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Amplitude Equations on Unstable Manifolds: singular behavior from neutral modes

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Abstract

Perturbation theories that expand in the amplitudes of the unstable modes are an important tool for analyzing the nonlinear behavior of a weak instability which saturates in a final state characterized by small mode amplitudes. If the unstable mode couples to neutrally stable modes, such expansions may be singular because nonlinear effects are very strong even in the regime of weak instability and small amplitudes. Two models are discussed that illustrate this behavior; in each case the unstable mode corresponds to a complex conjugate eigenvalue pair in the spectrum of the linearized dynamics. In the first model, there is only a single neutral mode corresponding to a zero eigenvalue. This example is first solved exactly and then using amplitude expansions. The Vlasov equation for a collisionless plasma is the second model; in this case there are an infinite number of neutral modes corresponding to the van Kampen continuous spectrum. In each of the two examples, the neutral modes sharply reduce the size of the resulting nonlinear oscillation. For the Vlasov instability, the amplitude of the saturated mode is predicted to scale like γ^2 where γ is the linear growth rate.

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These systems usually share another common feature in addition to weak growth rates and small mode amplitudes at saturation: there are no neutral modes (σ_c is empty). The importance of this latter feature is not always emphasized, but it plays a crucial role in the widespread success of amplitude equations in the analysis of weak instabilities. When the physical system has neutral modes, then one can find examples where nonlinear effects are *very strong* even when the unstable modes have arbitrarily small growth rates and saturate at arbitrarily small amplitudes.

The Vlasov equation for a collisionless plasma is a particularly interesting example of this circumstance but there are many other examples as well. In particular one can find very simple dynamical systems in finite dimensions where the dramatic effects of neutral modes can be analyzed in detail. Our interest in this issue originated in a study of weakly unstable modes in a Vlasov plasma,[1,2] but the subtleties of the nonlinear behavior can be best appreciated by first considering less complicated examples. In this discussion we concentrate primarily on a simple three-dimensional dynamical system where the consequences of a single neutral eigenvalue are similar in many respects to the consequences for the Vlasov equation of an infinite continuum of neutral modes.

In the Vlasov case we have focussed on the situation where σ_u contains only a simple conjugate pair of eigenvalues

$$\sigma_u = \{\gamma \pm i\omega\} \tag{1.4}$$

such that $\gamma > 0$, $\omega > 0$ and $0 < |\gamma/\omega| << 1$. One knows that there are small amplitude, fully nonlinear traveling waves (periodic orbits) in such a plasma[3] and numerical studies[4] show that these periodic orbits appear to describe the nonlinear saturation of the instability at least over moderately long time scales $0 < \omega t < 1200$. One question which has been of considerable theoretical and experimental interest is the dependence of the amplitude of the saturated wave on γ as γ is taken to zero.[4–7] In the absence of neutral modes, one commonly finds this dependence to be $\sqrt{\gamma}$. The presence of neutral modes can drastically reduce the size of the saturated wave leading to scalings of the form γ^p where $p = 1$ for our finite dimensional example and $p = 2$ for the Vlasov equation.

In the next section we analyze the equations describing one neutral mode and one unstable complex mode (1.4). Then in section III, we briefly formulate the corresponding one mode instability for the Vlasov equation and describe our results for the amplitude equation in that case.

II. A simple model: one neutral mode

Consider an n-dimensional flow (1.1a) so that $X \in \mathbf{R}^n (n \geq 3)$ and assume the linear matrix $\mathcal{L}$ has a pair of unstable eigenvalues (1.4). We further assume $\mathcal{L}$ has a simple real eigenvalue μ near zero and that all remaining eigenvalues belong to σ_s and are bounded away from the imaginary axis. For (μ, γ) near $(0,0)$, the time-asymptotic behavior of this flow near $X = 0$ is captured by a three-dimensional center manifold, and on this manifold the evolution equation reduces to a three-dimensional dynamical system which may be written

In more complicated problems, finding the exact solution for the periodic oscillation associated with such an instability will not be feasible because the number of neutral modes may be very large. This is the situation for the Vlasov equation. In the absence of exact results there have been several efforts to analyze the nonlinear saturation of such a Vlasov instability using amplitude expansions.[6] These efforts have not been particularly successful, and the origin of the difficulty can be understood by re-analyzing the present model (2.3) using such an amplitude expansion.

We base our derivation of the amplitude equation for the unstable mode on the two-dimensional unstable manifold associated with the two unstable eigenvalues.[8] As long as σ_u consists of a single complex conjugate pair, the unstable manifold will be two-dimensional even if there are many neutral modes. Hence this approach can also be readily applied to the Vlasov problem described in the next section.

The two-dimensional unstable manifold W^u in (2.1) appears as a one-dimensional unstable manifold in (2.3). As shown in Fig. 1, the unstable manifold is tangent to the (r, θ)-plane at $(r, z) = (0, 0)$, and near the origin we may describe the manifold as the graph of a function $h(r)$:

$$(r, z) \in W^u \quad \text{then} \quad (r, z) = (r, h(r)) \tag{2.7}$$

which satisfies

$$h(0) = h'(0) = 0 \ . \tag{2.8}$$

Given $h(r)$, the dynamics on the unstable manifold is obtained by replacing z with $h(r)$ in (2.3a):

$$\dot{r} = r[\gamma + a_1 h(r) + a_2 r^2] \ . \tag{2.9}$$

This is the amplitude equation for the unstable mode; it is expected to be valid for r sufficiently small since our representation of W^u in (2.7) will only hold in general near $(r, z) = (0, 0)$. To calculate the nonlinear oscillation, we seek $r_o > 0$ such that $\dot{r} = 0$ in (2.9)

$$\gamma + a_1 h(r_o) + a_2 r_o^2 = 0 \ . \tag{2.10}$$

This should determine $r_o(\gamma)$ from which the scaling behavior as $\gamma \to 0$ could be calculated.

Before (2.10) can be solved, we must find an expression for $h(r)$. The equation determining $h(r)$ follows from the fact that W^u is invariant under the dynamics (2.1). For a solution $(r(t), z(t)) \in W^u$ there are two ways to calculate $\dot{z}$: (i) from $z = h(r)$ and (2.9) we have

$$\dot{z} = \frac{dh}{dr} r[\gamma + a_1 h(r) + a_2 r^2] \tag{2.11a}$$

(ii) from $z = h(r)$ and (2.3b) we have

$$\dot{z} = \mu h(r) + b_1 r^2 + b_2 h(r)^2 \ . \tag{2.11b}$$

On W^u, these two calculations must agree; hence

$$\frac{dh}{dr} r[\gamma + a_1 h(r) + a_2 r^2] = \mu h(r) + b_1 r^2 + b_2 h(r)^2 \tag{2.12}$$

and

$$\frac{dH}{dx} = \frac{b_1 x^2 + b_2 H^2}{x[1 + a_1 H + a_2 \gamma x^2]}, \quad H(0) = 0 \tag{2.20}$$

respectively. The original system (2.3) is rescaled to

$$\dot{x} = \gamma x[1 + a_1\zeta + a_2\gamma x^2] \tag{2.21a}$$

$$\dot{\zeta} = \gamma[b_1 x^2 + b_2\zeta^2] \; . \tag{2.21b}$$

One could now solve (2.20) perturbatively $H(x) = \tilde{\alpha}_1 x^2 + \tilde{\alpha}_2 x^4 + \cdots$, and then attempt to find x_o from (2.19) by solving $1 + a_1 H(x_o) + a_2\gamma x_o^2 = 0$.

