DOE/ET-53088-384 IFSR #384

Nonlinear Behavior of Magnetohydrodynamic Modes Near Marginally Stable States II: Application to the Resistive Fast Interchange Mode

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July 1989

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Abstract

With the use of the general formulation developed in an earlier paper [N. Nakajima, IFS Report No. 373], the nonlinear evolution of the resistive fast interchange mode near the marginally stable state is obtained analytically. The nonlinear amplitude equation of the mode is shown to be of the Landau type. It is also shown that there is a stable equilibrium bifurcating from the initial equilibrium. Comparing this analytical result to numerical simulations, we confirm that the saturation level and the saturation time are well estimated by this Landau type of nonlinear amplitude equation.

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1. Introduction

In an earlier paper,[1] a general formulation was presented to obtain the nonlinear equation describing the time development of the mode near the marginally stable state. It was shown there that two different types of nonlinearity exist, depending upon the properties of the linear operator of the mode under consideration—one leads to a Hamiltonian-type equation for the amplitude, and the other leads to a Landau-type equation for the amplitude.[2] The former type of equation (which we will call the first type) may be obtained in the case where the linear operator is degenerate at the marginally stable state, i.e., when the linear dispersion relation has a double root for the frequency at the marginally stable state. The latter type of equation (or the second type) may be obtained in the case where the linear operator is nondegenerate, i.e., when the linear dispersion relation has a single root. In magnetohydrodynamics (MHD), the first type corresponds to nonresonant ideal modes, and the second type corresponds to resistive modes. In Ref. 1, the nonresonant kink mode in reversed field pinches (RFPs) and the quasi-interchange mode in tokamaks were examined by means of this general formulation.[1] These modes are nonresonant ideal MHD modes and the corresponding nonlinear amplitude equations are shown to be of the Hamiltonian type. As a result, we find that both the nonresonant kink mode and the quasi-interchange mode are nonlinearly stabilized, and new stable equilibria bifurcate from the initial equilibrium. This is a manifestation of nonlinear saturation of those modes.

In this paper, we examine the nonlinear behavior of the resistive fast interchange mode,[3] which will be shown to lead to the second type, or the Landau type, of amplitude equation, a different type from the one obtained for the nonresonant kink mode and the quasi-interchange mode in Ref. 1. Although our main goal in this paper is to present an example of the second type of amplitude equation, it should be noted that the nonlinear evolution of the resistive interchange mode plays an important role in terms of energy confinement in current fusion experiments.[4,5,6,7]

The rest of the paper is organized as follows. In Sec. 2 we derive the nonlinear amplitude equation for the resistive fast interchange mode, based upon the reduced nonlinear equations used in Ref. 4. Comparison between the analytical results and the numerical simulation results is made in Sec. 3, in order to show that the saturation level and the saturation time of this mode are well estimated by the analytical theory. Section 4 contains conclusions and discussion.

2. The Derivation of the Nonlinear Amplitude Equation

The general formulation in Ref. 1 is here applied to the resistive fast interchange mode in order to obtain the nonlinear amplitude equation near the marginally stable state. The reduced equations for the resistive fast interchange mode are as follows[4]:

$$\frac{\partial \psi}{\partial t} - x\frac{\partial \phi}{\partial y} - \frac{1}{S}\Delta_\perp \psi = -\{\phi, \psi\}, \tag{2.1}$$

$$\frac{\partial}{\partial t}\Delta_\perp \phi - x\frac{\partial}{\partial y}\Delta_\perp \psi + \frac{\partial p}{\partial y} - M\Delta_\perp^2 \phi = -\{\phi, \Delta_\perp \phi\} + \{\psi, \Delta_\perp \psi\}, \tag{2.2}$$

$$\frac{\partial p}{\partial t} + D\frac{\partial \phi}{\partial y} - \chi \Delta_\perp p = -\{\phi, p\}. \tag{2.3}$$

