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Reductive Perturbation Method for Quasi One-Dimensional Nonlinear Wave Propagation II: Applications to Magnetosonic Waves

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Abstract

By means of an application of the generic theory developed in the first paper of this series, the system of two-fluid magnetohydrodynamic equations is reduced to the three-dimensional Kadomtsev-Petviashvili equation. Stability conditions of the fast and slow magnetosonic solitons for transverse slow modulations are established for an isothermal plasma.

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1 Introduction

In a previous paper[1] by one of the present authors, which will hereafter be denoted as (I), a generic theory of quasi one-dimensional nonlinear wave propagation was studied by means of the reductive perturbation method. It dealt with an asymptotic method of reduction for waves of small but finite amplitude, and it was shown that weakly dispersive and dissipative systems can be reduced to the Kadomtsev-Petviashivili (K.P.) equation[2] and the Zablotskaya-Khokholov (Z.K.) equation,[3] respectively. Also, it was remarked that there exist exceptional cases. For example, in a weakly dispersive system, in general the dispersive terms are given by third-order derivatives with respect to the space and time variables; however, there exist exceptional systems with dispersive terms comprising second-order derivatives. A typical example is two-fluid magnetohydrodynamics without dissipation. By means of the reductive perturbation method this system was reduced to the K.P. equation by DeVito and Pantano[4] for a cold plasma and by Shah and Bruno[5] for a warm plasma. However, in both of these works a one-dimensional transverse perturbation was considered, which is a slow variation in the direction normal to the ambient magnetic field as well as the primary direction of propagation, and consequently two-dimensional (2D) K.P. equations were derived. Also, instabilities of solitons for transverse perturbations[2] were not studied. On the other hand, using a heuristic argument Kuznetsov and others[6,7] derived the three-dimensional (3D) K.P. equation for a cold plasma and showed that the nonlinear evolution of an instability of a fast magnetosonic soliton results in acoustic collapse, for which the calculation in 3D space is crucial.

In the present paper, as an application of the general theory to the exceptional case, the system is reduced to the 3D K.P. equation for a plasma of finite temperature, and stability conditions on the one- and two-dimensional magnetosonic solitons are established for both the fast and slow modes.

2 A Heuristic Derivation of the K.P. Equation

The two-fluid magnetohydrodynamic equations for an isothermal hydrogen plasma are given by[8,9]

$$\frac{\partial n}{\partial t} + \mathrm{div}\,(n\mathbf{v}_i) = 0 \; , \tag{1}$$

$$\frac{d\mathbf{v}_i}{dt} = -\beta(\,\mathrm{grad}\,n)/n + (1/n)\,\mathrm{curl}\,\mathbf{B}\times\mathbf{B} + \gamma\Big[\,\{(1/n)(\,\mathrm{curl}\,\mathbf{B}\cdot\mathrm{grad}\,)\}\,\mathbf{v}_i$$
$$+\frac{d}{dt}\,\{(1/n)\,\mathrm{curl}\,\mathbf{B}\}\Big] - \gamma(1+\gamma)\,[\{(1/n)\,\mathrm{curl}\,\mathbf{B}\cdot\mathrm{grad}\,\}\,\{(1/n)\,\mathrm{curl}\,\mathbf{B}\}] \; , \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}\,(\mathbf{v}_i\times\mathbf{B}) - \mathrm{curl}\,(d\mathbf{v}_i/dt) \; , \tag{3}$$

where n and $\mathbf{v}_i$ are, respectively, the density and the flow velocity of the ions, $\mathbf{B}$ is the magnetic flux vector, γ is the electron-to-ion mass ratio, β is the usual pressure ratio (i.e., a characteristic plasma pressure divided by a characteristic magnetic pressure), and $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_i \cdot \mathrm{grad}$. All the quantities are dimensionless, being normalized in units of a characteristic strength of $\mathbf{B}$, a characteristic density, the gyrofrequency of the ions, and the Alfvén velocity, the latter two quantities then being given in terms of the characteristic field strength and the density. The system of equations is obtained by assuming charge neutrality and then eliminating the electron flow velocity and the electric field vector by means of Ampére's law and the equation of motion for the electron fluid, respectively. In Eq. (2) the effects of the electron inertia appear in the last two terms on the right-hand side, whereas in Eq. (3) the last term on the right-hand side derives from the ion inertia. Both kinds of inertia terms involve second-order derivatives and are responsible for the dispersion of hydromagnetic waves. It is well known that for propagation normal to the magnetic field, the dispersion due to the ion inertia vanishes and the electron inertia is responsible for the dispersion of the fast wave. Here it may be noted that if the flow velocity of the plasma, $\mathbf{v}$, is used instead of the ion velocity, $\mathbf{v}_i$, we have

$$\frac{\partial n}{\partial t} + \mathrm{div}\,(n\mathbf{v}) = 0 \ , \tag{1'}$$

$$\frac{d\mathbf{v}}{dt} = -\beta(\,\mathrm{grad}\,n)/n + (1/n)\,\mathrm{curl}\,\mathbf{B}\times\mathbf{B} - \gamma\left[\{(1/n)\,\mathrm{curl}\,\mathbf{B}\cdot\mathrm{grad}\,\}\{(1/n)\,\mathrm{curl}\,\mathbf{B}\}\right] \ , \tag{2'}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}\,[\mathbf{v}\times\mathbf{B}] - (1-\gamma)\,\mathrm{curl}\,(d\mathbf{v}/dt) - \gamma\Big[\mathrm{curl}\,(d/dt)(\,\mathrm{curl}\,\mathbf{B}/n) + \mathrm{curl}\,\{(\,\mathrm{curl}\,\mathbf{B}/n)\cdot\mathrm{grad}\,\mathbf{v}\}\Big] \ . \tag{3'}$$

The last term in Eq. (2′) does not not contribute to the linear dispersion relation. Consequently, the contribution of the electron inertia to the linear dispersion relation is given by the last term in Eq. (3′) which in this case comprises third-order derivatives, whereas that of the ion inertia is given by second-order derivatives in the same way as in Eq. (3).