The limitations of this rescaled perturbation theory are most readily appreciated by first considering the exact system (2.21). The exact solution for the periodic orbit (2.4) is now given by

$$(x_o, \zeta_o) = \left(\sqrt{\frac{-b_2}{b_1 a_1^2}}, \frac{-1}{a_1}\right) \quad \text{as } \gamma \to 0 \; .$$

The stability of this solution is found by linearizing (2.21) about (x_o, ζ_o) and finding the eigenvalues $\Lambda_\pm$,

$$\Lambda_\pm = \frac{-b_2}{a_1} \pm \sqrt{\frac{b_2}{a_1}\left(\frac{b_2}{a_1} - 2\right)} + \mathcal{O}(\gamma) \; .$$

Since (2.2) implies $b_2/a_1 > 0$ there are essentially two possibilities:

(*i*) $2 < \frac{b_2}{a_1} < \infty$. For these parameter values, $\Lambda_\pm$ form a complex conjugate pair and $\mathbf{Re}\Lambda_\pm < 0$ so the solution (x_o, ζ_o) is stable.

(*ii*) $0 < \frac{b_2}{a_1} < 2$. For these parameter values, $\Lambda_\pm$ are real and negative. The periodic orbit (x_o, ζ_o) is again stable.

The global behavior of the unstable manifold of the origin is quite different for these two cases. The phase portraits are shown in Fig. 2 with the corresponding evolution of the mode amplitude for the flow on W^u. Note that since x_o is not small, a low order approximation to W^u, *i.e.* $H(x) = \tilde{\alpha}_1 x^2 + \mathcal{O}(x^4)$, will not in general lead to accurate results. When the eigenvalues $\Lambda_\pm$ are complex, there is an additional difficulty. In this case, the desired solution (x_o, ζ_o) is not located on the segment of W^u described by solving (2.20). Thus even if the perturbation series for $H(x)$ could be summed the resulting calculation of x_o would be wrong.

III. Collisionless one mode beam-plasma instability

The problem of a weakly unstable wave in a collisionless plasma provides a much less trivial example of singular behavior in an amplitude expansion.[6] Consider a one-dimensional plasma with mobile electrons, a neutralizing fixed background of positive charge density en_o

and $z = i\lambda/k$ satisfies

$$\Lambda(k,z) \equiv 1 + \int_{-\infty}^{\infty} \frac{dv\, \eta(k,v)}{v-z} = 0$$

with

$$\eta(k,v) = -\left(\frac{4\pi e^2 n_o}{mk^2}\right) \partial_v F_o(v) \ .$$

To analyze the amplitude equation on the unstable manifold, we introduce the complex mode amplitude $A = re^{i\theta}$ as before:

$$f(x,v,t) = A\psi(x,v) + \overline{A}\overline{\psi}(x,v) + S(x,v,t) \tag{3.4}$$

where S represents the components in the eigenfunction expansion for $\mathcal{L}$ orthogonal to ψ and $\overline{\psi}$. The evolution equation (3.2a) determines the equations for $\dot{A}$ and $\partial_t S$:

$$\dot{A} = \lambda A + < \tilde{\psi}, \mathcal{N}(f) > \tag{3.5a}$$

$$\partial_t S = \mathcal{L}S + \mathcal{N}(f) - [< \tilde{\psi}, \mathcal{N}(f) > \psi + \ c.c.] \tag{3.5b}$$

where $< \tilde{\psi}, \cdot >$ denotes the projection with the appropriate adjoint eigenfunction $\tilde{\psi}$. The amplitude equation for $r = |A|$ follows by restricting (3.5a) to the two-dimensional unstable manifold. This calculation will be discussed elsewhere[10]; the results can be easily summarized. From translation invariance it follows that the amplitude equation will take the form

$$\dot{r} = r[\gamma + g(r^2)] \tag{3.6}$$

where $g(x)$ is a function satisfying $g(0) = 0$. The amplitude expansion will, in principle, give the result

$$g(r^2) = \sum_{n=1}^{\infty} \alpha_n r^{2n} \ . \tag{3.7}$$

In practice the calculation of the coefficients is laborious and at present only the leading term $\alpha_1 r^2$ has been analyzed. In the limit of weak instability we find[10]

$$\alpha_1 = \frac{1}{\gamma^3}[b_o + \mathcal{O}(\gamma)] \quad \text{as } \gamma \to 0^+ \tag{3.8a}$$

with

$$b_o = -\frac{1}{12}\left(\frac{k^3}{4m|\Lambda'(k,r)|^2}\right)^2 \left[(\pi\eta'(k,r))^2 + 55\left(P\int_{-\infty}^{\infty} \frac{\eta'(k,v)\,dv}{v-r}\right)^2\right] \tag{3.8b}$$

where

$$\eta'(k,v) \equiv \frac{\partial}{\partial v}\eta(k,v)$$

$$\Lambda'(k,r) \equiv P\int_{-\infty}^{\infty} \frac{\eta'(k,v)\,dv}{v-r} + i\pi\eta'(k,r) \ .$$

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[11]At this conference, I learned of unpublished work by E. Larson which also predicts this scaling. Larson's theory uses asymptotic techniques to incorporate a boundary layer in velocity space at the linear phase velocity. He obtains an amplitude equation of the same form as (3.10).