With equilibrium quantities denoted by the subscript 0 and perturbed quantities by the subscript 1, the dependent variables in Eqs. (2.1)–(2.3) are defined by

$$\psi = \frac{1}{r_s B_{\theta 0} B_0}\psi_1, \qquad \phi = \frac{\sqrt{p_0}}{r_s B_{\theta 0} B_0}\phi_1,$$

and

$$p = \frac{2}{B_0^2 \left|\sigma_0\right|^{3/2}} p_1, \tag{2.4}$$

and the independent variables are defined by

$$x = |\sigma_0|^{1/2} \frac{r - r_s}{r_s}, \qquad y = \frac{B_{\theta_0} |\sigma_0|^{1/2}}{r_s B_0} (z - \mu_0 \theta),$$

and

$$t = \frac{B_{\theta_0} |\sigma_0|}{r_s \sqrt{p_0}} \tau. \tag{2.5}$$

Here, (r, θ, z) denote the polar coordinates of the cylinder in which the plasma is contained; τ denotes the time; $r = r_s$ is the radius of the resonant surface; $B_{\theta 0}$, B_{z0}, and B_0 indicate the azimuthal and longitudinal components and the absolute value of the equilibrium magnetic field $\mathbf{B}_0$, respectively; and p_i denotes the perturbed pressure. The perturbed magnetic field $\mathbf{B}_1$ and the perturbed velocity field $\mathbf{v}_1$ are expressed by $\mathbf{B}_1 = \nabla_\perp \psi_1 \times \mathbf{b}$ and $\mathbf{v}_1 = \nabla_\perp \phi_1 \times \mathbf{b}$, where $\mathbf{b} = \mathbf{B}_0 / B_0^2$. The parameters used here are defined by

$$\begin{aligned}
D &\equiv -2 r_s p_0' / B_0^2 \sigma_0^2, \qquad S \equiv r_s B_{\theta 0} / \eta \sqrt{p_0}, \\
M &\equiv \mu_\perp / r_s B_{\theta 0} \sqrt{p_0}, \qquad \chi \equiv (\Gamma - 1) \kappa_\perp / r_s B_{\theta 0} \sqrt{p_0}, \\
\mu_0 &\equiv r_s B_{Z0} / B_{\theta 0}, \qquad \sigma_0 \equiv B_{\theta 0} \mu_0' / B_0.
\end{aligned} \tag{2.6}$$

Here, a prime denotes d/dr; ρ_0 and p_0 are the equilibrium density and the equilibrium pressure, respectively; Γ is the ratio of the specific heats; and η, $\mu_\perp$, and $\kappa_\perp$ indicate the resistivity, the perpendicular viscosity, and the perpendicular heat conductivity, respectively. Note that all the equilibrium quantities are evaluated at the mode rational surface $r = r_s$. The parallel diffusion coefficients are ignored for simplicity. In writing Eqs. (2.1)–(2.3), we only take into account nonlinear interaction of the single helicity modes, the helicity of which is given by μ_0. Therefore, the parallel derivatives are expressed as $x \frac{\partial}{\partial y}$ in these equations. The Poisson bracket $\{\ ,\ \}$ and the perpendicular Laplacian are defined by

$$\begin{aligned}
\{u, v\} &\equiv \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}, \\
\Delta_\perp u &\equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.
\end{aligned} \tag{2.7}$$

The domain of the mode is defined as $|x| \le \delta_x$ and $|y| \le \delta_y$ with δ_x and δ_y being positive constants of order $S^{-1/2}$, which represents the narrow boundary layer within which the mode is considered to be localized. As in the case of the linear resistive fast interchange mode,[3] we assume that the mode decays rapidly away from its mode rational surface $x = 0$ and ψ, ϕ, and p are periodic in y with period $2\delta_y$.

