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Saturation of a Single Mode Driven by an Energetic Injected Beam I. Plasma Wave Problem

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Abstract

A formalism is established for calculating the saturation level of a discrete mode that is destabilized by the distribution function formed by a high-energy injected beam. The electrostatic plasma wave interaction is studied here for two problems. In one the distribution function is formed by injection of a source with a velocity spread and a steady-state bump-on-tail instability is established with only particle annihilation taken into account. In the second problem particle drag as well as particle annihilation is accounted for. In both problems the self-consistent distribution function in the presence of a finite amplitude wave needs to be calculated. By calculating the power transfer between particles and finite amplitude wave, the saturation level of discrete mode can be predicted. The drag problem with annihilation has the interesting feature that in steady state holes in phase space are formed for a large enough amplitude wave and the power transferred from particles to waves can be greatly enhanced due to the drag force on the population of holes.

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I. Introduction

In this series of papers[1,2] we investigate the saturation level of a plasma destabilized by a beam injected at high energy. The motivation for this problem is to develop a formalism to predict the saturation level of possible Alfvén wave instabilities[3,4,5] generated by alpha particles in an ignited tokamak. The excited Alfvén waves are likely to arise as a discrete well-defined single mode. To predict the saturated wave level for such a problem we have developed a method of analysis that can be applied to a class of beam injection problems with varying degrees of complexity.

We assume a steady-state high-energy neutral beam injected uniformly in space into a background plasma and ionized by atomic processes. Through classical transport mechanisms such as drag, pitch-angle scattering, and charge exchange losses, the beam particles are assumed to form a weakly destabilizing distribution function to a quantized wave of the background plasma whose wave number is fixed by external boundary conditions. In this paper we consider that the beam transport is due to drag and particle annihilation, and the background wave is an electrostatic plasma wave (or an ion acoustic wave). In the subsequent papers the method of analysis is extended to drift waves in a sheared magnetic field,[1] and then to Alfvén waves in a tokamak geometry.[2]

The treatment of a single mode has been studied previously by O'Neil[6] and Mazitov.[7] In the absence of collisions the distribution function was shown to flatten by particle trapping effects in the region of the resonance particles. The flatness of the final distribution resembles the prediction of the plateau in one-dimensional quasilinear theory.[8,9] In the plateau regime there is no exchange of energy between particles and waves. When collisions are accounted for, the flat distribution eventually obtains a finite slope leading to wave damping.[10]

The problem we formulate is somewhat more complex in that the classical transport processes establish an unstable distribution without any wave excitation. The excited waves resupplied and feed the resonance region.

To treat this problem we need to calculate the self-consistent shape of the hot particle distribution function in the presence of a finite amplitude wave. The wave will be assumed of low enough amplitude so that the structure of the linear eigenmode is valid. The principal nonlinear effect treated is how the wave-particle resonant interaction varies with increasing amplitude and therefore increasing distortion of the resonant particle distribution function. This wave particle interaction determines an energy transfer from particles to the wave. We also assume the presence of a linear background dissipation mechanism that remains undistorted in the presence of a finite (though small) amplitude wave. The tendency for the wave to flatten the resonant particle distribution ultimately weakens the instability drive and allows the background dissipation mechanisms to compete with the drive to determine the saturation level.

Our method of solution has a close resemblance to neoclassical theory.[11] The linear regime is similar to the plateau regime of neoclassical theory in that the wave energy transfer is independent of collisional processes. However, at finite wave amplitude particle wave trapping frequencies are faster than collisional frequencies and a theory similar to neoclassical "banana" perturbation theory can be used.[11] Part of our problem was treated by Mikhailovskii and Pyatak[12] who considered the influence of the particle source on the wave excitation. However, our work appears to be the first to treat the self-consistent case where there is present a source, particle collisions, and wave particle interactions to allow a steady state.

The structure of this paper is as follows. In Sec. II an overview of how the particle wave interaction is treated is given. In Sec. III we describe a model where the particle transport mechanism and incoming beam distribution excite a weak "bump-on-tail" instability.[8,9] This is the simplest of the genre of problems we have formulated. In Sec. IV we treat a weak bump-on-tail instability formed by a beam, injected at fixed speed, that slows down due to drag. In this problem the wave particle resonance region takes on a more complex nature. The method developed to solve this problem is particularly useful for the more complex problems treated in subsequent papers.[1,2] In Sec. V conclusions are presented.

Our perturbation method in the finite amplitude region at first appears standard. To lowest order the distribution function is a function of the constants of motion that can be obtained by solving for the particle trajectories in the absence of drag annihilation and source terms. To next order the source and sink terms must be taken into account. In the problem that treats annihilation alone (without drag) the perturbation method is quite straightforward and leads to an easily interpreted result, i.e. the power transfer between waves and particles is reduced by a factor $\nu_{\text{eff}}/\omega_b$, where ν_{eff} is the rate in which particles leave (or, by detail balance, enter) the resonance region and ω_b the typical circulation rate of particles in the trapping region ($\nu_{\text{eff}}/\omega_b$ is assumed less than unity; in fact this condition is the essential parameter of the nonlinear perturbation theory).

When the drag and annihilation problem is treated several surprises arise. There is a crucial physical effect that particles that are not trapped by the wave cannot enter the trapping region as they slow down. Consequently, if the injected velocity of the source is not in the trapping region (as is assumed in this paper) the particle source cannot feed the phase space region where particles are trapped. In fact in the steady state of the nonlinear problem the distribution function in the trapping region is zero. Thus, when annihilation is present, a large gradient in the distribution function must develop at the separatrix between trapped and passing regions. The perturbation theory we have developed assumes the bounce frequency is large. This assumption must necessarily fail at a critical region of phase space. To treat this region properly we have accounted for a boundary layer contribution that makes use of the steep phase space gradients at the separatrix. We then find that the power transfer rate is appreciably larger than linear theory for moderate ω_b. Even without particle annihilation large phase space gradients can develop at separatrices in more complicated geometry.

In Ref. 1 we show how a nonlinear drift wave problem exhibits a similar discontinuity and how the formalism developed here can be generalized to solve the drift wave problem. Further details and interpretations of the plasma wave problem are presented in the text and conclusion.

II. General Procedure

The general method of calculation for the class of problems considered in this and subsequent papers is as follows. We assume that there is a hot particle species injected at high energy. A steady-state distribution is established by either the slowing down of the injected particles to a low energy or by particle annihilation (for example by charge exchange). The resulting equilibrium distribution function is assumed to provide a weak instability source for the background plasma, where only the hot particles in the regions of phase space in resonance with the wave contribute to the instability. Except for the growth rate, the excited wave is assumed to have linear wave properties established by the background plasma. Only a single wave is assumed to be excited and this is a reasonable assumption for our ultimate application to Alfvén waves in a tokamak where the excited spectrum is discrete. In our applications to electrostatic waves our discreteness assumption is often artificial as normally a broad spectrum is excited. However, these problems are still of interest because they illustrate the general method rather simply without the more complicated algebraic manipulations required in the Alfvén wave problem. One can also envisage experimental situations where, due to finite spatial size, boundary conditions restrict waves to a discrete spectrum and the method developed here should be relevant. We also note in passing that our model may be applicable to cases where nonlinear mode locking arises so that conditions where a single mode is excited is effectively achieved.