For a cold plasma, it is well known that in the one-dimensional case a fast hydromagnetic wave propagating in a uniform plasma can be described by a Korteweg-deVries (KdV) equation.[3,10] Let the uniform density and magnetic field be specified as $n = 1$ and $B_z = \sin\theta$, $B_x = \cos\theta$, and $B_y = 0$ for a wave propagating in the (x, y)-plane with a wavenumber vector $\mathbf{k} = (k_x, k_y)$. Then, if $\gamma = 0$ is assumed, the linear dispersion relation of the system of Eqs. (1)–(3) [or Eqs. (1′)–(3′)] takes the simple form

$$(\omega^2 - k^2)\left[\omega^2 - (k_x\cos\theta)^2\right] - \omega^2 k^2 (k_x\cos\theta)^2 = 0 \ . \tag{4}$$

The first term on the left-hand side of Eq. (4) gives the phase velocities of the fast wave and the Alfvén wave in ideal magnetohydrodynamics (MHD), and the second term is responsible for the dispersion produced by the ion inertia. The dispersion vanishes for $\theta = \frac{\pi}{2}$, that is, for a wave propagating normally across the magnetic field, in which case the electron inertia is responsible for the dispersion. The critical angle beyond which the electron inertia becomes crucial is $\tan^{-1}(\gamma^{-1/2} - \gamma^{1/2})$. We now assume that $k_y \ll k_x \ll 1$, namely, the wave form varies slowly in the x-direction and even more slowly in the y-direction. This may be expressed by means of a small parameter ϵ as $k_y \sim \epsilon$, and $k_x \sim \epsilon^{1/2}$. Then Eq. (4) can be solved to give

$$\omega = k_x + \frac{1}{2}\left[(\cot\theta)^2 k_x^3 + (k_y^2/k_x)\right] + \cdots . \tag{5}$$

The second term on the right-hand side of Eq. (5), which is $\mathcal{O}(\epsilon^{3/2})$, gives the dispersion. The first term in the bracket results from the ion inertia. However, the second term originates from the expansion of the first term of Eq. (4), that is, the nondispersive term; consequently, it exists even in the ideal magnetohydrodynamic limit. Hence, the second term is independent of both the electron inertia and the ion inertia and takes the same form for all θ. It is obvious[2–4] that the coefficient of k_x^3 is in agreement with that of the dispersive term of the KdV equation for $\theta < \theta_c$ and $\gamma = 0$, while k_y^2/k_x is replaced by $\int \frac{\partial^2}{\partial y^2}\, dx$ so that the KdV equation is modified to the K.P. equation

$$\delta n_{,t} + \delta n_{,x} + \frac{3}{2}\,\delta n \delta n_{,x} + \nu \delta n_{,xxx} + \frac{1}{2}\int^x \delta n_{,\eta\eta}\, dx = 0 \ , \tag{6}$$

where δn is the density perturbation, and subscripts after a comma denote partial derivatives. The second term on the left-hand side of Eq. (6) may be eliminated by a Galilean transformation. Moreover, noticing that $\frac{\partial}{\partial x} \sim \epsilon^{1/2}$, $\frac{\partial}{\partial y} \sim \epsilon$ and $\frac{\partial}{\partial t} \sim \epsilon^{3/2}$, we find $\delta n \sim \epsilon$. Consequently upon introducing the coordinates ξ, η, and τ by $\xi = \epsilon^{1/2}(x - t)$, $\eta = \epsilon y$, $\tau = \epsilon^{3/2} t$, we can rewrite Eq. (6) in the usual form

$$\left(n^{(1)}_{,\tau} + \frac{3}{2}\,n^{(1)} n^{(1)}_{,\xi} + \nu n^{(1)}_{,\xi\xi\xi}\right)_{,\xi} + \frac{1}{2}\,n^{(1)}_{,\eta\eta} = 0 \tag{6'}$$

where $\delta n = \epsilon n^{(1)}$.

As is shown in Eq. (5), for $\theta < \theta_c$ we find that $\nu \sim -\frac{1}{2}\cot^2\theta$ is negative, whereas for all θ, ν takes the form[2,3]

$$\nu = (\gamma/2)\left\{1 - \left(\gamma^{-1/2} - \gamma^{1/2}\right)^2 \cot^2\theta\right\} \simeq (1/2)(\gamma - \cot^2\theta) \ , \tag{7}$$

and ν becomes positive for $\theta > \theta_c$ and vanishes at $\theta = \theta_c$. This expression for ν enables us to make an analogy with the shallow water wave, for which the K.P. equation takes the form[2]

$$\left[f_{,\tau} + \frac{3}{2} f f_{,\xi} + \frac{1}{2}\left(\frac{1}{3} - \widehat{T}\right) f_{,\xi\xi\xi}\right]_{,\xi} + \frac{1}{2} f_{,\eta\eta} = 0 \tag{8}$$

where f represents the surface elevation; τ, ξ, and η are the corresponding coordinates, all of which are normalized by the water depth and the linear speed of a water wave; and $\widehat{T}$ is a dimensionless surface tension, which in our case corresponds to the ion inertia term $\cot^2\theta$. A remarkable fact based on the results of Kadomtsev and Petviashvili is that if $1/3 - \widehat{T} < 0$ (i.e., very thin sheets of water), the one-dimensional soliton is unstable for long transverse perturbations. Therefore, we can conclude that the one-dimensional fast magnetosonic soliton is unstable for long transverse perturbations, unless almost perpendicular propagation is considered. On the other hand, in the latter case, ν becomes positive and one-dimensional solitons will be neutrally stable.

For a plasma of finite β, it is well known that in the limit of ideal MHD, the linear dispersion relation reduces to

$$\left[\omega^2 - (\mathbf{B}_0 \cdot \mathbf{k})^2\right]\left[\omega^4 - (1+\beta)\omega^2 k^2 + (\mathbf{B}_0 \cdot \mathbf{k})^2 k^2\right] = 0 \; . \tag{4$'$}$$

Solving Eq. (4′) for ω yields six real roots for the three modes, namely, the Alfvén wave, with $\pm\omega_A$, and the fast and slow magnetosonic waves with $\pm\omega_f$ and $\pm\omega_s$, where

$$\omega_A = \mathbf{B}_0 \cdot \mathbf{k} \tag{9}$$

$$\omega_{f,s} = k\left[\frac{1}{2}\left\{(1+\beta) \pm \left[(1+\beta)^2 - 4\beta(\mathbf{B}_0 \cdot \mathbf{k})^2/k^2\right]^{1/2}\right\}\right]^{1/2} \tag{10}$$

and where the $\pm$ signs correspond to the fast and slow waves, respectively. Then, assuming the same ordering enables us to expand ω_f/k_x and ω_s/k_x in inverse powers of k_x. If $\omega_{f,s}/k_x$ is denoted as $\Lambda_{of,s}$, we have

$$\omega_p \approx \Lambda_{op} k_x + Z_p k_z + \mu_{p2}(k_y^2/k_x) + \mu_{p3} k_z^2/k_x + \cdots \tag{5$'$}$$

where the suffix p denotes f or s. It may be noted that the second term in Eq. (5′), which is linear in k_z, derives from $\mathbf{B}_0 \cdot \mathbf{k}$, and consequently terms linear with respect to k_y do not exist. In other words, if the transverse perturbation is normal to the magnetic field, then $Z_p = 0$.[4,5]

On the other hand, it has been established that the fast and slow waves can be governed by respective KdV equations, for which explicit forms of the coefficients of the nonlinear terms and the dispersive terms have been obtained (see, for example, Ref. 10). Hence, it is readily anticipated that incorporating the KdV equations with the expansion of Eq. (5′) yields the K.P. equations for the fast and the slow waves,

$$\left(u_{,\tau} + \lambda_0' u u_{,\xi} + \nu u_{,\xi\xi\xi}\right)_{,\xi} + \mu_2 u_{,\eta\eta} + \mu_3 \mu_{,\zeta\zeta} = 0 \tag{11}$$

in which ξ, η, ζ, and τ are introduced by

$$\xi = \epsilon^{1/2}(x - \Lambda_{op} t) \tag{12a}$$

$$\eta = \epsilon y \tag{12b}$$

$$\zeta = \epsilon(z - Z_p t) \tag{12c}$$

$$\tau = \epsilon^{3/2} t.. \tag{12d}$$

The coefficient of the nonlinear term λ_0' is always positive[10] and the signs of ν, μ_2, and μ_3 are listed in Table 1. The stable angular domains of one-soliton solutions are shown in Fig. 1. Details will be explained in the next section.