We now consider the situation where the equilibrium pressure gradient, indicated by the parameter D, is slightly larger than its critical value D_c (which will be determined later as an eigenvalue of the linearized system) so that the plasma is linearly unstable. It is known that this unstable mode has no real frequency and that the functions ψ, ϕ, and p have odd, even, and even symmetry with respect to x, respectively.

We apply the general formulation in Ref. 1 to Eqs. (2.1)–(2.3) as follows. The expanded forms of ψ, ϕ, and p are given by

$$\begin{pmatrix} \psi \\ \phi \\ p \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \phi_1 \\ p_1 \end{pmatrix} + \lambda^2 \begin{pmatrix} \psi_2 \\ \phi_2 \\ p_2 \end{pmatrix} + \lambda^3 \begin{pmatrix} \psi_3 \\ \phi_3 \\ p_3 \end{pmatrix} + \cdots, \tag{2.8}$$

where λ is an ordering parameter. We introduce the following form for ψ_1 and also the multiple-time-scale method:

$$\psi_1 = A\psi_1(x) \cos ky, \tag{2.9}$$

$$\tau_1 = \lambda t, \qquad \tau_2 = \lambda^2 t, \ldots, \tag{2.10}$$

$$\frac{\partial}{\partial t} = \lambda \frac{\partial}{\partial \tau_1} + \lambda^2 \frac{\partial}{\partial \tau_2} + \cdots, \tag{2.11}$$

$$A = A(\tau_1, \tau_2, \ldots), \tag{2.12}$$

where A and $\psi_1(x)$ are real. In Eq. (2.11), we made use of the fact that the real frequency vanishes. Finally, we choose the mean pressure gradient D to be the parameter p in the general formulation, so that

$$D = D_c \pm \lambda^2, \tag{2.13}$$

where D_c denotes the critical value (or the linear stability limit) of D and the plus sign corresponds to the linearly unstable situation. Substituting Eqs. (2.8)–(2.13) into Eqs. (2.1)–(2.3) yields simultaneous equations for each order of λ, which we solve beginning with the lowest order.

Order λ:

This order corresponds to the marginally stable state, and we have the following linearized equations:

$$L\begin{pmatrix}\phi_1\\ p_1\end{pmatrix} = \begin{pmatrix}-Sx^2\frac{\partial^2\phi_1}{\partial y^2}+M\Delta_\perp^2\phi_1 & -\partial p_1/\partial y\\ \partial\phi_1/\partial y & -(\chi/D_c)\,\Delta_\perp p_1\end{pmatrix}=0, \tag{2.14}$$

and

$$\Delta_\perp\psi_1 = -Sx\frac{\partial\phi_1}{\partial y}. \tag{2.15}$$

Assuming that ψ_1 is given by Eq. (2.9), we may have the following type of linear solutions:

$$\begin{pmatrix}\psi_1\\ \phi_1\\ p_1\end{pmatrix} = A\begin{pmatrix}\psi_1(x)\cos ky\\ \phi_1(x)\sin ky\\ p_1(x)\cos ky\end{pmatrix}. \tag{2.16}$$

The real functions $\psi_1(x)$, $\phi_1(x)$, $p_1(x)$, and D_c are, respectively, the eigenfunctions and eigenvalue of the following eigenvalue problem:

$$L_1\begin{pmatrix}\phi_1(x)\\ p_1(x)\end{pmatrix}=0, \tag{2.17}$$

where $\phi_1(x)=p_1(x)=\phi_1'(x)=0$ at $|x|=\delta_x$, and

$$\left(\partial_x^2-k^2\right)\psi_1(x) = -Skx\phi_1(x), \tag{2.18}$$

with $\psi_1(x)=0$ at $|x|=\delta_x$. Here the linear operator L_1 is given by the $l=1$ case of the following operator:

$$L_l \equiv \begin{pmatrix} S(lkx)^2+M\left(\partial_x^2-(lk)^2\right)^2 & lk\\ lk & -(\chi/D_c)\left(\partial_x^2-(lk)^2\right)\end{pmatrix}. \tag{2.19}$$

It is seen that L_l is a Hermitian operator with respect to the following inner product:

$$(u_1, u_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \langle u_1 v_1 + u_2 v_2 \rangle, \tag{2.20}$$

where

$$\langle w \rangle \equiv \frac{1}{2\delta_x}\int_{-\delta_x}^{\delta_x} dx' w(t, x'). \tag{2.21}$$

We assume in what follows that the solutions of Eqs. (2.17) and (2.18) are unique.