The growth rate of the system is assumed small, so that in solving the Vlasov equation, the perturbed field amplitude can be treated as quasi-stationary. The slowly growing part of the field amplitude is determined by the wave energy equation which takes a form

$$\frac{\partial WE}{\partial t} + \mathcal{P}_h + \mathcal{P}_d = 0 \tag{1}$$

where WE is the wave energy, which includes the perturbed field energy and the background plasma's "sloshing" energy, $\mathcal{P}_h$ is the power transferred to the hot particles in resonance with the wave and $\mathcal{P}_d$ is the power transferred to background particles by dissipative mechanisms. The wave amplitude will be assumed to always be low enough that the quadratic forms found in linear theory for WE and $\mathcal{P}_d$ apply, viz;

$$WE \equiv G_w \left|\varphi\right|^2 \qquad \text{and} \qquad \mathcal{P}_d \equiv G_d \left|\varphi\right|^2$$

with φ the perturbed field. It will be assumed that $G_w > 0$ and $G_d > 0$.

In linear theory we have

$$\frac{\partial WE}{\partial t} = 2\gamma WE$$

where γ is the growth rate. In the absence of hot particles

$$\gamma = -G_d/2G_w \equiv -\gamma_d \tag{2}$$

and the wave is damped.

With hot particles, at low enough field amplitude, $\mathcal{P}_h$ has the form

$$\mathcal{P}_h \equiv G_h \left|\varphi\right|^2 \tag{3}$$

and we are interested in the case where $G_h < 0$. The growth rate is then

$$\gamma = -\frac{(G_h + G_d)}{G_w} = \gamma_h - \gamma_d \tag{4}$$

with $\gamma_h = -G_h/G_w$. Thus, instability arises if $-G_h > G_d$ (or $\gamma_h > \gamma_d$). The wave will grow until nonlinear effects alter the resonant particle response. Characteristically, at a finite wave amplitude, a nonlinear power transfer to the hot particles, $\mathcal{P}_{h\mathrm{NL}}$, can be expected to have the form

$$\mathcal{P}_{h\mathrm{NL}} \approx \frac{\nu_{\mathrm{eff}}}{\omega_b} G_h \, |\varphi|^2 \tag{5}$$

where ν_{eff} is the rate in which resonant particles leave the resonant region and ω_b the bounce frequency of a trapped particle at the bottom of the potential well. Hence stabilization would then arise when $\frac{\nu_{\mathrm{eff}}}{\omega_b} G_h + G_d \approx 0$, or when the field amplitude is large enough that

$$\frac{\omega_b}{\nu_{\mathrm{eff}}} \approx -G_h/G_d \, .$$

In practice we find such scaling sometimes occurs, as in Sec. III, but at other times the wave particle interaction causes steep phase space gradients that can alter the above inferred scaling, as in Sec. IV. Thus, detailed knowledge of the background wave properties including dissipation, and the saturation associated with resonant particle trapping, determines the nonlinear amplitude of the linearly unstable wave. In the next sections we calculate $\mathcal{P}_{h\mathrm{NL}}$ for several model problems.

III. Plasma Wave with Particle Source and Particle Annihilation

The simplest problem we can pose is to consider an electrostatic plasma wave (or acoustic wave) in one dimension. In addition to the background plasma we inject a hot species with an injection velocity distribution $Q(v)$. These particles are assumed to annihilate (through some physical mechanism such as charge exchange) at a rate ν_a. The kinetic equation for the distribution function, f, is

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} \mathcal{E} \frac{\partial f}{\partial v} = -\nu_a(v) f + Q(v) \tag{6}$$

where $\mathcal{E}$ is an electrostatic electric field, $\mathcal{E} = -\partial\varphi/\partial x$ with $\varphi = \varphi_0 \cos(kx - \omega t)$.

The power transfer is given by

$$\mathcal{P} = \frac{k}{2\pi} q \int_0^{2\pi/k} dx \int dv\, \mathcal{E} v f = \frac{k\omega}{2\pi} \varphi_0 q \int_0^{2\pi/k} dx \int dv \sin(kx - \omega t)(1 + ku/\omega) f \ . \tag{7}$$

with $v = \dfrac{\omega}{k} + u$. We anticipate that the contribution to $\mathcal{P}$ is from a narrow region in velocity space so that $|ku/\omega| \ll 1$ and can be neglected in Eq. (7).

In linear theory we have $f = f_0 + f_1$ with

$$f_0 = Q(v)/\nu_a(v) \tag{8}$$

and f_1 satisfies the linear equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f_1 + \nu_a f_1 = -\frac{q}{m} k\varphi_0 \sin(kx - \omega t)\frac{\partial f_0}{\partial v} \ . \tag{9}$$

We shall assume that $\dfrac{\partial}{\partial v}(Q/\nu_a) > 0$ so that instability can arise. It can be shown that if $|k\Delta v| \gg \nu_a$, where $\Delta v = f_0/(\partial f_0/\partial v)$, then $\nu_a f_1$ can be neglected in Eq. (9). The solution to Eq. (9) is then straightforwardly found to be

$$f_1 = \frac{q}{m} k\varphi_0 \frac{\partial f_0}{\partial v}\left[-\frac{P}{(\omega - kv)}\cos(kx - \omega t) - \pi\delta(\omega - kv)\sin(kx - \omega t)\right] \tag{10}$$

where P is the principle value.

By substituting Eq. (10) into Eq. (7), we obtain the linear power transfer $\mathcal{P}_{\rm L}$,

$$\mathcal{P}_{\rm L} = \frac{-q^2\pi}{2m} k\varphi_0^2\omega \int dv\, \delta(\omega - kv)\frac{\partial f_0}{\partial v} = \frac{-q^2\pi}{2m}\varphi_0^2 \frac{\omega k}{|k|}\frac{\partial}{\partial v}\left(\frac{Q(v)}{\nu_a(v)}\right)\Big|_{v=\omega/k} . \tag{11}$$

Note that instability can arise if

$$\frac{\omega}{k}\frac{\partial}{\partial v}\left(\frac{Q(v)}{\nu_a(v)}\right)\Big|_{v=\omega/k} > 0 \ .$$

For the nonlinear problem we assume that $f = f(\psi, u)$, with $\psi = kx - \omega t$, and $u = v - \omega/k$. Equation (6) then becomes,

$$ku\frac{\partial f}{\partial \psi} + \frac{qk\varphi_0 \sin\psi}{m}\frac{\partial f}{\partial u} = -\nu_a(v) f + Q(v) \ . \tag{12}$$

The characteristic equations for the left-hand side of Eq. (12) are

$$\dot{u} = \frac{qk\varphi_0}{m} \sin\psi \qquad \dot{\psi} = ku \; . \tag{13}$$

Equation (13) has a first integral determined from

$$\dot{\psi} \equiv \dot{u}\, \frac{\partial\psi}{\partial u} \equiv \frac{qk\varphi_0}{m} \sin\psi\, \frac{\partial\psi}{\partial u} \; . \tag{14}$$

Integrating Eq. (14) gives

$$\frac{u^2}{2} + \frac{q\varphi_0}{m} \cos\psi = E \tag{15}$$

with E (the energy in the wave frame) a constant. If we use Eq. (15) to transform from u to E, we find

$$\begin{aligned} \pm\, ku(E,\psi)\, \frac{\partial f^\pm}{\partial\psi} &= -\nu_a\left(\frac{\omega}{k} \pm u(E,\psi)\right) f^\pm + Q\left(\frac{\omega}{k} \pm u(E,\psi)\right) \\ &\doteq -\nu_a\left(\frac{\omega}{k}\right) f^\pm + Q\left(\frac{\omega}{k}\right) \pm u(E,\psi)\left[-\nu_a'\left(\frac{\omega}{k}\right) f^\pm + Q'\left(\frac{\omega}{k}\right)\right] \end{aligned} \tag{16}$$

with $u(E,\psi) = \sqrt{2}\,(E - A\cos\psi)^{1/2}$, $A = q\varphi_0/m$, the $\pm$ sign refers to positive and negative u-values, and $q\varphi_0$ is assumed positive.

