}_{\rm NL} \approx \frac{\nu_d \omega^2}{\omega_b^3} \mathcal{P}_{\rm L} \ . \tag{63}$$

We also observe that if the wave amplitude is sufficiently low, regime c is the appropriate regime. If $\mathcal{P}_d$ is the damping rate to the background plasma, saturation of the wave is determined from $\mathcal{P}_{\rm NL} = \mathcal{P}_d$, or using Eq. (63)

$$\omega_b = (\nu_d \omega^2)^{1/3} \left(\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d} \right)^{1/3} \ .$$

As regime (c) requires $\nu_d \omega / \nu \omega_b > 1$, consistency demands

$$\left(\frac{\mathcal{P}_d}{\mathcal{P}_{\rm L}} \right)^{1/3} \frac{\nu_d^{2/3} \omega^{1/3}}{\nu} > 1 \ .$$

In this regime there is no apparent enhancement of the nonlinear power transfer.

IV. Conclusion

In this paper we considered the saturation of an electrostatic wave driven by the density gradient free-energy established by high energy spatially dependent beam injected into a plasma in a sheared magnetic field. The distribution function was calculated self-consistently by taking into account several transport processes: drag and pitch-angle scattering due to the background plasma and simultaneously the particle-wave interaction with a finite amplitude wave. The power transfer between the beam particles and the wave can be calculated asymptotically in three regimes of collisionality. An interesting nonlinear effect occurs when pitch-angle diffusion can be neglected, i.e., when $\nu_d < \nu \omega_b^2/\omega^2$, (regime (a)) where ν_d is the pitch-angle diffusion rate, ν the drag rate, ω the frequency of the wave and $\omega_b = (c\varphi_0 \, v_\| \, k_y^2 / L_s \, B)^{1/2}$ the radial bounce frequency of a particle in resonance with a wave of amplitude φ_0. In this case an effect similar to the Ware pinch[7] arises. Passing particles oscillate in a direction transverse to the magnetic field with a small amplitude about their position of injection. However, particles trapped in the wave are constrained by particle adiabaticity to remain in resonance with the wave as they slow down. These trapped particles are then forced to cross field lines, and they can transverse a distance comparable to the width of the mode. The effect is somewhat like a conveyor belt. Passing particles are injected at different points x and slow down until they hit the points of resonance. If the separatrix width is increasing as one moves spatially outward, some of the passing particles will be engulfed by the separatrix, and these particles will then be conveyed across the field lines as they slow down further. When the separatrix width begins to diminish, the trapped particles near the separatrix are then released into the passing particle regions, now at a position remote from their position of injection.

The power transfer, $\mathcal{P}_{\rm NL}$, from particles to a nonlinear wave was calculated assuming $\omega_b^2 > \nu\omega$ and it was found to scale as

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} \approx \frac{\nu\omega^2}{\omega_b^3} ,$$

where $\mathcal{P}_{\rm L}$ is the power transfer predicted from linear theory. This scaling indicates that the power transfer rate is enhanced from linear theory if

$$\nu\omega < \omega_b^2 < \nu\omega^2/\omega_b .$$

A steady-state nonlinear wave can be established when the mode amplitude rises to a level where

$$\omega_b \approx (\nu\omega^2)^{1/3} \left(\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d}\right)^{1/3}$$

where $\mathcal{P}_d$ is the power transferred by dissipative processes to the background plasma.

For moderate pitch-angle scattering (regime (b) where $\nu_d\omega/\omega_b < \nu < \nu_d\omega^2/\omega_b^2$), trapped particles convect many separatrix widths, but pitch-angle scatter out of the trapping region in a distance that is small compared to a mode width. One finds if $\omega_b^2 > \nu\omega$

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} \approx \frac{\nu^2}{\nu_d\,\omega_b}\,.$$

Hence, the nonlinear power transfer is enhanced from linear theory in the regime,

$$\nu\omega < \omega_b^2 < \frac{\nu^2\,\omega_b}{\nu_d}$$

and stabilization arises for a field amplitude in which

$$\omega_b \approx \frac{\nu^2}{\nu_d}\,\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d}\,.$$

For stronger pitch-angle scattering (regime (c) where $\nu < \nu_d\,\omega/\omega_b$), a trapped particle scatters out of the trapped region before it can convect a separatrix width. One finds if $\omega_b^3 > \nu_d\,\omega^2$,

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} \approx \frac{\nu_d\,\omega^2}{\omega_b^3}$$

and saturation occurs when

$$\omega_b \approx (\nu_d\,\omega^2)^{1/3}\left(\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d}\right)^{1/3}\,.$$

In regime (c) we always find $\mathcal{P}_{\rm NL} < \mathcal{P}_{\rm L}$ in the region where the calculation of the power transfer can be self-consistently made.

One should also note that for a given ν, ν_d, and ω, it is the ratio of $\mathcal{P}_{\rm L}/\mathcal{P}_d$ that determines the regime in which the saturated state exists. If $1 < \frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d} < \frac{\nu_d^2\,\omega}{\nu^3}$ the saturated state is in regime (c); if $\frac{\nu_d^2\,\omega}{\nu^3} < \frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d} < \frac{\omega}{\nu_d}\left(\frac{\nu_d}{\nu}\right)^{3/2}$ the saturated state is in regime (b); if $\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d} > \frac{\omega}{\nu_d}\left(\frac{\nu_d}{\nu}\right)^{3/2}$ the saturated state is in regime (a).

These considerations make regime (c) the most likely regime for a thermonuclear plasma where for alpha particles slowing down and scattering on the background plasma, $\nu_d/\nu \approx 0.1$ and typically, $\omega/\nu \gtrsim 10^6$. Hence in Ref. 2, where the formalism for Alfvén waves is presented, detailed calculations will only be given for regime (c). However, systems can be envisioned where a high-energy beam is injected into a cold plasma and a low frequency oscillation excited. In that case regimes (a) and (b) can be of interest.

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Figure Captions

1. Particle trajectories in x, x^*, ψ space. Helical lines show trapped particle convection across field lines. Passing trajectories of particles far from resonance (trajectories a, c, d, and f) and close to resonance (trajectories b and e) are shown. Note that the two patterns of resonance are centered about different x positions (separated by the x-distance between points A and B) and the trapped particles convection connects corresponding phases of the two patterns.

2. Trajectory of particles that transform from passing to trapped particles, with passing particles trajectories in the background.