Order λ^2:

The equations in this order are given by

$$L\begin{pmatrix} \phi_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} \{\phi_1, \Delta_\perp \phi_1\} - Sx\frac{\partial}{\partial y}\{\phi_1, \psi_1\} - \{\psi_1, \Delta_\perp \psi_1\} + \frac{\partial}{\partial \tau_1}\Delta_\perp \phi_1 - Sx\frac{\partial^2}{\partial \tau_1 \partial y}\psi_1 \\ -\frac{1}{D_c}\left[\{\phi_1, p_1\} + \frac{\partial p_1}{\partial \tau_1}\right] \end{pmatrix}, \tag{2.22}$$

and

$$\Delta_\perp \psi_2 = -Sx\frac{\partial \phi_2}{\partial y} + S\{\phi_1, \psi_1\} + S\frac{\partial \psi_1}{\partial \tau_1}. \tag{2.23}$$

From the inhomogeneous terms of Eqs. (2.22)–(2.23), we see that ψ_2, ϕ_2, and p_2 have components with the same phase in y as the order-λ solutions: $\psi_{21}(x)\cos ky$, $\phi_{21}(x)\sin ky$, and $p_{21}(x)\cos ky$. Here the functions $\psi_{21}(x)$, $\phi_{21}(x)$, and $p_{21}(x)$ are given by

$$L_1\begin{pmatrix} \phi_{21}(x) \\ p_{21}(x) \end{pmatrix} = \frac{\partial A}{\partial \tau_1}\begin{pmatrix} (\partial_x^2 - k^2)\phi_1(x) + Skx\psi_1(x) \\ -\frac{1}{D_c}p_1(x) \end{pmatrix}, \tag{2.24}$$

with $\phi_{21}(x) = p_{21}(x) = \phi_{21}'(x) = 0$ at $|x| = \delta_x$, and

$$\left(\partial_x^2 - k^2\right)\psi_{21}(x) = -Skx\phi_{21}(x) + \frac{\partial A}{\partial \tau_1}S\psi_1(x), \tag{2.25}$$

with $\psi_{21}(x) = 0$ at $|x| = \delta_x$. From the solvability condition of Eq. (2.24), we obtain

$$\frac{\partial A}{\partial \tau_1}\left\langle -\left[(\partial_x \phi_1(x))^2 + (k\phi_1(x))^2\right] + \left[(\partial_x \psi_1(x))^2 + (k\psi_1(x))^2\right] - \frac{1}{D_c}p_1^2(x)\right\rangle = 0. \tag{2.26}$$

The averaged quantity $\langle \cdots \rangle$ in Eq. (2.26) does not vanish generally, and hence we have

$$\frac{\partial A}{\partial \tau_1} = 0. \tag{2.27}$$

Then, the inhomogeneous terms of Eqs. (2.24)–(2.25) vanish, and so we have the solutions proportional to the order-λ solutions. According to the general formulation, however, these solutions could be transferred into the order-λ ones by redefinition of the coefficient A, and therefore we put $\psi_{21}(x) = \phi_{21}(x) = p_{21}(x) = 0$. Then, from Eqs. (2.22)–(2.23) we have

$$\begin{pmatrix} \psi_2 \\ \phi_2 \\ p_2 \end{pmatrix} = A^2 \begin{pmatrix} \psi_{20}(x) \\ \phi_{20}(x) \\ p_{20}(x) \end{pmatrix} + A^2 \begin{pmatrix} \psi_{22}(x)\cos 2ky \\ \phi_{22}(x)\sin 2ky \\ p_{22}(x)\cos 2ky \end{pmatrix}, \tag{2.28}$$