3 Derivation of the K.P. Equation by Means of the Reductive Perturbation Method

We first assume that $\theta < \theta_c$ so that γ is equated to zero. Then, for the configuration under consideration, the system of Eqs. (1′)–(3′) is written in matrix form:

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{3} A^j \frac{\partial U}{\partial x_j} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{d}{dt} K^j U = 0 \,. \tag{13}$$

Here, $x_1 = x$, $x_2 = y$, and $x_3 = z$; U is a column vector with seven components,

$$U = \begin{bmatrix} n \\ v_x \\ v_y \\ v_z \\ B_x \\ B_y \\ B_z \end{bmatrix} ; \tag{14}$$

and A^j and K^j are 7×7 matrices, as follows:

$$A^1 = \begin{bmatrix} v_x & n & 0 & 0 & 0 & 0 & 0 \\ \beta/n & v_x & 0 & 0 & 0 & B_y/n & B_z/n \\ 0 & 0 & v_x & 0 & 0 & -B_x/n & 0 \\ 0 & 0 & 0 & v_x & 0 & 0 & -B_x/n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & -v_y & v_x & 0 \\ 0 & B_z & 0 & -B_x & -v_z & 0 & v_x \end{bmatrix} \tag{15a}$$

$$A^2 = \begin{bmatrix} v_y & 0 & n & 0 & 0 & 0 & 0 \\ 0 & v_y & 0 & 0 & -B_y/n & 0 & 0 \\ \beta/n & 0 & v_y & 0 & B_x/n & 0 & B_z/n \\ 0 & 0 & 0 & v_y & 0 & 0 & -B_y/n \\ 0 & -B_y & B_x & 0 & v_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_y & 0 \\ 0 & 0 & B_z & -B_y & 0 & 0 & v_y \end{bmatrix} \tag{15b}$$

$$
A^3 = \begin{bmatrix}
v_z & 0 & 0 & n & 0 & 0 & 0 \\
0 & v_z & 0 & 0 & -B_z/n & 0 & 0 \\
0 & 0 & v_z & 0 & 0 & -B_z/n & 0 \\
\beta/n & 0 & 0 & v_z & B_x/n & B_y/n & 0 \\
0 & -B_z & 0 & B_x & v_z & 0 & 0 \\
0 & 0 & -B_z & B_y & 0 & v_z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_z
\end{bmatrix} \tag{15c}
$$

$$
K^1 = \begin{bmatrix}
\multicolumn{7}{c}{\text{all the elements are zero}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{16a}
$$

$$
K^2 = \begin{bmatrix}
\multicolumn{7}{c}{\text{all the elements are zero}} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{16b}
$$

$$K^3 = \begin{bmatrix} & & & \\ & \text{all the elements are zero} & & \\ & & & \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{16c}$$

K^1 is responsible for the dispersion in the x-direction, which is associated with an interchange of v_y and v_z, arising from $(\nabla \times v)_x$, whereas K^2 and K^3 yield the dispersion effects in the y- and z-directions, which are higher order than that given by K^1, but, nevertheless will not be discarded. Equation (13) must be supplemented by the subsidiary condition $\operatorname{div} \mathbf{B} = 0$, which perpetuates if it is valid initially.

A. Linear Dispersion Characteristics

The linear dispersion relation is expressed as

$$-\omega \delta U + \left(\sum k_j A_0^j - i \sum \omega k_j K^j \right) \delta U = 0 , \tag{17}$$

where $A_0^j \equiv A^j(U_0)$ and

$$U_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cos\theta \\ 0 \\ \sin\theta \end{pmatrix} . \tag{18}$$

Following the ordering so far assumed, we write $k_1 = \epsilon^{1/2} \overline{k}_x$, $k_2 = \epsilon \overline{k}_y$, $k_3 = \epsilon \overline{k}_z$, and $\omega = \epsilon^{1/2} \varpi$. Then Eq. (17) takes the form

$$\left[-\varpi I + \overline{k}_x A_0^1 + \epsilon^{1/2} \left(\overline{k}_y A_0^2 + \overline{k}_z A_0^3 - i\varpi \overline{k}_x K^1 \right) + \epsilon(-i\varpi \overline{k}_y) K^2 + \epsilon(-i\varpi \overline{k}_z) K^3 \right] \delta U = 0 . \tag{19}$$

As was shown in (I), this equation may be written in a form familiar to the usual perturbation method,

$$\left(H_0 + \epsilon^{1/2} H' + \epsilon H''\right) \Psi = \Lambda \Psi \tag{20}$$

where δU is written as $\Psi, \Lambda \equiv \overline{\omega}/\overline{k}_x$, and H_0 is the unperturbed part

$$H_0 \equiv A_0^1 \tag{21}$$

whereas H' and H'' are perturbation terms

$$H' = (\overline{k}_y/\overline{k}_x)A_0^2 + (\overline{k}_z/\overline{k}_x)A_0^3 - i\Lambda\overline{k}_x K^1 \tag{22}$$

$$H'' = -i\Lambda\overline{k}_y K^2 - i\Lambda\overline{k}_z K^3 \ . \tag{23}$$

The eigenvalue problem of Eq. (20) can be solved by a primitive expansion in powers of $\epsilon^{1/2}$:

$$\Psi = \Psi_0 + \epsilon^{1/2}\Psi_1 + \epsilon\Psi_2 + \epsilon^{3/2}\Psi_3 + \cdots \ , \tag{24}$$

$$\Lambda = \Lambda_0 + \epsilon^{1/2}\Lambda_1 + \epsilon\Lambda_2 + \epsilon^{3/2}\Lambda_3 + \cdots \ . \tag{25}$$