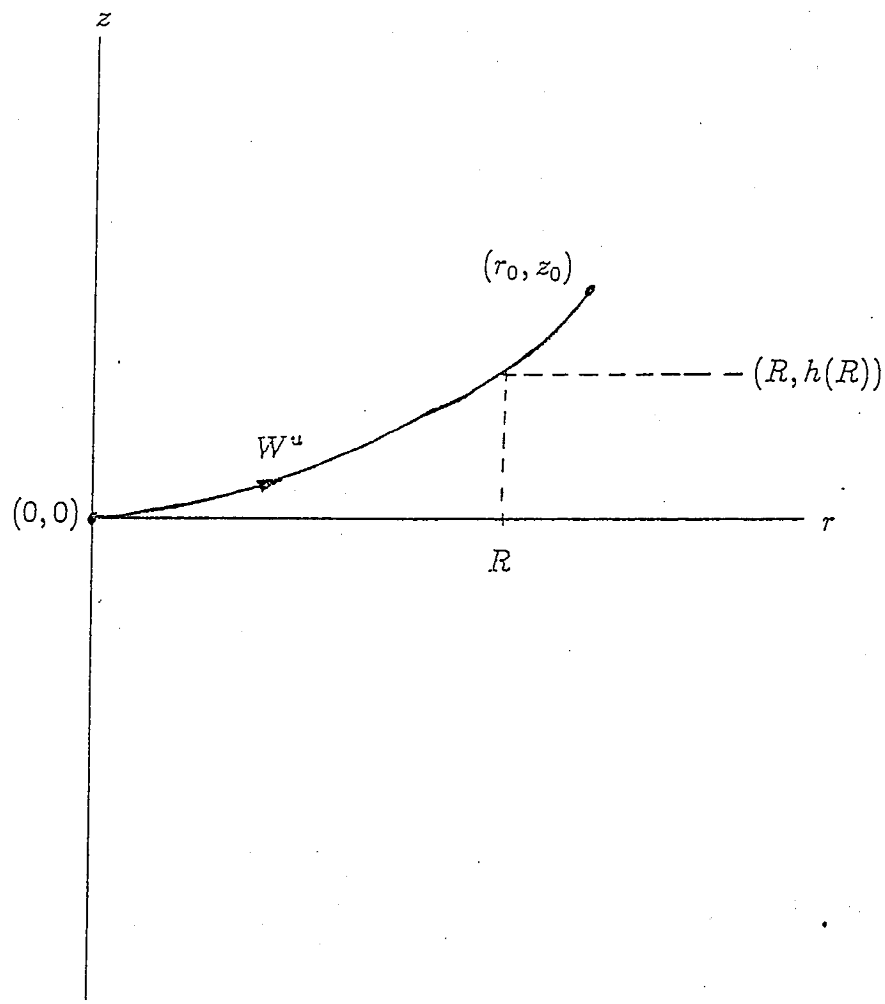
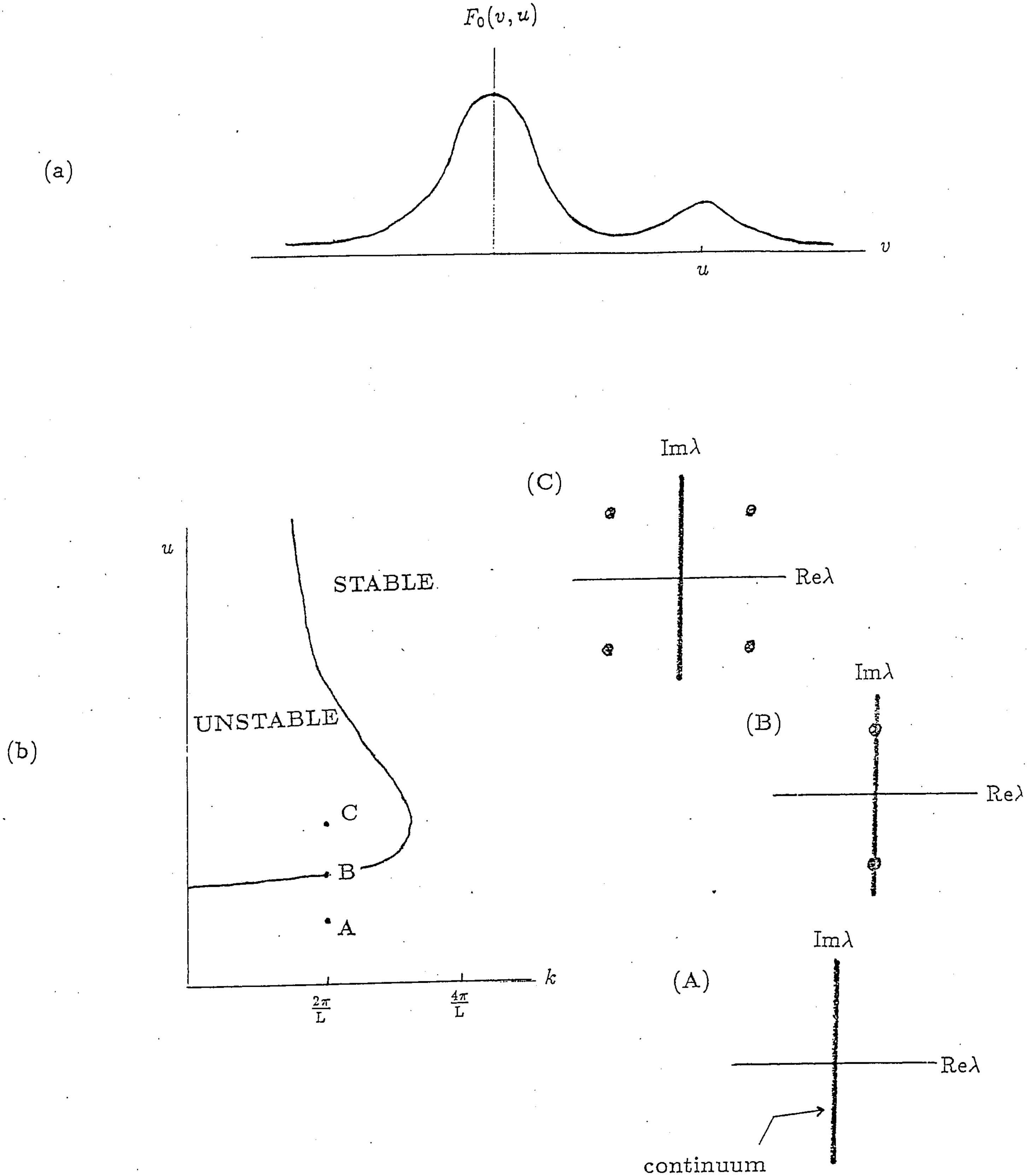


Figure 1



$F_0(v,u)$
(a)
v
u
(b)
u
STABLE
UNSTABLE
C
B
A
k
$\frac{2\pi}{L}$
$\frac{4\pi}{L}$
(C)
Imλ
Reλ
(B)
Imλ
Reλ
(A)
Imλ
Reλ
continuum

Figure 3

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I. Introduction

The circumstance that leads to an amplitude equation with singular behavior may be roughly formulated as follows. A physical system has an equilibrium state X_o and the dynamics of any other state $X_o + X$ may be described by an evolution equation of the form

$$\frac{dX}{dt} = \mathcal{L}X + \mathcal{N}(X), \quad X \in M, \tag{1.1a}$$

where $\mathcal{L}$ is a linear operator that depends on X_o and $\mathcal{N}(X)$ represents nonlinear terms in X. In practice (1.1a) can be either finite dimensional (an o.d.e.) or infinite dimensional (a p.d.e.) depending on the dimension of M, the phase space of the system.

For X near X_o it is common to concentrate first on the linearized dynamics, ignoring in a first approximation the nonlinear terms $\mathcal{N}(X)$. Often the spectrum of $\mathcal{L}$, denoted by σ, determines the solutions to the linear problem:

$$\frac{dX}{dt} = \mathcal{L}X \ . \tag{1.1b}$$

One first finds the spectrum by analyzing the eigenvalue problem

$$\mathcal{L}\Psi = \lambda\Psi \tag{1.2}$$

(or more precisely from the properties of the resolvent $(\mathcal{L} - \lambda)^{-1}$); then, for example, a nondegenerate eigenvalue λ_1 with eigenfunction Ψ_1 implies that $\exp\{\lambda_1 t\}\Psi_1$ is a solution to (1.1b). Whether such a solution grows or decays in time depends on the sign of $\mathbf{Re}\lambda_1$; this distinction makes it useful to partition the spectrum σ into three subsets $\sigma = \sigma_s \cup \sigma_c \cup \sigma_u$ where

$$\sigma_s = \{\lambda \in \sigma | \mathbf{Re}\lambda < 0\} \tag{1.3a}$$
$$\sigma_c = \{\lambda \in \sigma | \mathbf{Re}\lambda = 0\} \tag{1.3b}$$
$$\sigma_u = \{\lambda \in \sigma | \mathbf{Re}\lambda > 0\} \tag{1.3c}$$

denote the stable spectrum, center spectrum and unstable spectrum, respectively.