where $\psi_{2j}(x)$, $\phi_{2j}(x)$, and $p_{2j}(x)$ $(j = 0, 2)$ may be obtained from the following equations:

$$L_0 \begin{pmatrix} \phi_{20}(x) \\ p_{20}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{k}{2D_c}\partial_x\left(\phi_1(x)p_1(x)\right) \end{pmatrix}, \tag{2.29}$$

with $\phi_{20}(x) = p_{20}(x) = \phi'_{20}(x) = 0$ at $|x| = \delta_x$,

$$\partial_x^2\psi_{20}(x) = -\frac{kS}{2}\partial_x\left(\phi_1(x)\psi_1(x)\right), \tag{2.30}$$

with $\psi_{20}(x) = 0$ at $|x| = \delta_x$, and

$$L_2 = \begin{pmatrix} \phi_{22}(x) \\ p_{22}(x) \end{pmatrix} = \begin{pmatrix} \frac{k}{2}f \\ -\frac{k}{2D_c}p_1^2(x)\partial_x\left(\frac{\phi_1(x)}{p_1(x)}\right) \end{pmatrix}, \tag{2.31}$$

with $\phi_{22}(x) = p_{22}(x) = \phi'_{22}(x) = 0$ at $|x| = \delta_x$,

$$\left(\partial_x^2 - (2k)^2\right)\psi_{22}(x) = -2Skx\phi_{22}(x) + \frac{kS}{2}\psi_1^2(x)\partial_x\left(\frac{\phi_1(x)}{\psi_1(x)}\right), \tag{2.32}$$

with $\psi_{22}(x) = 0$ at $|x| = \delta_x$, where

$$\begin{aligned} f \;\equiv\; & -\phi_1^2(x)\partial_x\left(\frac{\left(\partial_x^2 - k^2\right)\phi_1(x)}{\phi_1(x)}\right) + 2Sx\psi_1^2(x)\partial_x\left(\frac{\phi_1(x)}{\psi_1(x)}\right) \\ & -\psi_1^2(x)\partial_x\left(\frac{\left(\partial_x^2 - k^2\right)\psi_1(x)}{\psi_1(x)}\right). \end{aligned} \tag{2.33}$$

Here we note that $\phi_{20}(x) \equiv 0$.

Order λ^3:

In order to obtain the nonlinear equation for A, we consider the solutions ψ_{31}, ϕ_{31}, and p_{31} which have the same phase in y as that of the order-λ solutions, i.e., $\psi_{31}(x)\cos ky$, $\phi_{31}(x)\sin ky$, and $p_{31}(x)\cos ky$. The functions $\psi_{31}(x)$, $\phi_{31}(x)$, and $p_{31}(x)$ are the solutions of the following equations:

$$L_1\begin{pmatrix}\phi_{31}(x)\\ p_{31}(x)\end{pmatrix}=\begin{pmatrix}A^3 f_1+\frac{\partial A}{\partial\tau_2}\left[\left(\partial_x^2-k^2\right)\phi_1(x)+Skx\psi_1(x)\right]\\ -A^3\frac{g_1}{D_c}-A\frac{(\pm 1)k}{D_c}\phi_1(x)-\frac{\partial A}{\partial\tau_2}\frac{1}{D_c}p_1(x)\end{pmatrix}, \tag{2.34}$$

with $\phi_{31}(x)=p_{31}(x)=\phi_{31}'(x)=0$ at $|x|=\delta_x$, and

$$\left(\partial_x^2-k^2\right)\psi_{31}(x)=-Skx\phi_{31}(x)+A^3Sh_1+\frac{\partial A}{\partial\tau_2}S\psi_1(x), \tag{2.35}$$