To proceed further, we assume $\nu_a/\omega_b \ll 1$, where $\omega_b = (qk^2\varphi_0/m)^{1/2}$ is the trapped particle radian bounce frequency. Note that ω_b is also an estimate of the spatially periodic transit frequency of passing particles that are near the separatrix. We then expand $f^\pm$ in powers of ν_a/ω_b (formally we treat $ku \sim \omega_b$ which is appropriate near resonance) and consider the equations

$$\pm\, k\, \frac{\partial f_0^\pm}{\partial\psi} = 0 \tag{17a}$$

$$\pm\, k\, \frac{\partial f_1^\pm}{\partial\psi} = \frac{-\nu_a\left(\frac{\omega}{k}\right) f_0^\pm + Q\left(\frac{\omega}{k}\right)}{u(E,\psi)} \pm \left[-\nu_a'\left(\frac{\omega}{k}\right) f_0^\pm + Q'\left(\frac{\omega}{k}\right)\right] \; . \tag{17b}$$

First we consider passing particles with $E > A$, so that f is a periodic function of ψ with period 2π. From Eq. (17a), it follows that to lowest order f_0 is independent of ψ, i.e., $f_0^\pm = f_0^\pm(E)$. We can determine $f_0^\pm$ from the next order equation, by integrating Eq. (17b) in ψ from 0 to 2π and demanding $f_1^\pm(\psi = 0) = f_1^\pm(\psi = 2\pi)$. We then find

$$f_0^\pm = \frac{Q\left(\frac{\omega}{k}\right) \pm \left\langle \frac{1}{u(E,\psi)} \right\rangle^{-1} Q'\left(\frac{\omega}{k}\right)}{\nu_a\left(\frac{\omega}{k}\right) \pm \nu_a'\left(\frac{\omega}{k}\right)\left\langle \frac{1}{u(E,\psi)} \right\rangle^{-1}} \doteq \frac{Q\left(\frac{\omega}{k}\right)}{\nu_a\left(\frac{\omega}{k}\right)} \pm \left\langle \frac{1}{u} \right\rangle^{-1} \left(\frac{Q}{\nu_a}\right)' \tag{18}$$

where $\langle g(\psi) \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, g(\psi)$. For trapped particles $-A < E < A$ and ψ is restricted to the region $\psi_{\rm min} < \psi < \psi_{\rm max}$ with $\psi_{\rm min} = \cos^{-1} E/A$ and $\psi_{\rm max} = 2\pi - \psi_{\rm min}$. To lowest order Eq. (17a) still gives a ψ-independent solution for f_0. Then using the boundary condition $f^+(E,\psi_{\rm min}) = f^-(E,\psi_{\rm min})$, gives $f_0^+(E) = f_0^-(E) \equiv f_0(E)$. In next order we integrate Eq. (17b) from $\psi_{\rm min}$ to $\psi_{\rm max}$ and find

$$f_1^\pm(\psi_{\rm max}) - f_1^\pm(\psi_{\rm min}) = \frac{\pm 2\pi}{k}\left\{\left[-\nu_a\left(\frac{\omega}{k}\right) f_0 + Q\left(\frac{\omega}{k}\right)\right]\left\langle \frac{1}{u(E,\psi)} \right\rangle \right.$$

$$\left. \pm\left[-\nu_a'\left(\frac{\omega}{k}\right) f_0 + Q'\left(\frac{\omega}{k}\right)\right]\left(\frac{\psi_{\rm max} - \psi_{\rm min}}{2\pi}\right)\right\}$$

with now $\left\langle \frac{1}{u(E,\psi)} \right\rangle = \frac{1}{2\pi}\int_{\psi_{\rm min}}^{\psi_{\rm max}} \frac{d\psi}{u(E,\psi)}$. Now subtracting the f^+ equation from the f^- equation, and using that $f^+ = f^-$ at the endpoints, yields the equation for f_0

$$f_0 = Q\left(\frac{\omega}{k}\right) \Big/ \nu_a\left(\frac{\omega}{k}\right) . \tag{19}$$

Observe that f_0 is continuous at the separatrix between passing and trapped particles, but has discontinuous derivatives.

We are now in a position to construct the nonlinear power transfer $\mathcal{P}_{\rm NL}$. In Eq. (7) we transform from v and x to E and ψ and find

$$\mathcal{P}_{\rm NL} = \frac{m\omega}{k}\int_{-A}^{\infty} dE \left\langle \frac{kA\sin\psi}{u(E,\psi)}(f^+ + f^-)\right\rangle\left[1 + \mathcal{O}\left(\frac{ku}{\omega}\right)\right]$$

$$= -\frac{\omega}{k} m \int_{-A}^{\infty} dE \left\langle u(E,\psi) \frac{k\partial(f^+ + f^-)}{\partial\psi} \right\rangle \tag{20}$$

where the last expression is obtained from an integration by parts. Now from Eqs. (17b), (18) and (19), we obtain $\frac{\partial}{\partial\psi}(f^+ + f^-)$ in terms of the known f_0. We then find

$$\mathcal{P}_{\rm NL} = \frac{-2m\omega\nu_a}{k}\left(\frac{Q}{\nu_a}\right)' \left\{ \int_A^{\infty} dE \left[\langle u(E,\psi)\rangle - \left\langle \frac{1}{u(E,\psi)} \right\rangle^{-1} \right] \right.$$

$$\left. + \int_{-A}^{A} dE \, \langle u(E,\psi)\rangle \right\} = \frac{-2\sqrt{2}\, q^2\omega\varphi_0^2}{m} \frac{\nu_a}{\omega_b} \left(\frac{Q}{\nu_a}\right)' I \, \mathrm{sgn}(k) \tag{21}$$

and

$$I = \int_1^{\infty} dy \left[\left\langle (y-\cos\psi)^{1/2} \right\rangle - \left\langle \frac{1}{(y-\cos\psi)^{1/2}} \right\rangle^{-1} \right]$$

$$+ \int_{-1}^{1} dy \int_{\psi_{\rm min}}^{\psi_{\rm max}} \frac{d\psi}{2\pi} (y-\cos\psi)^{1/2} = 0.291 + \frac{16\sqrt{2}}{9\pi} = 1.09 \, .$$

Observe that trapped particles contribute directly to the power transfer.