In zeroth order it yields

$$H_0\Psi_0 = \Lambda_0\Psi_0 \ . \tag{26}$$

As is shown by Eqs. (9) and (10), in the limit of ideal magnetohydrodynamics, besides the null eigenvalue (for B_x), there are six real eigenvalues of H_0, corresponding to the fast and slow magnetosonic waves and the Alfvén waves, that is,

$$\Lambda_{on} = \pm\Lambda_{of} \ , \quad \pm\Lambda_{os} \ , \quad \pm\cos\theta \ , \quad \text{and} \quad 0 \ , \tag{27}$$

where $\Lambda_{of} \approx 1 + \frac{\beta}{2}\sin^2\theta$ and $\Lambda_{os} \approx \sqrt{\beta}\cos\theta\left(1 - \frac{\beta}{2}\sin^2\theta\right)$, if $\beta \ll 1$. The respective eigenvectors $\Psi_{on} = (\Psi_f^\pm, \Psi_s^\pm, \Psi_A^\pm, \Psi_{B_x})$ will be represented as

$$\Psi_f^\pm = \begin{bmatrix} 1 \\ \pm\Lambda_{of} \\ 0 \\ v_{zf}^\pm \\ 0 \\ 0 \\ B_{zf}^\pm \end{bmatrix} \tag{28a}$$

$$\Psi_s^\pm = \begin{bmatrix} 1 \\ \pm\Lambda_{os} \\ 0 \\ v_{zs}^\pm \\ 0 \\ 0 \\ B_{zs}^\pm \end{bmatrix} \tag{28b}$$

$$\Psi_A^\pm = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \mp 1 \\ 0 \end{bmatrix} \tag{28c}$$

$$\Psi_{B_x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \tag{28d}$$

Here $v_{zf}^\pm$ etc. are defined by

$$v_{zf}^\pm = \mp(\cos\theta\sin\theta)\Lambda_{of}/(\Lambda_{of}^2 - \cos^2\theta) \approx -\cot\theta\left[1 - \frac{\beta}{2}(1+\cos^2\theta)\right] \tag{29}$$

$$B_{zf}^\pm = \sin\theta\Lambda_{of}^2(\Lambda_{of}^2 - \cos^2\theta) \approx \mathrm{cosec}\theta(1 - \beta\cos^2\theta) . \tag{30}$$

Also, $v_{zs}^{\pm}$ and $B_{zs}^{\pm}$ are given by the same expressions, but with Λ_{of} replaced by Λ_{os}; consequently we find $v_{zs}^{\pm} \approx \pm\sqrt{\beta}\sin\theta\left[1+\frac{\beta}{2}(1+\cos^2\theta)\right]$ and $B_{zs}^{\pm} \approx -\beta\sin\theta(1+\beta\cos^2\theta)$.

In the first place, the fast wave propagating in the x-direction will be considered an unperturbed state, and hence it is given by Ψ_f^+. Hereafter, the superscript + will be omitted unless otherwise stated. From the representations of A_0^2, A_0^3, and K^1 it is readily seen that

$$A_0^2\Psi_f = \begin{bmatrix} 0 \\ 0 \\ \sin\theta\, B_{zf} + \beta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{31a}$$

$$A_0^3\Psi_f = \begin{bmatrix} v_{zf} \\ 0 \\ 0 \\ \beta \\ -\sin\theta\Lambda_{of} + \cos\theta\, v_{zf} \\ 0 \\ 0 \end{bmatrix} \tag{31b}$$

$$K^1\Psi_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -v_{zf} \\ 0 \end{bmatrix}. \tag{31c}$$

Equations (31a) and (31c) mean that the y-components of the flow velocity and the magnetic field are excited by the variation in the y-direction and the dispersion in the x-direction, respectively. This may be regarded as a rotation of the polarization of the velocity and the magnetic field; consequently, the state so obtained belongs to the Alfvén wave mode. On the other hand, Eq. (31b) implies that the magnetosonic wave with a compression of B_x is excited due to the variation in the z-direction. From these representations we find

$$(\Phi_f H' \Psi_f) \equiv \langle f|H'|f\rangle = \left\langle f|(\bar{k}_z/\bar{k}_x)\, A_0^3|f\right\rangle \;, \tag{32}$$

where Φ_{on} is an adjoint eigenvector to Ψ_{on}, introduced by the equation

$$\Phi_{on}\, H_0 = \pm\Lambda_{on}\, \Phi_{on} \;, \tag{33}$$

such that the orthonormality condition

$$(\Phi_{om}\, \Psi_{on}) = \delta_{mn} \tag{34}$$

holds. Specifically, we have $\Phi_{f,s} = \frac{1}{N_{f,s}}\, \Psi_{f,s}^{\dagger}\, S$ and $\Phi_A = \frac{1}{2}\, \Psi_A^{\dagger}$. Here, the 7×7 matrix S is given by

$$S = \begin{bmatrix} \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & & \\ 0 & & & I & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{bmatrix} \tag{35}$$

which makes SH_0 symmetric; also, $\dagger$ denotes the transpose, and $N_{f,s} = (\Psi_{f,s}^{\dagger}\, S\Psi_{f,s}) = \beta + \Lambda_{of,s}^2 + (v_{zf,s})^2 + (B_{zf,s})^2$. Equation (32) yields $\Lambda_{1f} = 2\beta\, v_{zf}/N_f \approx -(\beta/2)\sin 2\theta$, while Ψ_{1f} is given by Eq. (14) of (I) as

$$\Psi_{1f} = \sum_n{}' \, \frac{\langle n|H_0'|f\rangle}{\Lambda_{of} - \Lambda_{on}}\, \Psi_{on} + c_f\, \Psi_f \;, \tag{36}$$

with the definition $H_0' \equiv H'(\Lambda = \Lambda_{of})$.

In the next order, Λ_{2f} is obtained as

$$\Lambda_{2f} = \sum_n{}' \, \frac{\langle f|H_0'|n\rangle\, \langle n|H_0'|f\rangle}{\Lambda_{of} - \Lambda_{on}} + \langle f|H_0''|f\rangle \equiv -\left\langle f \left| H_0' \, \frac{1}{H_0 - \Lambda_{of}}\, H_0' \right| f \right\rangle + i\Lambda_{of}^2\, \bar{k}_y\, B_{zf} \tag{37}$$

where $\langle f|K^3|f\rangle = 0$ has been used. Introducing the expression for H' into Eq. (37) and noting that $\langle f^-|A_0^3|f^+\rangle = \langle A^\pm|A_0^3|f\rangle = 0$, we have

$$\Lambda_{2f} \;=\; (\bar{k}_y/\bar{k}_x)^2 \sum_{A^\pm} \frac{\langle f|A_0^2|n\rangle\, \langle n|A_0^2|f\rangle}{\Lambda_{of} - \Lambda_{on}} + (\bar{k}_z/\bar{k}_x)^2 \sum_{B_x, s^\pm} \frac{\langle f|A_0^3|n\rangle\, \langle n|A_0^3|f\rangle}{\Lambda_{of} - \Lambda_{on}}$$