When there are eigenvalues $\lambda \in \sigma_u$, if an initial condition has components along the corresponding unstable modes then these modes will grow exponentially until the nonlinear terms $\mathcal{N}(X)$ are strong enough to arrest the growth and saturate the instability. One often finds that if the initial growth rate $\mathbf{Re}\lambda$ is weak then the nonlinear effects saturate the growth of the mode at a small amplitude. When the new nonlinear state involves small amplitude modes, it is natural to expect that nonlinear effects are weak. Under these circumstances, an effective theoretical description of the growth and saturation of the unstable modes may be obtained by treating the nonlinear effects perturbatively and solving (1.1a) using an expansion in powers of the unstable mode amplitudes. This approach has been successfully used to study a wide variety of systems where one finds weakly unstable modes.

These systems usually share another common feature in addition to weak growth rates and small mode amplitudes at saturation: there are no neutral modes (σ_c is empty). The importance of this latter feature is not always emphasized, but it plays a crucial role in the widespread success of amplitude equations in the analysis of weak instabilities. When the physical system has neutral modes, then one can find examples where nonlinear effects are *very strong* even when the unstable modes have arbitrarily small growth rates and saturate at arbitrarily small amplitudes.

The Vlasov equation for a collisionless plasma is a particularly interesting example of this circumstance but there are many other examples as well. In particular one can find very simple dynamical systems in finite dimensions where the dramatic effects of neutral modes can be analyzed in detail. Our interest in this issue originated in a study of weakly unstable modes in a Vlasov plasma,[1,2] but the subtleties of the nonlinear behavior can be best appreciated by first considering less complicated examples. In this discussion we concentrate primarily on a simple three-dimensional dynamical system where the consequences of a single neutral eigenvalue are similar in many respects to the consequences for the Vlasov equation of an infinite continuum of neutral modes.

In the Vlasov case we have focussed on the situation where σ_u contains only a simple conjugate pair of eigenvalues

$$\sigma_u = \{\gamma \pm i\omega\} \tag{1.4}$$

such that $\gamma > 0$, $\omega > 0$ and $0 < |\gamma/\omega| << 1$. One knows that there are small amplitude, fully nonlinear traveling waves (periodic orbits) in such a plasma[3] and numerical studies[4] show that these periodic orbits appear to describe the nonlinear saturation of the instability at least over moderately long time scales $0 < \omega t < 1200$. One question which has been of considerable theoretical and experimental interest is the dependence of the amplitude of the saturated wave on γ as γ is taken to zero.[4−7] In the absence of neutral modes, one commonly finds this dependence to be $\sqrt{\gamma}$. The presence of neutral modes can drastically reduce the size of the saturated wave leading to scalings of the form γ^p where $p = 1$ for our finite dimensional example and $p = 2$ for the Vlasov equation.

In the next section we analyze the equations describing one neutral mode and one unstable complex mode (1.4). Then in section III, we briefly formulate the corresponding one mode instability for the Vlasov equation and describe our results for the amplitude equation in that case.

II. A simple model: one neutral mode

Consider an n-dimensional flow (1.1a) so that $X \in \mathbf{R}^n (n \geq 3)$ and assume the linear matrix $\mathcal{L}$ has a pair of unstable eigenvalues (1.4). We further assume $\mathcal{L}$ has a simple real eigenvalue μ near zero and that all remaining eigenvalues belong to σ_s and are bounded away from the imaginary axis. For (μ, γ) near $(0,0)$, the time-asymptotic behavior of this flow near $X = 0$ is captured by a three-dimensional center manifold, and on this manifold the evolution equation reduces to a three-dimensional dynamical system which may be written in normal form[8] as

$$\begin{aligned}
\dot{\theta} &= \omega + \mathcal{O}(|r,z|^2) && (2.1a)\\
\dot{r} &= r[\gamma + a_1 z + a_2 r^2 + \mathcal{O}(z^2, |r,z|^3)] && (2.1b)\\
\dot{z} &= \mu z + b_1 r^2 + b_2 z^2 + \mathcal{O}(r^2 z, z^3, |r,z|^4)\ . && (2.1c)
\end{aligned}$$

In this notation, the complex amplitude of the unstable mode A has been expressed in polar variables $A = re^{i\theta}$, and z denotes the amplitude of the real mode. We assume $\mu \leq 0$ and $\gamma > 0$ so that there is only one unstable mode and the real mode z is either stable ($\mu < 0$) or neutral ($\mu = 0$). The coefficients a_1, a_2, b_1 and b_2 are assumed to satisfy

$$a_1 b_1 < 0 \qquad b_1 b_2 < 0 \tag{2.2}$$

but are otherwise arbitrary.

In these variables, the equations (2.1) are independent of θ and the properties of (2.1b,c) can be analyzed without considering the phase. If we ignore (2.1a) and neglect the indicated higher terms in (2.1b,c) we obtain the two-dimensional system

$$\begin{aligned}
\dot{r} &= r[\gamma + a_1 z + a_2 r^2] && (2.3a)\\
\dot{z} &= \mu z + b_1 r^2 + b_2 z^2\ . && (2.3b)
\end{aligned}$$

The nonlinear periodic orbit produced by the linear instability in (2.1b) is obtained by finding the equilibrium solution (r_o, z_o) to (2.3) with the property that $r_o \neq 0$ and $(r_o, z_o) \to (0,0)$ as $\gamma \to 0$. This solution corresponds to a periodic orbit

$$A_o(t) = r_o e^{i\theta_o(t)} \tag{2.4}$$

if the phase dynamics (2.1a) are reinstated. A simple calculation shows $z_o = -(\gamma + a_2 r_o^2)/a_1$ and

$$r_o^2 = \frac{\gamma(\mu a_1 - \gamma b_2)}{b_1 a_1^2}[1 + \mathcal{O}(\mu, \gamma)] \tag{2.5}$$

for (μ, γ) near $(0,0)$.

The effect of the neutral mode is clear. If $\mu < 0$ (and fixed) so that σ_c is empty then

$$r_o^2 = \left(\frac{\mu}{a_1 b_1}\right)\gamma \quad \text{as } \gamma \to 0 \tag{2.6a}$$

but when $\mu = 0$ and a neutral mode is present then

$$r_o^2 = \left(\frac{-b_2}{b_1 a_1^2}\right)\gamma^2 \quad \text{as } \gamma \to 0\ . \tag{2.6b}$$

The difference between these two scalings is dramatic; for small γ, the saturated mode in (2.6a) is much larger in amplitude than the saturated mode in (2.6b).