The ratio of $\mathcal{P}_{\rm NL}/\mathcal{P}_{\rm L}$ is

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} = 1.9\,\frac{\nu_a}{\omega_b} \, . \tag{22}$$

Suppose a positive background dissipative power transfer, $\mathcal{P}_d$, is present with $-\mathcal{P}_{\rm L} > \mathcal{P}_d$, so that there is linear instability. Then saturation arises when $\mathcal{P}_{\rm NL} = \mathcal{P}_d$ or when the field amplitude reaches the level

$$\omega_b \equiv \left(\frac{q\varphi_0 k^2}{m}\right)^{1/2} = 1.9\nu_a \left(\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d}\right) \, . \tag{23}$$

IV. Plasma Wave Problem with Drag and Annihilation

We now consider a somewhat more complicated problem. We assume we have a high energy particle injection source sharply peaked about a speed v_0 and that the injected particles are annihilated at a rate ν_a. In addition, the fast particles are assumed to slow down due to drag with the background plasma. Because of the annihilation a distribution f is formed with $\partial f/\partial v > 0$, so that an energy source is available to self-excite electrostatic plasma waves.

The one-dimensional kinetic equation for the distribution f is taken as

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + \frac{qk\varphi_0}{m}\sin(kx-\omega t)\,\frac{\partial f}{\partial v} = -\nu_a f + Q_0\delta(v-v_0) + a\,\frac{\partial f}{\partial v}\,. \tag{24}$$

We note that the drag term enters formally the same way as a dc electric field. Hence our problem is related to the dc resistivity of a periodic system.[13]

The power transfer function is still given by Eq. (7), where again the contribution to the power transfer comes from a region where $v-\omega/k \ll \omega/k$.

In linear theory we have $f = f_0 + f_1$ with

$$f_0 = \frac{Q_0}{a}\exp\left[\frac{\nu_a(v-v_0)}{a}\right]\Theta(v_0-v) \tag{25}$$

and f_1 satisfies the linear equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x}\right) f_1 + \nu_a f_1 - a\,\frac{\partial f_1}{\partial v} = -\frac{q}{m}\,k\varphi_0\sin(kx-\omega t)\,\frac{\partial f_0}{\partial v}\,. \tag{26}$$

If $k\Delta v \gg \max\left[\nu_a, a/\Delta v\right]$, with $\Delta v = f_0\left(\dfrac{\partial f_0}{\partial v}\right)^{-1}$, then the terms $\nu_a f_1$ and $a\,\partial f_1/\partial v$ can be neglected in Eq. (26). Equation (10) then gives the expression for f_1, and the linear power transfer is found to be

$$\mathcal{P}_{\rm L} = -\frac{\pi m}{2}\left(\frac{q\varphi_0}{m}\right)^2 \frac{\omega k}{|k|}\,\frac{\nu_a}{a^2}\,Q_0\exp\left(\frac{-\nu_a(v_0-\omega/k)}{a}\right)\Theta\left(v_0-\frac{\omega}{k}\right)\,. \tag{27}$$

For $\dfrac{\omega}{k} < v_0$, $\mathcal{P}_{\rm L} < 0$, an instability arises if $-\mathcal{P}_{\rm L} > \mathcal{P}_d$, where $\mathcal{P}_d$ is the dissipative power transfer rate to the background plasma.

For the nonlinear problem we assume that $f = f(\psi, u)$ with $\psi = kx - \omega t$ and $u = v - \omega/k$. Equation (24) then becomes

$$ku\,\frac{\partial f}{\partial \psi} + \frac{qk\varphi_0}{m}\sin\psi\,\frac{\partial f}{\partial u} = -\nu_a(u+\omega/k)f + Q_0\delta\Big(u + \frac{\omega}{k} - v_0\Big) + a\,\frac{\partial f}{\partial u}\,. \tag{28}$$

We note that the coefficient of $\dfrac{\partial f}{\partial u}$ in Eq. (28) is $-\partial V(x)/\partial x$ where $V(x) = A\cos kx + ax$, with $A = q\phi_0/m$ for particle motion that takes into account the drag. $V(x)$ can be viewed as an effective potential per unit mass. With this potential, an exact constant of motion is $\mathcal{E} = \dfrac{u^2}{2} + V(x)$. In Fig. 1 the solid curve indicates this potential which has maxima and minima if $kA/a > 1$ (when these extrema exist trapped particles arise). The value $u^2/2$ can be graphically obtained in Fig. 1 as the difference of the horizontal line $\mathcal{E} = \mathcal{E}_q$ and $V(x)$. When a particle passes a local peak in the potential $V = V_{\max}$, it will be reflected near the next peak if $E - V_{\max} < 2\pi a/k$. In steady state the passing particles only change their directions as they slow down; they cannot be trapped in the well. If we imagine an adiabatic buildup of a wave from zero to finite amplitude, initially there are particles in the trapping region and if $\nu_{\text{eff}}/\omega_b \gtrsim 1$, passing particles will penetrate into the trapping region. However, as the wave continues to grow in amplitude, so that $\nu_{\text{eff}}/\omega_b < 1$, the motion becomes adiabatic and new particles can no longer penetrate the trapping region when they slow down past the separatrix region. Then the old particles in the trapped region eventually disappear because of the finite annihilation rate. Thus we expect that the structure of the steady-state distribution function dramatically changes as the finite amplitude wave grows. When $kA/a > 1$ a separatrix appears between passing and trapped particles. As particles cannot penetrate this region and the original particles in this region are annihilated, the distribution of trapped particles will be zero. This structure needs to be accounted for in the nonlinear calculation.

We shall solve Eq. (28) in the limit that the right-hand side is considered small and the left-hand side is formally taken to be $\mathcal{O}(1)$. The characteristics of the left-hand side of this equation are determined by Eq. (13). The first integral of these equations give the expression for the energy per unit mass, E, of the particle in the wave frame

$$E = \frac{u^2}{2} + A\cos\psi \tag{29}$$

with $-A < E < \infty$. For $-A < E < A$ particle orbits are trapped in the wave. The energy E is drawn in Fig. 1 as the dotted straight line. If a is sufficiently small, E is close to $\mathcal{E}$ for many periods and this is the basis of our approximate theory. However, E eventually diverges from $\mathcal{E}$ as the particle slows down.

If we now transform from u to E, Eq. (28) becomes

$$\pm\, ku(E,\psi)\,\frac{\partial f^{\pm}}{\partial \psi} = -\nu_a\, f^{\pm} + g_{\pm} Q_0\, u(E,\psi)\delta\Big(E - A\cos\psi - u_0^2/2\Big) \pm a\, u(E,\psi)\,\frac{\partial f^{\pm}}{\partial E} \quad (30)$$

with $u_0 = v_0 - \omega/k$, $g_+ = 1$, $g_- = 0$, and otherwise the same notation is used here as in Eq. (16).