$$-(\Lambda_{of}\,\bar{k}_x)^2 \sum_{A^\pm} \frac{\langle f|K^1|n\rangle \langle n|K^1|f\rangle}{\Lambda_{of}-\Lambda_{on}} , \tag{38}$$

in which the contribution from the cross term

$$-i\bar{k}_y\,\Lambda_{of} \sum_n \left[\left\langle f|A_0^2|n\right\rangle \left\langle n|K^1|f\right\rangle + \left\langle f|K^1|n\right\rangle \left\langle n|A_0^2|f\right\rangle\right] / (\Lambda_{of}-\Lambda_{on}) \tag{39}$$

cancels the second term of Eq. (37) so that Λ_{2f} becomes real, as was anticipated by the result in the previous section. The dispersion matrix SK_0^1 is not antisymmetric whereas SA_0^2 is symmetric; hence without the contribution from $\langle f|H_0''|f\rangle \propto \langle f|K^2|f\rangle$, which originates from the dispersion in the y-direction, the frequency shift does not become real. This situation is rather special from a generic view, for example, as was assumed in a previous paper[11] since if

$$\left\langle f|K^1|f\right\rangle = \left\langle f|K^2|f\right\rangle = 0 \tag{40}$$

is assumed, ω becomes, in general, complex. We thus find that the present system belongs to the exceptional case defined in (I).

B. Nonlinear Derivation

The reductive perturbation method incorporates the expansion of the linear dispersion relation with the expansion in amplitude, which is assumed to be $\mathcal{O}(\varepsilon)$. Namely, U is expanded as

$$U = U_0 + \epsilon U_1 + \epsilon^{3/2} U_{3/2} + \epsilon^2 U_2 + \cdots , \tag{41}$$

which corresponds to Eq. (24), where $\delta U \propto \Psi_0$ should be $\mathcal{O}(\varepsilon)$. Then the coordinates ξ, η, ζ, and τ are introduced by the extended Gardner-Morikawa transformation as was given in Eq. (6′) [see, e.g., Eqs. (17)–(19) in (I)]:

$$\xi = \epsilon^{1/2}(x-\lambda_0 t) , \tag{42a}$$

$$\eta = \epsilon y , \tag{42b}$$

$$\zeta = \epsilon[z - 2\beta(v_{zf}/N_f)t] \ , \tag{42c}$$

$$\tau = \epsilon^{3/2} t \ . \tag{42d}$$

Here $\lambda(U)$ is one of the seven eigenvalues $(\lambda_{f\pm}, \lambda_{s\pm}, \lambda_{A\pm}, 0)$ of $A^1(U)$, and λ_0 is taken to be $\lambda_{f+}(U_0)$, which is equal to Λ_{of}. (Note that λ and Λ are different quantities.)

Introducing Eq. (41) and the above transformation into Eq. (13), in the first order of expansion, that is, in the order $\epsilon^{3/2}$, we have

$$(A_0^1 - \Lambda_{of})\, U_{1,\xi} = 0 \tag{43}$$

which, in view of the identity $A_0^1 \equiv H_0$, admits a solution

$$U_1 = \Psi_f\, u \tag{44}$$

where u is a scalar function to be determined later. In the next order (the order ϵ^2), Eq. (41) yields

$$(H_0 - \Lambda_{of})\, U_{3/2,\xi} + A_0^2\, U_{1,\eta} + \left(A_0^3 - \left\langle f|A_0^3|f\right\rangle\right) U_{1,\zeta} - \Lambda_{of} K^1\, U_{1,\xi\xi} = 0 \ . \tag{45}$$

Substituting Eq. (44) into Eq. (45) and noticing that $\langle f|A_0^2|f\rangle = \langle f|K^1|f\rangle = 0$, we can solve Eq. (45) to obtain

$$U_{3/2,\xi} = \frac{-1}{H_0 - \Lambda_{of}} \left[A_0^2\, \Psi_f\, u_{,\eta} + A_0^3\, \Psi_f\, u_{,\zeta} - \Lambda_{of}\, K^1\, \Psi_f\, u_{,\xi\xi}\right] + v\Psi_f \ , \tag{46}$$

where v is an arbitrary scalar function. It is obvious that Eq. (36) is a Fourier transformation of Eq. (46). The nonlinear term appears in the next higher order $\epsilon^{5/2}$, which now takes the form

$$(H_0 - \Lambda_{of})\, U_{2,\xi} + U_{1,\tau} + (\nabla_u\, A_0^1 \cdot U_1)\, U_{1,\xi} + A_0^2\, U_{3/2,\eta} + A_0^3\, U_{3/2,\zeta}$$

$$-\Lambda_{of}\left[K^1\, U_{3/2,\xi\xi} + K^2\, U_{1,\xi\eta} + K^3\, U_{1,\xi\zeta}\right] = 0 \ , \tag{47}$$

where ∇_u is defined by the gradient with respect to the seven components of $U\{u_j\}$, i.e., $\nabla_u A_0^1 U_1 = \sum_j \left(\frac{\partial A^1}{\partial u_j}\right)_{U_0} \Psi_{oj} u$. Multiplying this equation by Φ_{of} from the left, and substituting Eq. (46) we obtain the K.P. equation

$$(u_{,\tau} + \lambda_0' u\, u_{,\xi} + \nu u_{,\xi\xi\xi})_{,\xi} + \mu_2\, u_{,\eta\eta} + \mu_3\, u_{,\zeta\zeta} = 0 \ . \tag{48}$$

Here ν and μ_i are given by

$$\nu \ = \ \Lambda_{of}^2 \left\langle f \left| K^1 \frac{1}{H_0 - \Lambda_{of}} K^1 \right| f \right\rangle , \tag{49a}$$

$$\mu_i \ = \ \left\langle f \left| A_0^i \frac{1}{H_0 - \Lambda_{of}} A_0^i \right| f \right\rangle , \quad (i = 2, 3) , \tag{49b}$$

which, by means of Eqs. (28) and (31), take the forms

$$\nu \ = \ -(v_{zf}^2/N_f) \left[\Lambda_{of}^3/(\Lambda_{of}^2 - \cos^2\theta)\right] , \tag{50}$$

$$\mu_2 \ = \ (\beta + \sin\theta\, B_{zf})^2 \left[(\Lambda_{of}/N_f)/(\Lambda_{of}^2 - \cos^2\theta)\right] , \tag{51}$$

$$\begin{aligned} \mu_3 \ = \ & (\beta^2/N_f\, N_s) \left[\frac{(v_{zf}^+ + v_{zs}^+)^2}{\Lambda_{of} - \Lambda_{os}} + \frac{(v_{zf}^+ + v_{zs}^-)^2}{\Lambda_{of} + \Lambda_{os}}\right] \\ & + (-\sin\theta\, \Lambda_{of} + \cos\theta\, v_{zf})^2/(N_f\, \Lambda_{of}) , \end{aligned} \tag{52}$$

while

$$\lambda_0' \ = \ (\nabla_u \lambda_f)_0 \Psi_f = \left(\frac{\partial\lambda_f}{\partial n}\right)_0 + \left(\frac{\partial\lambda_f}{\partial v_x}\right)_0 \Lambda_{of} + \left(\frac{\partial\lambda_f}{\partial B_z}\right)_0 B_{zf} \ . \tag{53}$$

From Eq. (50) we find that ν is negative; also it vanishes for $\theta = \pi/2$, as v_{zf} does, where the electron inertia becomes crucial. On the other hand, Eqs. (51) and (52) show that μ_2 and μ_3 are both positive and do not vanish for $\theta = \pi/2$. We also note that the origin of the coordinate ζ moves in the negative z-direction because $v_{zf}^+ < 0$.