In more complicated problems, finding the exact solution for the periodic oscillation associated with such an instability will not be feasible because the number of neutral modes may be very large. This is the situation for the Vlasov equation. In the absence of exact results there have been several efforts to analyze the nonlinear saturation of such a Vlasov instability using amplitude expansions.[6] These efforts have not been particularly successful, and the origin of the difficulty can be understood by re-analyzing the present model (2.3) using such an amplitude expansion.

We base our derivation of the amplitude equation for the unstable mode on the two-dimensional unstable manifold associated with the two unstable eigenvalues.[8] As long as σ_u consists of a single complex conjugate pair, the unstable manifold will be two-dimensional even if there are many neutral modes. Hence this approach can also be readily applied to the Vlasov problem described in the next section.

The two-dimensional unstable manifold W^u in (2.1) appears as a one-dimensional unstable manifold in (2.3). As shown in Fig. 1, the unstable manifold is tangent to the (r, θ)-plane at $(r, z) = (0, 0)$, and near the origin we may describe the manifold as the graph of a function $h(r)$:

$$(r, z) \in W^u \quad \text{then} \quad (r, z) = (r, h(r)) \tag{2.7}$$

which satisfies

$$h(0) = h'(0) = 0 \ . \tag{2.8}$$

Given $h(r)$, the dynamics on the unstable manifold is obtained by replacing z with $h(r)$ in (2.3a):

$$\dot{r} = r[\gamma + a_1 h(r) + a_2 r^2] \ . \tag{2.9}$$

This is the amplitude equation for the unstable mode; it is expected to be valid for r sufficiently small since our representation of W^u in (2.7) will only hold in general near $(r, z) = (0, 0)$. To calculate the nonlinear oscillation, we seek $r_o > 0$ such that $\dot{r} = 0$ in (2.9)

$$\gamma + a_1 h(r_o) + a_2 r_o^2 = 0 \ . \tag{2.10}$$

This should determine $r_o(\gamma)$ from which the scaling behavior as $\gamma \to 0$ could be calculated.

Before (2.10) can be solved, we must find an expression for $h(r)$. The equation determining $h(r)$ follows from the fact that W^u is invariant under the dynamics (2.1). For a solution $(r(t), z(t)) \in W^u$ there are two ways to calculate $\dot{z}$: (i) from $z = h(r)$ and (2.9) we have

$$\dot{z} = \frac{dh}{dr} r[\gamma + a_1 h(r) + a_2 r^2] \tag{2.11a}$$

(ii) from $z = h(r)$ and (2.3b) we have

$$\dot{z} = \mu h(r) + b_1 r^2 + b_2 h(r)^2 \ . \tag{2.11b}$$

On W^u, these two calculations must agree; hence

$$\frac{dh}{dr} r[\gamma + a_1 h(r) + a_2 r^2] = \mu h(r) + b_1 r^2 + b_2 h(r)^2 \tag{2.12}$$

provides the desired equation for $h(r)$. We solve (2.12) for $h(r)$ using the amplitude expansion

$$h(r) = \sum_{n=1}^{\infty} \alpha_n r^{2n} \tag{2.13a}$$

and find

$$\alpha_1 = \frac{b_1}{(2\gamma - \mu)} \tag{2.13b}$$

with

$$\alpha_n = \frac{-2a_2(n-1)\alpha_{n-1} + \sum_{j=1}^{n-1}(b_2 - 2ja_1)\alpha_j \alpha_{n-j}}{2n\gamma - \mu} \tag{2.13c}$$

for $n \geq 2$. From this solution the amplitude equation (2.9) is given by the expansion

$$\dot{r} = r[\gamma + (a_1\alpha_1 + a_2)r^2 + \alpha_2 r^4 + \cdots] \ . \tag{2.14}$$

At leading order in r^2 this yields an approximate solution to (2.10)

$$r_o^2 = \frac{-\gamma}{a_1\alpha_1 + a_2} = \frac{-\gamma(2\gamma - \mu)}{a_1 b_1 + a_2(2\gamma - \mu)} \ . \tag{2.15}$$

For $\mu < 0$, (2.15) predicts

$$r_o^2 \sim \left(\frac{\mu}{a_1 b_1}\right)\gamma \quad \text{as } \gamma \to 0 \tag{2.16a}$$

in agreement with (2.6a), but for $\mu = 0$ (2.15) predicts

$$r_o^2 \sim \left(\frac{-2}{a_1 b_1}\right)\gamma^2 \tag{2.16b}$$

which is *not* correct although the dependence on γ agrees with the exact result (2.6b).

It is easy to see why (2.15) is wrong when $\mu = 0$. From (2.13b,c), the coefficents in our perturbation theory are singular when $\mu = 0$:

$$|\alpha_n| \sim \frac{1}{\gamma^{2n-1}} \quad \text{as } \gamma \to 0, \tag{2.17}$$

and this means that higher order terms in (2.14) are not negligible.

We can construct a perturbation theory free of this singular behavior by rescaling our amplitude variable appropriately. If we define (x, ζ) by

$$r = \gamma x \qquad z = \gamma\zeta \tag{2.18}$$

then (2.13a) becomes $h(r) = \gamma \sum_n (\alpha_n \gamma^{2n-1}) x^{2n}$ and the coefficients $\alpha_n \gamma^{2n-1}$ are now well-behaved as $\gamma \to 0$. This motivates the additional definition $h(r) \equiv \gamma H(x)$ in terms of which (2.9) and (2.12) become

$$\dot{x} = \gamma x[1 + a_1 H(x) + a_2 \gamma x^2] \tag{2.19}$$

and

$$\frac{dH}{dx} = \frac{b_1 x^2 + b_2 H^2}{x[1 + a_1 H + a_2 \gamma x^2]}, \quad H(0) = 0 \tag{2.20}$$

respectively. The original system (2.3) is rescaled to

$$\dot{x} = \gamma x[1 + a_1 \zeta + a_2 \gamma x^2] \tag{2.21a}$$

$$\dot{\zeta} = \gamma[b_1 x^2 + b_2 \zeta^2] \ . \tag{2.21b}$$

One could now solve (2.20) perturbatively $H(x) = \tilde{\alpha}_1 x^2 + \tilde{\alpha}_2 x^4 + \cdots$, and then attempt to find x_o from (2.19) by solving $1 + a_1 H(x_o) + a_2 \gamma x_o^2 = 0$.

The limitations of this rescaled perturbation theory are most readily appreciated by first considering the exact system (2.21). The exact solution for the periodic orbit (2.4) is now given by

$$(x_o, \zeta_o) = \left(\sqrt{\frac{-b_2}{b_1 a_1^2}}, \frac{-1}{a_1} \right) \quad \text{as } \gamma \to 0 \ .$$

The stability of this solution is found by linearizing (2.21) about (x_o, ζ_o) and finding the eigenvalues $\Lambda_\pm$,

$$\Lambda_\pm = \frac{-b_2}{a_1} \pm \sqrt{\frac{b_2}{a_1}\left(\frac{b_2}{a_1} - 2\right)} + \mathcal{O}(\gamma) \ .$$

Since (2.2) implies $b_2/a_1 > 0$ there are essentially two possibilities:

(*i*) $2 < \frac{b_2}{a_1} < \infty$. For these parameter values, $\Lambda_\pm$ form a complex conjugate pair and $\mathbf{Re}\Lambda_\pm < 0$ so the solution (x_o, ζ_o) is stable.