We also construct the nonlinear power transfer. The method is identical with the previous section and we find (see Eq. (20))

$$\mathcal{P}_{\mathrm{NL}} = -m\omega \int_{-A}^{\infty} dE \left\langle u(E,\psi)\, \frac{\partial (f^+ + f^-)}{\partial \psi} \right\rangle . \quad (31)$$

To proceed further, we will also assume for convenience that the wave amplitude is small enough so that the bounce frequency of trapped particles (as well as the transit frequency of passing particles near the separatrix) is less than the external frequency; i.e. $\omega_b/\omega \ll 1$. Also, we restrict ourselves to a "smooth" distribution where $\dfrac{v}{f_0}\dfrac{\partial f_0}{\partial v} \approx \mathcal{O}(1)$ which requires from Eq. (25) $\nu_a \approx a/v$. It then follows that

$$\nu_{\text{eff}} \equiv \frac{ak}{\omega_b} \gg \nu_a \ .$$

Physically, ν_{eff} can be interpreted as follows. The rate in which a particle changes its speed by an amount Δv due to drag is $a/\Delta v$. The velocity width of the separatrix region is $\Delta v = \omega_b/k$, therefore the rate in which drag causes the speed to change by this separatrix width is $ak/\omega_b = \nu_{\text{eff}}$. If we estimate the order of magnitude of the terms in Eq. (30), we find for $ku \approx \omega_b$, $\dfrac{\partial f}{\partial \psi} \approx f$, $\dfrac{\partial f}{\partial E} \approx f/u^2 \approx f/A$, that the right-hand side is less than the left-hand side by a factor $\nu_{\text{eff}}/\omega_b$, which is the basic parameter of our perturbation theory. Now expanding f, we find

$$\pm k \, \frac{\partial f_0^{\pm}}{\partial \psi} = 0$$

$$\pm k \, \frac{\partial f_1^{\pm}}{\partial \psi} = -\frac{\nu_a f_0^{\pm}}{u(E,\psi)} + g_{\pm} Q_0 \delta\Big(E - A\cos\psi - \frac{u_0^2}{2}\Big) \pm a \, \frac{\partial f_0^{\pm}}{\partial E} \, . \tag{32}$$

To properly analyze Eq. (32) it is essential to note that the ordering breaks down near the separatrix between passing and trapped particles as the bounce frequency, $\tilde{\omega}_b(E)$, for particles close to the separatrix approaches zero. Nonetheless, the solution to Eq. (32) is adequate for obtaining the energy transfer. We note that except for right near the separatrix, f can be solved both in the trapping region and the passing region, giving solutions that are generally discontinuous around the separatrix. To obtain the power transfer, we can integrate Eq. (31) between $A(1-\delta) < E < A(1+\delta)$ with $\delta \ll 1$. If we substitute Eq. (30) into Eq. (31), we have

$$\int_{A(1-\delta)}^{A(1+\delta)} dE \left\langle u(E,\psi) \, \frac{\partial (f^+ + f^-)}{\partial \psi} \right\rangle$$

$$= \frac{a}{k} \int_{A(1-\delta)}^{A(1+\delta)} dE \left\langle u(E,\psi) \, \frac{\partial (f^+ + f^-)}{\partial E} \right\rangle \Big(1 + \mathcal{O}(\delta)\Big)$$

$$= \frac{a}{k} \left\langle u(A,\psi) \left[f^+\Big(A(1+\delta)\Big) + f^-\Big(A(1+\delta)\Big) - f^+\Big(A(1-\delta)\Big) - f^-\Big(A(1-\delta)\Big) \right] \right\rangle \, . \tag{33}$$

We need to choose δ such that at the energy $E = A(1 \pm \delta)$ the bounce (transit) frequency $\tilde{\omega}_b\Big(A(1 \pm \delta)\Big)$ is larger than $\nu_{\rm eff}$, and yet have $\delta \ll 1$ so that the E integration in Eq. (33) is in a narrow layer. If $\tilde{\omega}_b(-A) \gg \nu_{\rm eff}$ it readily follows that such a value of δ can be found. It may also be observed that Eq. (33) can be generalized to the case $\nu_{\rm eff}/\omega_b \approx \mathcal{O}(1)$. The E integration should then be taken across the separatrix boundary, and the contribution from the discontinuity is then determined if $f(E,\psi)$ just above and just below the separatrix are known. We have argued that below the separatrix the trapped particle distribution is zero. Above the separatrix one can choose f to sufficient accuracy if $\omega/\omega_b \ll 1$ by taking the unperturbed value of f at the resonant velocity as given by Eq. (25). This calculation will be presented towards the end of this section. However, if better accuracy is desired, f above resonance must be calculated more accurately. This is only easily done in the limit $\nu_{\rm eff}/\omega_b \ll 1$, where the value of the distribution function can be consistently calculated at an energy close to the separatrix; i.e. at $E = A(1+\delta)$. The method of evaluation is indicated below.

Returning to our discussion of the basic perturbation method for $\nu_{\rm eff}/\omega_b \ll 1$ we see that in evaluating the power transfer given by Eq. (31), we need to integrate over nearly all of the E region where f is given by the solutions of Eq. (32), and then add the contribution from Eq. (33), which accounts for the rapid change of f around the separatrix.

Now in solving Eq. (32) we first consider passing particles with $E > A$. To lowest order $\partial f_0^\pm/\partial\psi = 0$, and therefore $f_0^\pm = f_0^\pm(E)$. The form for $f_0^\pm(E)$ is obtained by integrating the $\partial f_1^\pm/\partial\psi$ equation in Eq. (32) between $0 < \psi < 2\pi$. Then, as $f_1^\pm(0) = f_1^\pm(2\pi)$, the f_1 term cancels, and we find that $f_0^\pm(E)$ satisfies the equation,

$$a\frac{\partial f_0^\pm}{\partial E} \mp \nu_a f_0^\pm \left\langle \frac{1}{u(E,\psi)} \right\rangle = -g_\pm Q_0 \left\langle \delta\left(E - A\cos\psi - \frac{u_0^2}{2}\right)\right\rangle . \tag{34}$$

Except for E near $u_0^2/2$, the right-hand side of Eq. (34) vanishes, and the solution to Eq. (34), with the boundary condition $f^+(A) = f^-(A)$, is

$$f_0^\pm = C\,\exp\left[\pm\frac{\nu_a}{a}\int_A^E dE' \left\langle \frac{1}{u(E',\psi)}\right\rangle\right] . \tag{35}$$

To determine the constant C we note that $f_0^+(E\to\infty) = 0$. If $u_0^2 \gg A$, $f_0^+(E)$ remains zero up to the source energy $u_0^2/2$. If we now integrate Eq. (34), in the interval $\frac{u_0^2}{2}(1+\delta) < E < \frac{u_0^2}{2}(1-\delta)$, we find $f_0^+\left(E = \frac{u_0^2}{2}(1-\delta)\right) = Q_0/a$. Then matching Eq. (35) to this result gives

$$\begin{aligned} C &= \frac{Q_0}{a}\exp\left[-\frac{\nu_a}{a}\int_A^{u_0^2/2} dE' \left\langle\frac{1}{u(E',\psi)}\right\rangle\right] \\ &\doteq \frac{Q_0}{a}\exp\left(-\frac{\nu_a u_0}{a}\right)\left[1 + \frac{\nu_a\langle u(A,\psi)\rangle}{a} + \frac{1}{2}\frac{\nu_a^2}{a^2}\langle u(A,\psi)\rangle^2 + \mathcal{O}\left(\frac{\nu_a^3}{\nu_{\rm eff}^3}\right) + \mathcal{O}\left(\frac{\omega_b^2}{\omega^2}\right)\right] . \end{aligned} \tag{36}$$