For small values of β, the angular dependences of the coefficients can be shown explicitly. Namely, if $\beta \ll 1$, then $N_f \approx 2\mathrm{cosec}^2\theta + 2\beta(1 - 2\cot^2\theta)$ and $N_s \approx 2\beta$; hence, they may be approximated as

$$\nu \approx -\frac{1}{2}\cot^2\theta\left[1-\beta\left(1+\frac{1}{2}\sin^2\theta\right)\right] , \tag{54}$$

$$\mu_2 \approx \frac{1}{2}+\frac{\beta}{2}\left(1-\frac{1}{2}\sin^2\theta\right) , \tag{55}$$

$$\mu_3 \approx \frac{1}{2}+\frac{\beta}{2}\left(1-\frac{3}{2}\sin^2\theta\right) , \tag{56}$$

$$\lambda_0' \approx \frac{3}{2}+\frac{\beta}{4}\sin^2\theta , \tag{57}$$

while ζ is expressed as

$$\zeta \approx \epsilon\left[z+\left(\frac{\beta}{2}\sin 2\theta\right)t\right] . \tag{58}$$

The approximate expressions of Eqs. (54)–(57) imply that the coefficients may be well approximated by those of the cold plasma; however, it should be noted that to order β, a difference between μ_2 and μ_3 appears, implying an anisotropy in the (y, z)-plane. Here the physical meaning of μ_3 may be noted: The first term on the right-hand side of Eq. (52) results from the intermediate state comprising the slow waves, whereas the second term is due to compression of B_x, and in the small-β approximation the dominant term is the latter one. Hence, for the fast wave, the compression of the x-component of the magnetic field is responsible for perturbations in the z-direction.

For the slow wave we likewise obtain the K.P. equation (48), the coefficients of which may be given by Eq. (50)–(53), provided the subscripts f and s are interchanged. It then follows from Eqs. (50) and (51) that for $\frac{\pi}{2} > \theta > 0$, ν is positive and μ_2 is negative. However, in Eq. (52), the first term can be negative; consequently μ_3 changes sign depending on the values of β and θ. Also, noticing that $(\lambda_{s,v_x})_0\,\Lambda_{os} + (\lambda_{s,n})_0 > 0$, one sees that $\lambda'_{so} > 0$. These results may be seen by expanding in β:

$$\nu \approx \frac{1}{2}\beta^{3/2}\sin^2\theta\cos^2\theta , \tag{59}$$

$$\mu_2 \approx -\frac{1}{2}\beta^{3/2}\cos^3\theta , \tag{60}$$

$$\mu_3 \approx -\frac{1}{2}\beta^{3/2}\cos 3\theta , \tag{61}$$

$$\lambda_0' \approx \sqrt{\beta}\cos\theta , \tag{62}$$

while $\zeta = \epsilon[z - 2\beta(v_{zs}/N_s)t]$ is approximated as

$$\zeta \approx \epsilon[z - \sqrt{\beta}(\sin\theta)t] . \tag{63}$$

In Eq. (61) it is exhibited that in the domain $\frac{\pi}{2} \geq \theta \geq 0$, μ_3 is negative for $\frac{\pi}{6} > \theta \geq 0$ and positive for $\frac{\pi}{2} > \theta > \frac{\pi}{6}$. This implies that there is a critical angle near $\pi/6$, beyond which μ_3 changes sign from negative to positive as θ increases. An explicit form of μ_2 was given by Shah and Bruno.[5] We can easily check that for $\beta \ll 1$, it leads to the same expressions as those in Eqs. (55) and (60). The approximate expressions for μ_2 and μ_3 — viz., Eqs. (55), (56), (60) and (61) — can be shown to be in agreement with those given by the heuristic argument in the previous section. Thus its validity is demonstrated. The merit of the analytical approach in this section is that physical pictures underlying the reduction to the K.P. equation can be visualized. For example, the coefficients (except λ_0') can be given by the diagrams illustrated in Fig. 2, for the fast wave. The same diagrams may be used for the slow wave, in which case the solid lines represent the slow waves, and s^+ and s^- in Fig. 2(c) should read f^+ and f^-, respectively.

It has been well established[10] that for almost normal propagation, the electron inertia is responsible for dispersion and, for the fast wave, the sign of ν changes from positive to negative beyond a critical angle θ_c. However, for the slow wave, this does not take place; namely, ν is positive and decreases to zero as θ increases to $\pi/2$ (whereas, it diverges as $\theta \to 0$).[10] Since the K.P. equation (48) is obtained at order $\epsilon^{5/2}$, the electron inertia terms in Eqs. (2′) can be neglected as being higher order, whereas in Eq. (3′) they are order of $\epsilon^{5/2}$

and higher. Consequently Eq. (13) may be modified to

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{3} A^j \frac{\partial U}{\partial x_j} + (1-\gamma) \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{d}{dt} K^j U + \gamma \frac{\partial^2}{\partial x_1^2} \frac{\partial}{\partial t} K^e U = 0 \tag{64}$$

where the last term derives from $\gamma\,\mathrm{curl}\,\frac{d}{dt}(\,\mathrm{curl}\,\mathbf{B}/n)$ and K^e is represented as

$$K^e = \begin{bmatrix} & & & & & & \\ & & \text{all the elements are zero} & & & & \\ & & & & & & \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} . \tag{65}$$

Hence, $\langle f|K^e|f\rangle$ does not vanish but is equal to $-B_{zf}^2/N_f$ (namely, the system is not exceptional.)

Therefore, for the fast wave propagating almost normally across the magnetic field, ν is approximated as

$$\nu \approx \gamma \Lambda_{of}\, B_{zf}^2/N_f \approx \frac{\gamma}{2}\left[1 - \beta\left(\cos^2\theta + 1 - 2\cot^2\theta - \frac{1}{2}\sin^2\theta\right)\right] . \tag{66}$$

For the slow wave the contribution of the electron inertia is given by

$$\gamma \Lambda_{os}\, B_{zf}^2/N_s \approx \frac{\gamma}{2}\,\beta^{3/2}\sin^2\theta\cos\theta . \tag{67}$$

In comparison with Eq. (59) this is smaller than the contribution from the ion inertia by the factor γ. Consequently, for the slow wave the effect of the electron inertia may be neglected even at $\theta = \frac{\pi}{2}$ where ν vanishes. (This can be seen from the exact expression for ν.[10]) Since λ_s also vanishes for $\theta = \frac{\pi}{2}$, the wave does not propagate in this direction.