(*ii*) $0 < \frac{b_2}{a_1} < 2$. For these parameter values, $\Lambda_\pm$ are real and negative. The periodic orbit (x_o, ζ_o) is again stable.

The global behavior of the unstable manifold of the origin is quite different for these two cases. The phase portraits are shown in Fig. 2 with the corresponding evolution of the mode amplitude for the flow on W^u. Note that since x_o is not small, a low order approximation to W^u, *i.e.* $H(x) = \tilde{\alpha}_1 x^2 + \mathcal{O}(x^4)$, will not in general lead to accurate results. When the eigenvalues $\Lambda_\pm$ are complex, there is an additional difficulty. In this case, the desired solution (x_o, ζ_o) is not located on the segment of W^u described by solving (2.20). Thus even if the perturbation series for $H(x)$ could be summed the resulting calculation of x_o would be wrong.

III. Collisionless one mode beam-plasma instability

The problem of a weakly unstable wave in a collisionless plasma provides a much less trivial example of singular behavior in an amplitude expansion.[6] Consider a one-dimensional plasma with mobile electrons, a neutralizing fixed background of positive charge density en_o

and periodic boundary conditions in $x \in [0, L)$. The electron distribution function $F(x, v, t)$ satisfies the Vlasov equation

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial F}{\partial v} = 0 \tag{3.1a}$$

$$-\frac{\partial^2 \phi}{\partial x^2} = 4\pi e n_o[1 - \int_{-\infty}^{\infty} F\, dv] \tag{3.1b}$$

where the electrostatic potential $\phi(x, t)$ is determined by Poisson's equation (3.1b). Let $F_o(v, u)$ denote a spatially uniform equilibrium ($\phi_o \equiv 0$) depending on a parameter u and define f by $f(x, v, t) \equiv F(x, v, t) - F_o(v, u)$. Then (3.1) may be rewritten as an evolution equation for $f(x, v, t)$

$$\frac{\partial f}{\partial t} = \mathcal{L}f + \mathcal{N}(f) \tag{3.2a}$$

where

$$\mathcal{L}f = -v\,\partial_x f - \frac{e}{m}\partial_x\phi\,\partial_v F_o \tag{3.2b}$$

$$\mathcal{N}(f) = -\frac{e}{m}\partial_x\phi\,\partial_v f \tag{3.2c}$$

and

$$\partial_x^2\phi = 4\pi e n_o \int_{-\infty}^{\infty} f\, dv \ . \tag{3.2d}$$

The spectrum of $\mathcal{L}$ depends on $F_o(v, u)$.[1,9] For a beam-plasma equilibrium ($n_o = n_p + n_b$)

$$n_o F_o(v, u) = \frac{n_p}{\sqrt{\pi\sigma_p^2}} e^{\frac{-v^2}{\sigma_p^2}} + \frac{n_b}{\sqrt{\pi\sigma_b^2}} e^{\frac{-(v-u)^2}{\sigma_b^2}} \tag{3.3}$$

we may regard the beam velocity u as a bifurcation parameter and fix the other parameters, see Fig. 3a. Then if the plasma length L is chosen appropriately only the electrostatic wave with maximum wavelength $k = 2\pi/L$ will become unstable as u is increased above the critical velocity u_c. A detailed description of how the spectrum of $\mathcal{L}$ varies near the onset of this instability has been given elsewhere.[1] In Fig. 3b we show schematically how the spectrum appears for $u < u_c$, $u = u_c$ and $u > u_c$. In the weakly unstable regime ($u > u_c$), there is a single conjugate pair of eigenvalues in the unstable spectrum ($\lambda = \gamma + i\omega$):

$$\mathcal{L}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} \lambda & \psi \\ \bar{\lambda} & \bar{\psi} \end{pmatrix}$$

where

$$\psi(x, v) = e^{ikx}\left(-\frac{\eta(k, v)}{v - z}\right)$$

and $z = i\lambda/k$ satisfies

$$\Lambda(k,z) \equiv 1 + \int_{-\infty}^{\infty} \frac{dv\,\eta(k,v)}{v-z} = 0$$

with

$$\eta(k,v) = -\left(\frac{4\pi e^2 n_o}{mk^2}\right) \partial_v F_o(v) \ .$$

To analyze the amplitude equation on the unstable manifold, we introduce the complex mode amplitude $A = re^{i\theta}$ as before:

$$f(x,v,t) = A\psi(x,v) + \overline{A}\,\overline{\psi}(x,v) + S(x,v,t) \tag{3.4}$$

where S represents the components in the eigenfunction expansion for $\mathcal{L}$ orthogonal to ψ and $\overline{\psi}$. The evolution equation (3.2a) determines the equations for $\dot{A}$ and $\partial_t S$:

$$\begin{aligned} \dot{A} &= \lambda A + < \tilde{\psi}, \mathcal{N}(f) > & (3.5a) \\ \partial_t S &= \mathcal{L}S + \mathcal{N}(f) - [< \tilde{\psi}, \mathcal{N}(f) > \psi + \ c.c.] & (3.5b) \end{aligned}$$

where $< \tilde{\psi}, \cdot >$ denotes the projection with the appropriate adjoint eigenfunction $\tilde{\psi}$. The amplitude equation for $r = |A|$ follows by restricting (3.5a) to the two-dimensional unstable manifold. This calculation will be discussed elsewhere[10]; the results can be easily summarized. From translation invariance it follows that the amplitude equation will take the form

$$\dot{r} = r[\gamma + g(r^2)] \tag{3.6}$$

where $g(x)$ is a function satisfying $g(0) = 0$. The amplitude expansion will, in principle, give the result

$$g(r^2) = \sum_{n=1}^{\infty} \alpha_n r^{2n} \ . \tag{3.7}$$

In practice the calculation of the coefficients is laborious and at present only the leading term $\alpha_1 r^2$ has been analyzed. In the limit of weak instability we find[10]

$$\alpha_1 = \frac{1}{\gamma^3}[b_o + \mathcal{O}(\gamma)] \quad \text{as } \gamma \to 0^+ \tag{3.8a}$$

with

$$b_o = -\frac{1}{12}\left(\frac{k^3}{4m|\Lambda'(k,r)|^2}\right)^2 \left[(\pi\eta'(k,r))^2 + 55\left(P\int_{-\infty}^{\infty} \frac{\eta'(k,v)\,dv}{v-r}\right)^2\right] \tag{3.8b}$$

where

$$\begin{aligned} \eta'(k,v) &\equiv \frac{\partial}{\partial v}\eta(k,v) \\ \Lambda'(k,r) &\equiv P\int_{-\infty}^{\infty} \frac{\eta'(k,v)\,dv}{v-r} + i\pi\eta'(k,r) \ . \end{aligned}$$

Note that b_o is less than zero independent of the detailed shape of $F_o(v,u)$.