In order to consider trapped particles in the region $-A < E < A$ we use the boundary conditions

$$f^+(\psi_{\rm max}) = f^-(\psi_{\rm max})$$

$$f^+(\psi_{\rm min}) = f^-(\psi_{\rm min}) \ .$$

As in the previous section, it then follows that $f_0^+ = f_0^- = f_0(E)$. Now, we integrate the $\partial f_1^\pm/\partial\psi$ equation in Eq. (32) from $\psi_{\rm min}$ to $\psi_{\rm max}$ and obtain the expression for $f^\pm(\psi_{\rm max}) - f_1^\pm(\psi_{\rm min})$. We eliminate f_1 terms by subtracting the f_1^+ equation from the f_1^- equation. We then find (assuming that there is no direct source term in the trapping region),

$$\nu_a \int_{\psi_{\rm min}}^{\psi_{\rm max}} d\psi \, \frac{(f_0^+ + f_0^-)}{u(E,\psi)} = a \int_{\psi_{\rm min}}^{\psi_{\rm max}} d\psi \, \frac{\partial (f_0^+ - f_0^-)}{\partial E} \ .$$

As $f_0^+ = f_0^- = f_0(E)$, it follows that $f_0(E) = 0$. This formally confirms our expectation that the trapped particle distribution vanishes.

We can now calculate the power transfer given by Eq. (31) assuming that there are no particles injected at resonance. We use Eq. (32) to obtain $\dfrac{\partial f_1}{\partial \psi}$ in terms of f_0 and Q, and Eq. (34) to eliminate the source term Q. We find, including the boundary layer contribution given by Eq. (33), and using $f_1 = 0$ in the trapping region,

$$\begin{aligned}\mathcal{P}_{\rm NL} = &-\frac{m\omega}{k} a \Big\langle \Big(u(A), \psi \Big) \Big\rangle \Big[f_0^+ \Big(A(1+\delta) \Big) + f_0^- \Big(A(1+\delta) \Big) \Big] \\ &- \frac{m\omega\nu_a}{k} \int_{A(1+\delta)}^{\infty} dE \, (f_0^+ - f_0^-) \Big(\langle u(E,\psi) \rangle \Big\langle \frac{1}{u(E,\psi)} \Big\rangle - 1 \Big)\end{aligned} \qquad (37)$$

where δ is taken sufficiently large so that $f_0\Big(A(1+\delta)\Big)$ is independent of phase. For $f_0^+ - f_0^-$ we use Eqs. (35) and (36), and that the contribution to the integral in Eq. (37) comes from $E \approx A \ll u_0^2$, to find

$$f_0^+ - f_0^- = \frac{Q_0}{a} \exp\left(-\frac{\nu_a u_0}{a}\right) \left[\exp\left(\frac{\nu_a}{a} \int_A^E dE' \Big\langle \frac{1}{u(E',\psi)} \Big\rangle \right) \right.$$

$$- \exp\left(-\frac{\nu_a}{a}\int_A^\infty dE'\left\langle\frac{1}{u(E',\psi)}\right\rangle\right)\Bigg]$$

$$\doteq \frac{2Q_0\nu_a}{a^2}\exp\left(-\frac{\nu_a u_0}{a}\right)\int_A^E dE'\left\langle\frac{1}{u(E',A)}\right\rangle$$

$$= \frac{2Q_0\nu_a}{a^2}\exp\left(-\frac{\nu_a u_0}{a}\right)\left[\langle u(E,\psi)\rangle - \langle u(A,\psi)\rangle\right] .$$

We also use

$$f^+\Big(A(1+\delta)\Big) \underset{\delta\to 0}{\longrightarrow} f^-\Big(A(1+\delta)\Big) \underset{\delta\to 0}{\longrightarrow} \frac{Q_0}{a}\exp\left(-\frac{\nu_a u_0}{a}\right) .$$

As a result, we have

$$\mathcal{P}_{\rm NL} = -\frac{2m\omega\nu_a^2 Q_0}{ka^2}\exp\left(-\frac{\nu_a u_0}{a}\right)$$

$$\cdot \int_A^\infty dE\left(\langle u(E,\psi)\rangle\left\langle\frac{1}{u(E,\psi)}\right\rangle - 1\right)\left(\langle u(E,\psi)\rangle - \langle u(A,\psi)\rangle\right)$$

$$- \frac{2m\omega}{k}Q_0\exp\left(-\frac{\nu_a u_0}{a}\right)\left[\langle u(A,\psi)\rangle + \frac{\nu_a}{a}\langle u(A,\psi)\rangle^2 + \frac{\nu_a^2}{2a^2}\left\langle u(A,\psi)^2\right\rangle\right]$$

$$= -\frac{2\sqrt{2}m\omega\nu_a^2}{ka^2}Q_0\exp\left(-\frac{\nu_a u_0}{a}\right)A^{3/2}I - \frac{8}{\pi}\frac{m\omega}{k}A^{1/2}Q_0\exp\left(-\frac{\nu_a u_0}{a}\right)\left(1 + \frac{4\nu_a A^{1/2}}{\pi a} + \frac{8\nu_a^2 A}{\pi^2 a^2}\right) \tag{38}$$

with

$$I = \int_1^\infty dy\left(\left\langle\sqrt{y-\cos\psi}\right\rangle\left\langle\frac{1}{\sqrt{y-\cos\psi}}\right\rangle - 1\right)\left(\left\langle\sqrt{y-\cos\psi}\right\rangle - \left\langle\sqrt{1-\cos\psi}\right\rangle\right)$$

$$= \frac{2^{3/2}}{9\pi}\left(\frac{24}{\pi^2} - 1\right) = .143 .$$

Equation (38) can be written as

$$\mathcal{P}_{\rm NL} = \mathcal{P}_{\rm L}\left\{\alpha_1\frac{\nu_a}{\omega_b} + \alpha_2\frac{\nu_{\rm eff}}{\omega_b} + \alpha_3\frac{\nu_{\rm eff}^2}{\nu_a\,\omega_b}\left[1 + \mathcal{O}\left(\frac{\nu_{\rm eff}}{\omega_b}\right)\right]\right\} \tag{39}$$

with $\alpha_1 = 4\sqrt{2}\,I/\pi + 64/\pi^3 \doteq 2.34$, $\alpha_2 = 64/\pi^3$, $\alpha_3 = 16/\pi^2$. The first term in the bracket gives the same scaling as the power transfer calculated in the previous problem (see Eq. (21).

The second term in the bracket has the scaling expected from the simple dimensional argument given by Eq. (5); i.e. for this term the linear power transfer is reduced by the rate in which particles enter the resonant region divided by the transit rate of particles near resonance. The second term is intrinsically larger than the first term by the factor ν_{eff}/ν_a. However, the most dominant contribution comes from the third term in the bracket and it is due to a new effect, the interaction of the drag force with the discontinuity between passing and trapped particles at the separatrix that arises from a finite amplitude wave. One should also note that the correction to the α_3 term can exceed the α_1 and α_2 terms.