with $\psi_{31}(x)=0$ at $|x|=\delta_x$, where

$$\begin{aligned}
f_1 \;\equiv\;& k\Big\{-\partial_x\phi_1(x)\cdot\left(\partial_x^2-(2k)^2\right)\phi_{22}(x)+\phi_{22}(x)\cdot\left(\partial_x^2-k^2\right)\phi_1(x)\\
&-\frac{1}{2}\phi_1(x)\cdot\partial x\left(\partial_x^2-(2k)^2\right)\phi_{22}(x)+\frac{1}{2}\partial_x\phi_{22}(x)\cdot\left(\partial_x^2-k^2\right)\phi_1(x)\\
&+\partial_x\psi_1(x)\cdot\left(\partial_x^2-(2k)^2\right)\psi_{22}(x)-\psi_{22}(x)\cdot\left(\partial_x^2-k^2\right)\psi_1(x)\\
&+\frac{1}{2}\psi_1(x)\cdot\partial x\left(\partial_x^2-(2k)^2\right)\psi_{22}(x)-\frac{1}{2}\partial_x\psi_{22}(x)\cdot\left(\partial_x^2-k^2\right)\psi_1(x)\\
&-\psi_1(x)\partial_x^3\psi_{20}(x)+\partial_x\psi_{20}(x)\cdot\left(\partial_x^2-k^2\right)\psi_1(x)+Sxk_1\Big\}
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
g_1 \;\equiv\;& -k\Big\{\phi_1(x)\cdot\partial_x p_{20}(x)+\partial_x\phi_1(x)p_{22}(x)+\phi_{22}(x)\cdot\partial_x p_1(x)\\
&+\frac{1}{2}\phi_1(x)\cdot\partial_x p_{22}(x)+\frac{1}{2}\partial_x\phi_{22}(x)\cdot p_1(x)\Big\}
\end{aligned} \tag{2.37}$$

$$\begin{aligned}
h_1 \;\equiv\;& -k\Big\{\phi_1(x)\partial_x\psi_{20}(x)+\partial_x\phi_1(x)\cdot\psi_{22}(x)+\phi_{22}(x)\cdot\partial_x\psi_1(x)\\
&+\frac{1}{2}\phi_1(x)\cdot\partial_x\psi_{22}(x)+\frac{1}{2}\partial_x\phi_{22}(x)\cdot\psi_1(x)\Big\}.
\end{aligned} \tag{2.38}$$

In Eq. (2.34), the term involving the factor of ± 1 comes from Eq. (2.13). The solvability condition of Eq. (2.34) yields the following nonlinear equation, as was proved in the general formulation[1] in the nondegenerate case:

$$d_0 \frac{\partial A}{\partial \tau_2} \pm d_1 A + d_3 A^3 = 0, \tag{2.39}$$

where

$$d_0 \equiv \left\langle - \left[(\partial_x \phi_1(x))^2 + (k\phi_1(x))^2 \right] + (\partial_x \psi_1(x))^2 + (k\psi_1(x))^2 - \frac{1}{D_c} p_1^2(x) \right\rangle, \tag{2.40}$$

$$d_1 \equiv \frac{\chi}{D_c^2} \left\langle (\partial_x p_1(x))^2 + (k p_1(x))^2 \right\rangle, \tag{2.41}$$

$$d_3 \equiv \left\langle \phi_1(x) f_1 - \frac{1}{D_c} p_1(x) g_1 \right\rangle. \tag{2.42}$$

Rewriting Eq. (2.39) in terms of the original variables, i.e., $t = \tau_2/\lambda^2$ and $\mathcal{A} = \lambda A$, we have the nonlinear amplitude equation:

$$d_0 \frac{d\mathcal{A}}{dt} + (D - D_c)\, d_1 \mathcal{A} + d_3 \mathcal{A}^3 = 0, \tag{2.43}$$

where we used Eq. (2.13). Defining

$$\sigma \equiv -\frac{(D - D_c)\, d_1}{d_0}, \tag{2.44}$$

$$l \equiv \frac{2 d_3}{d_0}, \tag{2.45}$$

we obtain

$$\frac{d\mathcal{A}^2}{dt} = 2\sigma \mathcal{A}^2 - l \mathcal{A}^4. \tag{2.46}$$