The singularity in (3.8a) at $\gamma = 0$ indicates that the appropriate rescaling of the amplitude[11] is

$$r = \gamma^2 x \tag{3.9}$$

so that (3.6) becomes

$$\dot{x} = \gamma x[1 + b_o x^2 + \sum_{n=2}^{\infty}(\gamma^{4n-1}\alpha_n)x^{2n}] \ . \tag{3.10}$$

The strength of the singularities in α_n $(n \geq 2)$ as $\gamma \to 0$ remains to be studied, but one can speculate on the behavior of the higher order terms in (3.10). There are at least three possibilities:

(a) $\gamma^{4n-1}\alpha_n \to 0$ as $\gamma \to 0$ for $n \geq 2$; then (3.10) yields

$$\dot{x} = \gamma x[1 + b_o x^2] \quad \text{as } \gamma \to 0$$

and $x_o^2 = -1/b_o$ determines the saturation amplitude. Also the saturated state x_o is approached in a monotonic fashion as in case (*ii*) of the simple model.

(b) $\gamma^{4n-1}\alpha_n \to \infty$ as $\gamma \to 0$ for some $n \geq 2$; then the rescaling (3.9) is too weak.

(c) $|\alpha_n| \sim 1/\gamma^{4n-1}$ as $\gamma \to 0$ for all $n \geq 2$; this would be similar to the simple model. In this case the rescaling (3.9) is correct, but to determine the saturated state x_o would in general not be practical using the perturbative amplitude equation (3.10).

Numerical results by J. Denavit show the saturation of this instability with a scaling given by (3.9), and the approach to saturation is a decaying oscillation (trapping oscillations) similar to case (*i*) of the simple model.[4] These results strongly suggest that it is (c) rather than (a) or (b) that applies to the higher order terms in (3.10). The trapping oscillations further indicate that the unstable manifold probably approaches the periodic orbit (BGK mode) in a spiral fashion.

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[11]At this conference, I learned of unpublished work by E. Larson which also predicts this scaling. Larson's theory uses asymptotic techniques to incorporate a boundary layer in velocity space at the linear phase velocity. He obtains an amplitude equation of the same form as (3.10).

Figure Captions

1. Phase portrait for system (2.3) showing the periodic orbit (r_o, z_o), the unstable manifold W^u of the origin, and local description of W^u by the graph $z = h(r)$.

2. Dynamics for the rescaled system (2.21) showing the two possible global behaviors of the unstable manifold and the corresponding behavior of the unstable mode as described by the dynamics on the unstable manifold.

3. (a) Form of a beam-plasma distribution; (b) Linear stability of the distribution $F_o(v, u)$ in the beam velocity (u) vs. wavenumber (k) plane. The spectrum of the linear operator $\mathcal{L}$ in (3.2b) is sketched at the points A, B and C. At criticality (B) there is a complex conjugate pair of imaginary eigenvalues embedded in the continuum of neutral modes. In the weakly unstable regime (C) we find a quadruplet of eigenvalues: a stable conjugate pair and an unstable conjugate pair.