We now discuss the problem of obtaining the dominant power transfer when $\nu_{\text{eff}}/\omega_b \approx \mathcal{O}(1)$ as long as $\omega_b/\omega \ll 1$. In the integral for the power transfer given in Eq. (31), a large contribution to the energy integral will come from the discontinuity in $\partial f^{\pm}/\partial E$ that exists at the separatrix (if $\nu_{\text{eff}}/\omega_b \leq 1$) which occurs at

$$E = E_s(\psi) \equiv \frac{a}{k}\left[\sin^{-1}\left(\frac{\nu_{\text{eff}}}{\omega_b}\right) + \left(\frac{\omega_b^2}{\nu_{\text{eff}}^2} - 1\right)^{1/2}\right] - \frac{a}{k}\psi \, .$$

Now integrating Eq. (30) across this separatrix gives

$$\int_{E_s^-(\psi)}^{E_s^+(\psi)} dE\, u(E,\psi)\, \frac{(f^+ + f^-)}{\partial\psi}$$

$$= k\,\frac{\partial V_d(\psi)}{\partial\psi}\, u(E_s(\psi))\left[f^+(E_s^+(\psi)) + f^-(E_s^+) - f^+(E_s^-) - f^-(E_s^-)\right] \qquad (40)$$

where the $+(-)$ superscript on $E_s(\psi)$ means incrementally greater (less) than $E_s(\psi)$ and $V_d(\psi) = a\psi/k$ is the effective potential for the drag. Now, f vanishes for trapped particles and therefore $f^{\pm}(E_s^-) = 0$. To obtain $f^{\pm}(E_s^+)$, we note that it is essentially the same as the unperturbed case when $\omega_b/\omega \ll 1$. For example, this can be seen explicitly in comparing Eqs. (35) and (36) to Eq. (25). They only differ by a small term $\mathcal{O}(\nu_a/\nu_{\text{eff}})$. The explicit correction term we have retained in Eq. (36) is only correctly calculated in the limit $\nu_{\text{eff}}/\omega_b \ll 1$ and it gives the next correction to the dominant power transfer term which was calculated in Eq. (39). However, when there is a separatrix the dominant power transfer can now be obtained for arbitrary $\nu_{\text{eff}}/\omega_b$ as $f^+(E_s^+)$ is known sufficiently accurately, and we find

$$\mathcal{P}_{\text{NL}} \doteq -2\sqrt{2}\,\frac{m\omega}{k}\,Q_0 \exp\left(-\frac{\nu_a u_0}{a}\right) \int_{\psi_{\min}}^{\psi_{\max}} \frac{d\psi}{2\pi}\,\left[E_s(\psi) - A\cos\psi\right]^{1/2} \tag{41}$$

with $\psi_{\min} = \sin^{-1}\left(\frac{\nu_{\text{eff}}}{\omega_b}\right)$ and $\psi_{\max}$ is determined from the condition that $u(E, \psi_{\max}) = 0$, which gives

$$\frac{a}{k}\,\sin^{-1}\left(\frac{\nu_{\text{eff}}}{\omega_b}\right) + \frac{a}{k}\left(\frac{\omega_b^2}{\nu_{\text{eff}}^2} - 1\right)^{1/2} = A\cos\psi_{\max} + \frac{a}{k}\,\psi_{\max}\,.$$

When $\nu_{\text{eff}}/\omega_b \ll 1$, Eq. (41) reduces to lowest order to the α_3 term in Eq. (39). If desired Eq. (41) can be used to calculate the higher order corrections to the α_3 coefficient of Eq. (39). Recently, the power transfer has been calculated[14] from a direct integration of Eq. (28) in the limit $\omega_b/\omega \ll 1$ but with $\nu_{\text{eff}}/\omega_b$ arbitrary. The dominant contribution given by Eq. (40) is reproduced and numerically evaluated, and the correction term is also evaluated for arbitrary $\nu_{\text{eff}}/\omega_b$. In the limit $\nu_{\text{eff}}/\omega_b \ll 1$, the correction term calculated in Ref. 14 reproduces the $\alpha_2\,\nu_{\text{eff}}/\omega_b$ term in Eq. (39).

It is also interesting to note that the enhanced power transfer that arises when there is a hole in phase space seems to have the following physical explanation. The total number of particles, N_h, evacuated from the hole is

$$N_h = \frac{\sqrt{2}}{k}\int_{\psi_{\min}}^{\psi_{\max}} d\psi\,\left[E_s(\psi) - A\cos\psi\right]^{1/2}(f^+ + f^-)$$

where f^+ and f^- are the values of the distribution function next to the hole. From Eqs. (35), (36), and (41) it follows that, as $\nu_a \to 0$, that the power transfer from waves to particles can be written as

$$P_{\text{NL}} = -\left\langle N_h\,F_{\text{drag}}\,v_{\text{wave}}\right\rangle$$

where $F_{\text{drag}} = ma$ is the drag force and $v_{\text{wave}} = \omega/k$ is the wave speed. In other words the work (per unit time) the drag force exerts on the holes converts into wave energy.

The ratio of the nonlinear to linear power transfer is from Eqs. (27) and (39),

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} = \frac{16}{\pi^2}\frac{a^2k^2}{\nu_a\omega_b^3} = \frac{16}{\pi^2}\frac{\nu_{\rm eff}^2}{\nu_a\omega_b} . \tag{42}$$

If φ is large enough, the nonlinear energy transfer rate is lower than the linear transfer rate, and saturation arises when $\mathcal{P}_{\rm NL}$ becomes less than $\mathcal{P}_d$, the background dissipation rate. This arises when

$$\omega_b^2 \equiv \frac{k^2 q\varphi_0}{m} > \left(\frac{16}{\pi^2}\right)^{2/3}\left(\frac{a^2k^2}{\nu_a}\frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d}\right)^{2/3} . \tag{43}$$

We also note that because of the enhanced contribution from the discontinuity at the separatrix $\mathcal{P}_{\rm NL}$ exceeds $\mathcal{P}_{\rm L}$ when

$$\nu_{\rm eff} \lesssim \omega_b < \left(\frac{16}{\pi^2}\right)^{1/3}\left(\frac{a^2}{\nu_a^2 v_r^2}\right)^{1/3}(\omega^2\nu_a)^{1/3} .$$

This enhancement of the nonlinear power transfer turns off when $\nu_{\rm eff}/\omega_b \to 1$, when the separatrix disappears. For $\nu_{\rm eff}/\omega_b > 1$ the power transfer is comparable to the linear level and is numerically calculated in Ref. 14. Our steady-state distribution function has an extreme shape due to the absence of trapped particles. This probably gives rise to side band instabilities. The investigation of the side band instability is beyond the scope of this study. We note, however, that if boundary conditions preclude neighboring $\mathbf{k}$ values from being excited, then the single mode saturation described here may be a realistic equilibrium. The present description of nonlinear saturation serves as a prototype for the application to the saturation of alpha particle driven Alfvén wave instabilities, which is studied in an accompanying paper.[2]

V. Conclusion

We have investigated the saturated state of plasma waves excited in a driven system with particle injection at high energy. A "bump-on-tail" distribution is formed by a particle annihilation process that can arise from charge exchange.