The solitary wave of the 3D K.P. equation (48) is readily obtained as

$$u = (6/\lambda_0')(\nu/|\nu|)2\kappa^2 \mathrm{sech}^2\left[\kappa/\sqrt{|\nu|}\,\right)\left\{\xi + P_2\eta + P_3\zeta - \left(\mu_2 P_2^2 + \mu_3 P_3^2 + 4(\nu/|\nu|)\kappa^2\right)\tau\right\}\right] , \tag{68}$$

where κ, P_2, and P_3 are arbitrary constants. For $P_2 = P_3 = 0$, it reduces to the one-soliton solution of the KdV equation, and it is unstable if ν and μ_2 (or μ_3) take different signs, namely, $\nu\mu_2 < 0$ or $\nu\mu_3 < 0$. For $P_2 = 0$ or $P_3 = 0$, we have a one-soliton solution of the 2D K.P. equation which is obtained by putting $\mu_2 = 0$ or $\mu_3 = 0$ in Eq. (48). In this case, if $\nu\mu_3 < 0$ or $\nu\mu_2 > 0$, the one-soliton solution is unstable for perturbations in the y- or the z-direction, respectively. These results are summarized in Table 1 and illustrated in Fig. 1.

4 Conclusions and Remarks

In the present paper, stability conditions on magnetosonic solitary waves are established. For the fast wave, one-soliton solutions of the KdV equation and the 2D K.P. equation are stable only for nearly normal propagation across the magnetic field. The characteristic length of the solitons is the skin depth, c/ω_{pe}. It should, however, be remarked that for finite β, if a disturbance is imposed in the z-direction of the unperturbed magnetic field, a soliton propagating in the (x, y)-plane begins to move in the $\mp z$-direction for the $\pm$ mode of the fast wave, respectively. This suggests that a cylindrical solitary wave propagating radially inward or outward should be examined in detail. (The same is valid for the shock wave.) For the slow mode, one KdV soliton is always unstable for transverse perturbations, and only the 2D K.P. soliton propagating in the (x, y)-plane at an angle $\frac{\pi}{2} \geq \theta \geq \theta_0$ is stable for transverse perturbations that are in the z-direction. The characteristic length is, of course, given by the normalization $v_A/\omega_{ci} = c/\omega_{pi}$. However, in this case the coefficient μ_3 of the second-order z-derivative term becomes negative, and consequently it should be examined as to whether the initial value problem of the 3D K.P. equation (48) is well-posed or not.

It has been shown by Kuznetsov and Musher[6,7] that during the nonlinear evolution of the instabilities, collapse occurs, leading to an acceleration of ions. On the other hand, it has been shown by Ohsawa[10] that resonant ions are accelerated by the magnetosonic solitons and that the acceleration is remarkably efficient for the fast magnetosonic soliton propagating almost normally across the magnetic field, which is stable with respect to transverse perturbations. Incorporating these two results implies that ions are accelerated in all directions.

Finally, we note that the perturbed solutions in lowest order, $U^{(1)}$, is proportional to Ψ_{of} or Ψ_{os}, and therefore if $B^{(1)}$ vanishes, then $v^{(1)}$ and $n^{(1)}$ vanish simultaneously and, by means of the generalized Ohm's law, the perturbation of the electric field $E^{(1)}$ also vanishes. Hence, without a magnetic perturbation, the plasma does not move. In other words, any displacement that does not accompany a magnetic perturbation is excluded. That means that even if β is small, the so-called low-β approximation is not considered, for which the electrostatic motion given by the $\mathbf{E} \times \mathbf{B}_0$ drift (where $E = -\nabla\varphi$) is crucial.

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The Fast Wave

Table 1

	$0 < \theta < \theta_c$	$\theta = \theta_c \sim \frac{\pi}{2} - \sqrt{\gamma}$	$\theta_c < \theta \leq \frac{\pi}{2}$
ν	< 0	0	> 0
μ_2	> 0	> 0	> 0
μ_3	> 0	> 0	> 0
1D soliton in x	unstable		stable
2D soliton in xy	unstable		stable
2D soliton in xz	unstable		stable
stability	unstable		stable

The Slow Wave

Table 2

θ	$0 < \theta < \theta_0$	$\theta = \theta_0 \sim \frac{\pi}{6} \, (\beta \ll 1)$	$\theta_0 < \theta \leq \frac{\pi}{2}$
ν	> 0	> 0	> 0
μ_2	< 0	< 0	< 0
μ_3	< 0	0	> 0
1D soliton in x	unstable		unstable
2D soliton in xy	unstable		stable
2D soliton in xz	unstable		unstable

Figure Captions

1. Phase velocity surfaces for $\beta = 0.5$ and $0 \leq \theta \leq \pi/2$, illustrating the angular stability regions of the one-soliton solutions (cf. Tables 1 and 2); the critical angles θ_o and θ_c shown here are approximate.

2. Diagrammatic representation of the linear frequency shifts for the fast wave: the solid lines represent the fast waves; the other lines represent the intermediate states specified as $A^{\pm}$ (Alfvén waves), $s^{\pm}$ (slow waves), and $B_x(\Psi_{Bx})$; and A_0^2, A_0^3, K^1, and K^2 are the vertices. Diagrams (a), (b), and (c) yield ν, μ_2, and μ_3, respectively, while (d) shows how the cross terms are cancelled by $\langle f|K^2|f\rangle$.