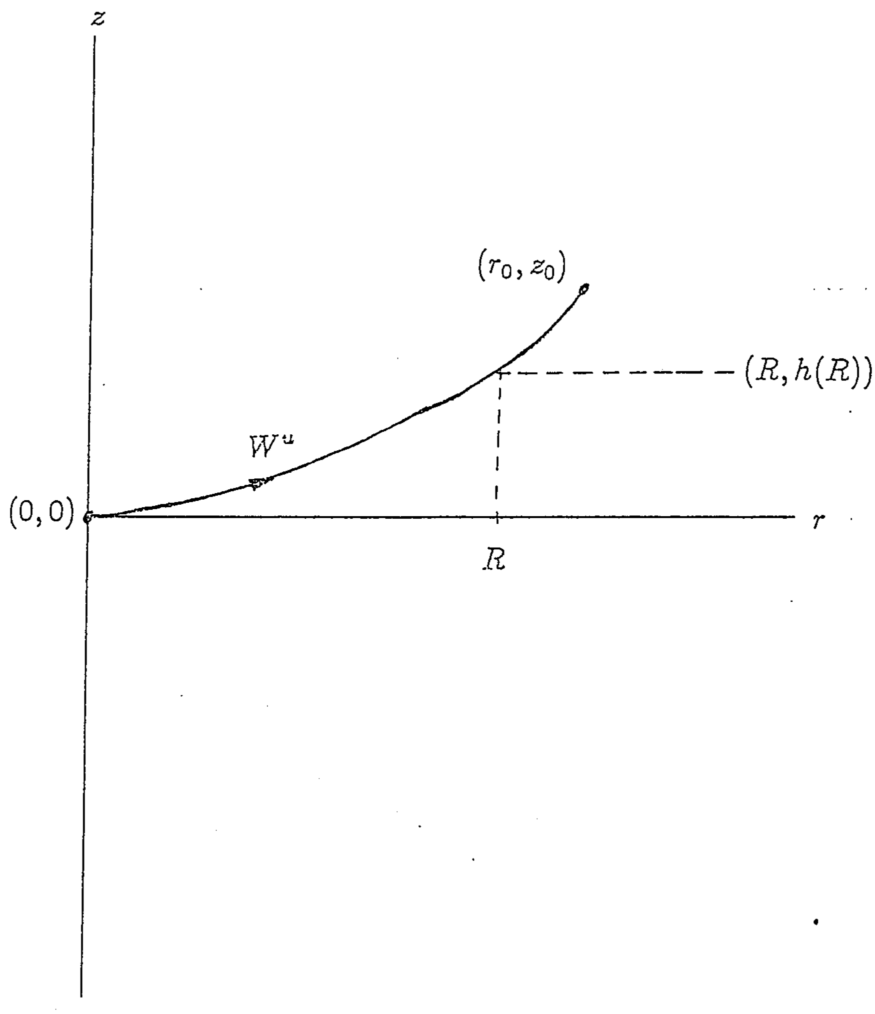


Figure 1

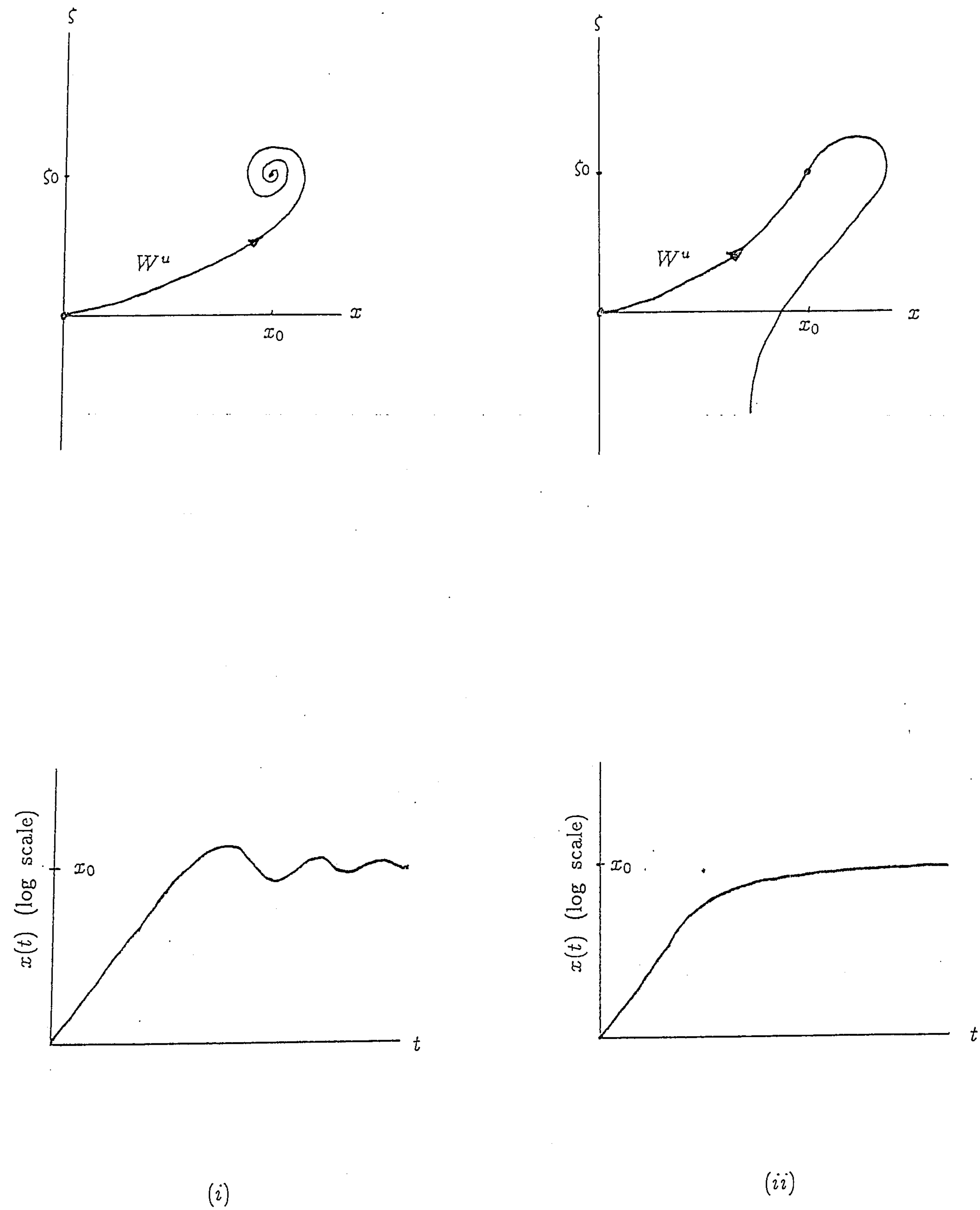
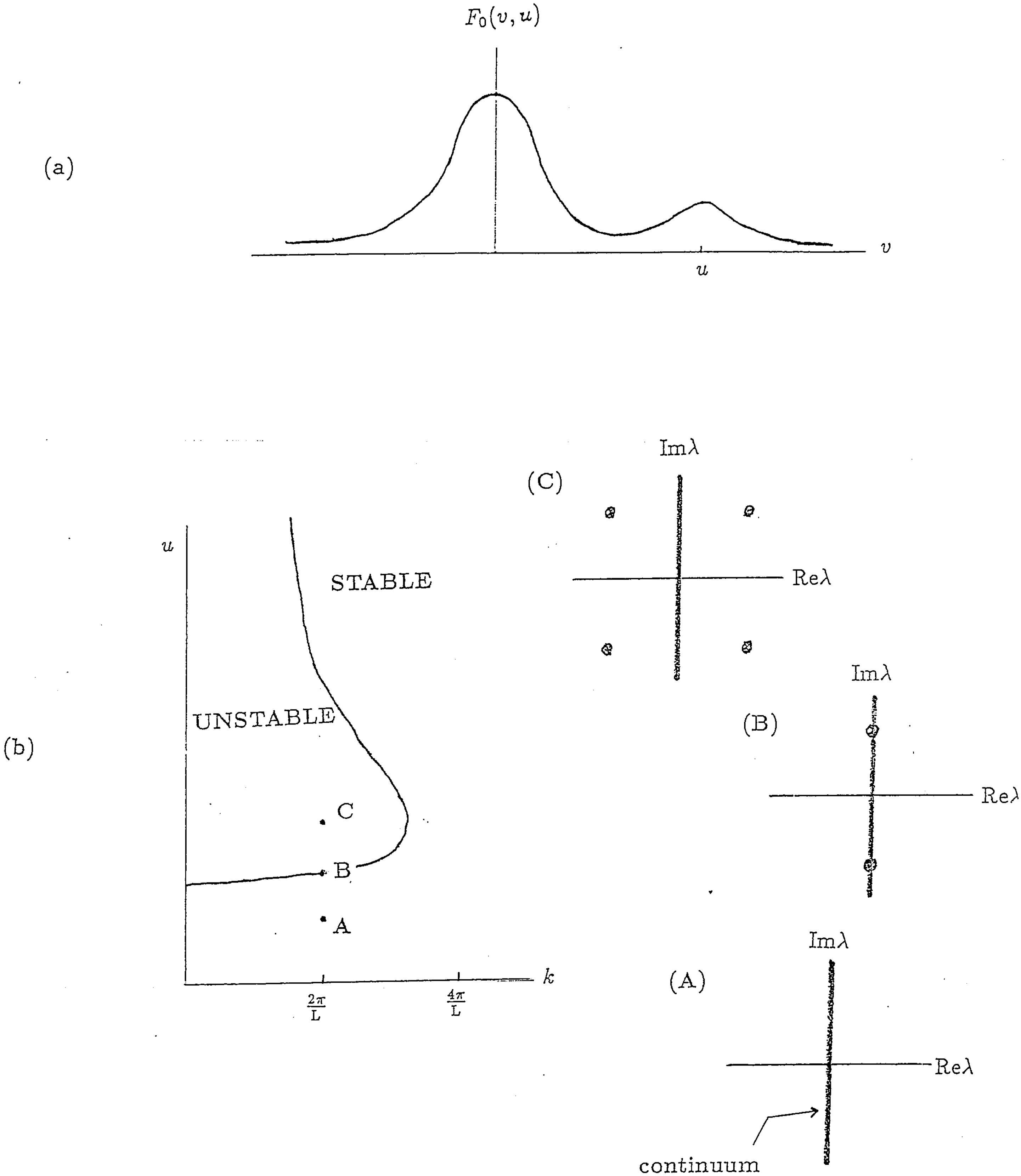


Figure 2



(a)
$F_0(v,u)$
v
u
(b)
u
STABLE
UNSTABLE
C
B
A
k
$\frac{2\pi}{L}$
$\frac{4\pi}{L}$
(C)
Imλ
Reλ
(B)
Imλ
Reλ
(A)
Imλ
Reλ
continuum

Figure 3