In the simplest model treated, a particle annihilation rate ν_a is the only transport process accounted for. In this case the resonant particles are formed from direct injection of the neutral particle source into the resonance region. The resonant particle wave interaction of an individual resonant particle only terminates when the particle is annihilated by the charge exchange process. For sufficiently large wave amplitudes, where $\omega_b > \nu_a$, the ratio of the nonlinear power transfer (of beam particles to the wave), $\mathcal{P}_{\mathrm{NL}}$, to the linear power transfer, $\mathcal{P}_{\mathrm{L}}$, scales as

$$\frac{\mathcal{P}_{\mathrm{NL}}}{\mathcal{P}_{\mathrm{L}}} \approx \frac{\nu_a}{\omega_b}$$

where $\omega_b = \left(\frac{q\,k^2\varphi_0}{m}\right)^{1/2}$ is the bounce frequency of a particle trapped in the wave. Thus with increasing field amplitude the relative power transfer of particles to the wave reduces from the prediction of linear theory. If the background dissipative mechanisms remain linear and the power transfer of waves to the background plasma is given by $\mathcal{P}_d$, with $\mathcal{P}_d/\mathcal{P}_{\mathrm{L}} < 1$, then the amplitude for wave saturation occurs when

$$\frac{\omega_b}{\nu_a} \approx \frac{\mathcal{P}_{\mathrm{L}}}{\mathcal{P}_d} .$$

The second problem is somewhat more complex. A particle drag force, $-ma$, is added as a transport mechanism and the injection speed is higher than the phase velocity of the wave. In order for particles to interact resonantly they must slow down and go through the resonant region. In steady state, when the excited field reaches a steady amplitude, the injected particle cannot penetrate the trapping region. Then the trapping region is empty as the particle annihilation process removes particles that may have originally been trapped. Hence, there is a discontinuity in the distribution function between passing and trapped particles. This discontinuity dominates the nonlinear power transfer between particles and waves. We have presented a nonlinear calculation of the power transfer that is valid when

$$\omega_b > (ka)^{1/2}$$

where k is the wave number. We find

$$\frac{\mathcal{P}_{\rm NL}}{\mathcal{P}_{\rm L}} \approx \frac{k^2\, a^2}{\nu_a\, \omega_b^3}\ .$$

This result indicates that the nonlinear power transfer, $\mathcal{P}_{\rm NL}$, from particles to waves can exceed the linear power transfer, $\mathcal{P}_{\rm L}$, if

$$\omega_b < \left(\frac{k^2\, a^2}{\nu_a}\right)^{1/3}\ .$$

Hence a growth rate enhanced from linear theory is predicted for moderate field amplitudes. For large enough field amplitudes we find $\mathcal{P}_{\rm NL} < \mathcal{P}_{\rm L}$, and saturation is predicted when

$$\omega_b \approx \left(\frac{k^2\, a^2}{\nu_a}\, \frac{\mathcal{P}_{\rm L}}{\mathcal{P}_d}\right)^{1/3}\ .$$

These nonlinear scaling results indicate that $\mathcal{P}_{\rm NL}/\omega_b^4$ increases as ω_b decreases. However, this scaling is only valid if $\omega_b > (ka)^{1/2}$. It follows from Eq. (40) that as $\dfrac{ka}{\omega_b^2} \equiv \nu_{\rm eff}/\omega_b \to 1$, that the nonlinear power transfer decreases rapidly since the width of the separatrix is becoming small. The precise evaluation of the power transfer for $\nu_{\rm eff}/\omega_b \approx 1$ is given in Ref. 14.

Finally, we emphasize that we have restricted our discussion to systems with well-separated discrete modes. This assumption is physically appropriate to our ultimate application to Alfvén waves in tokamaks. This discreteness is assumed to preclude side-band instabilities that are known to be important when trapped particle effects arise.[14,15]

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Figure Caption

1. Particle Potential and Energy Diagram

   The oscillating and linearly increasing curve is the effective potential $V(x) \equiv \frac{q\varphi_0}{m} \cos kx + ax$. The horizontal dashed line, $V = \mathcal{E}_q$ is the particle energy including drag which is a constant of motion. The distance between the oscillatory curve and a horizontal line is one half the square of the particle speed, $u^2/2$. Note that particles exist in the region where $\mathcal{E} - V(x) > 0$, and that a passing particle, such as the one with $\mathcal{E} = \mathcal{E}_q$, changes its velocity from positive to negative at the turning point t. However, such a passing particle cannot penetrate into the trapping region that exists between the points a and b. The trapping regions are indicated in the shaded area. The dotted line indicates the quantity $E = \frac{q\varphi_0}{m} \cos kx + \frac{u^2}{2}$ which changes linearly due to drag as a particle moves in x.

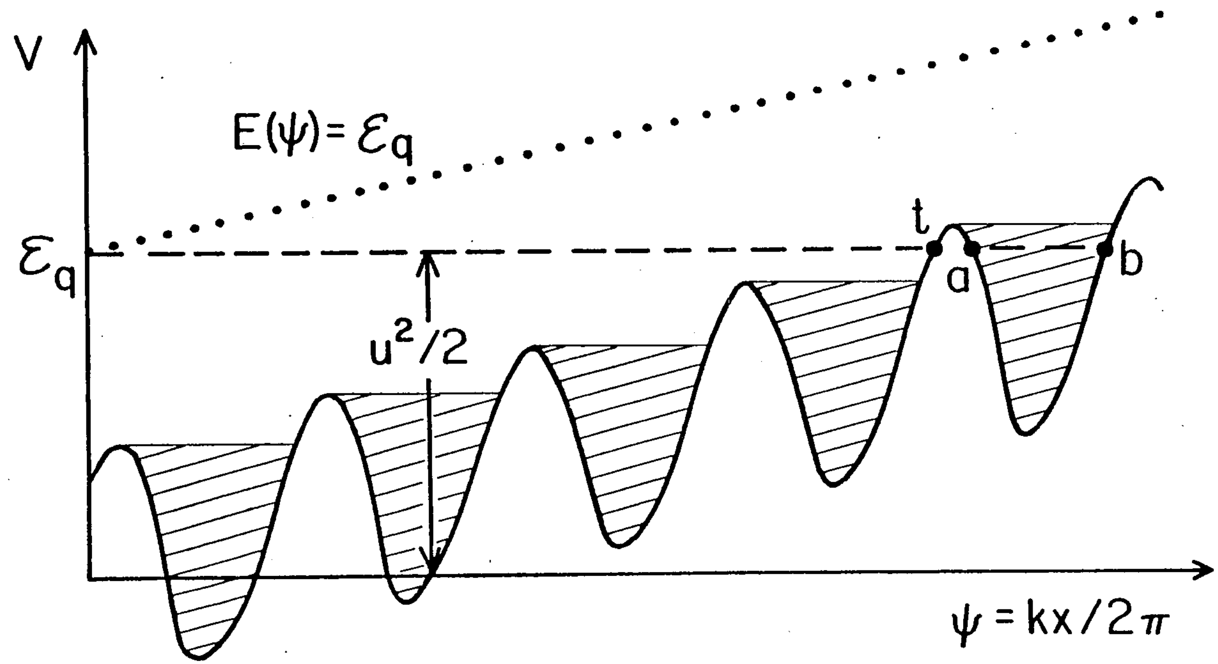

Fig. 1