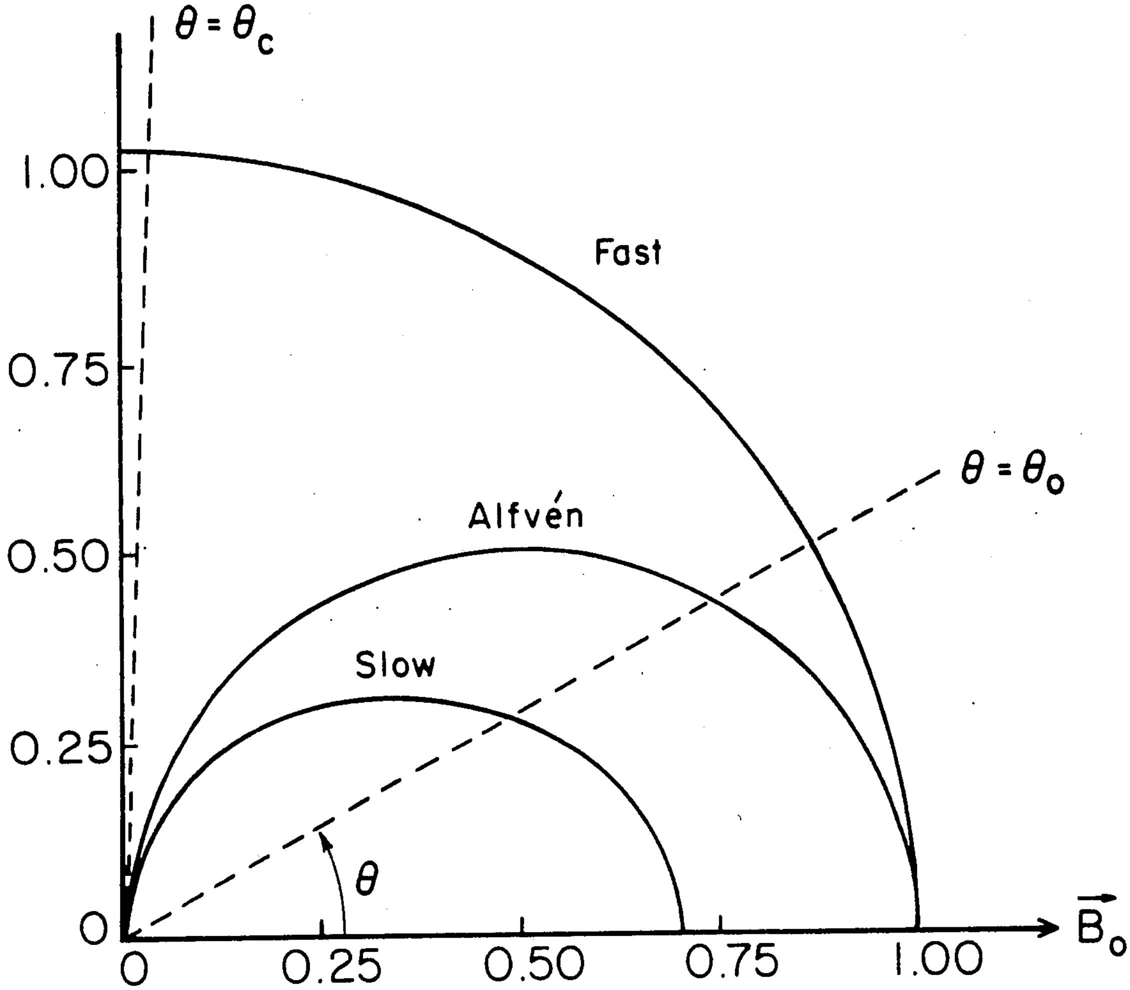
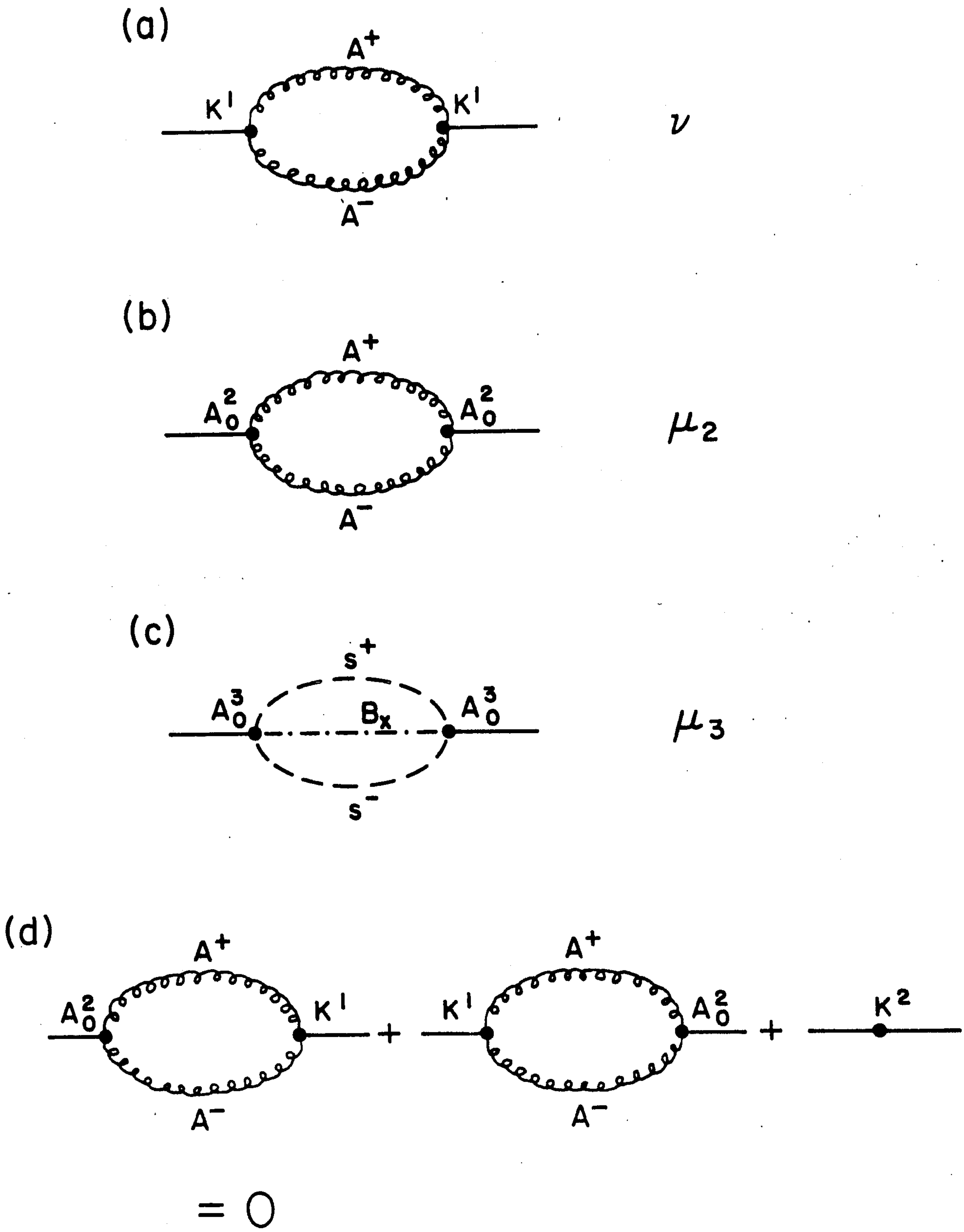


Figure 1



(a)
A^+
K^1
K^1
A^-
ν
(b)
A^+
A_0^2
A_0^2
A^-
μ_2
(c)
s^+
A_0^3
B_x
A_0^3
s^-
μ_3
(d)
A^+
A_0^2
K^1
+
K^1
A^+
A_0^2
+
K^2
A^-
A^-
= 0

Figure 2