This equation is the Landau-type equation well known in fluid dynamics,[2] the solution of which is expressed analytically as

$$\mathcal{A}^2 = \frac{\mathcal{A}_0^2}{\frac{l}{2\sigma}\mathcal{A}_0^2 + \left(1 - \frac{l}{2\sigma}\mathcal{A}_0^2\right) e^{-2\sigma t}}, \tag{2.47}$$

where $\mathcal{A}_0$ is the initial amplitude. It should be noted that σ gives the linear growth rate of the mode.

The solution (2.47) exhibits wide classes of nonlinear phenomena, depending upon the signs of the coefficients σ and l. For example, consider the linearly unstable case, i.e., $\sigma > 0$. In this case, if the mode under consideration cannot be nonlinearly stabilized by the second term on the right-hand side of Eq. (2.46), i.e., $l < 0$, the solution becomes unbounded in time, so that none of the higher order terms may be truncated and there is a fast transition to turbulence. On the other hand, if the mode is nonlinearly stabilized, i.e., $l > 0$, then a new stable equilibrium bifurcating from the initial equilibrium is obtained. In this case, the amplitude $\mathcal{A}$ asymptotically approaches its saturation level $\mathcal{A}_e$ given by

$$\mathcal{A}_e = \sqrt{\frac{2\sigma}{l}} = \sqrt{-\frac{(D - D_c)\, d_1}{d_3}} \tag{2.48}$$

for any positive initial amplitude $\mathcal{A}_0 > 0$.

It is easy to show that this saturation amplitude $\mathcal{A}_e$ is identical to the one given in Ref. 4, which was obtained by a somewhat different nonlinear analysis. In the perturbation method presented in this paper, however, we also obtain a nonlinear time evolution of the mode from the initial amplitude $\mathcal{A}_0$, which was not assessed in Ref. 4. Such time evolution of the mode at the initial stage is characterized by the saturation time $\Delta\tau$, or the time that elapses from $\mathcal{A} = 0.1\mathcal{A}_e$ to $\mathcal{A} = 0.9\mathcal{A}_e$, which is given by

$$\Delta\tau \simeq \frac{3}{\sigma}. \tag{2.49}$$

3. Comparison Between Theory and Numerical Simulations

In this section, we present sample calculations of the nonlinear evolution of the resistive fast interchange mode, using both theory [Eq. (2.46)] and direct numerical simulations of the nonlinear system Eqs. (2.1)–(2.3). The coefficients σ and l in Eq. (2.46), given by

Eqs. (2.44) and (2.45), respectively, are calculated from the solutions of the linear equations (2.17), (2.18), and (2.29)–(2.32). The details of the numerical simulations of Eqs. (2.1)–(2.3) may be found in Ref. 4. Following Ref. 4, we use the scale transformation

$$\begin{aligned} x &\rightarrow S^{-1/2}x, \qquad y \rightarrow S^{-1/2}y, \\ \psi &\rightarrow S^{-1}\psi, \qquad \phi \rightarrow S^{-1}\phi, \\ p &\rightarrow S^{-1/2}p, \end{aligned} \tag{3.1}$$

and introduce new parameters $M_s = SM$ and $\chi_s = S\chi$ in order to eliminate the explicit dependence of the system on the parameter S. The following values were chosen for the parameters in our calculations:

$$M_s = 1.0, \qquad \chi_s = 0.1, \qquad \delta_x S^{1/2} = 25, \qquad \text{and} \qquad \delta_y S^{1/2} = \pi. \tag{3.2}$$

In the numerical simulations of the equations (2.1)–(2.3), 150 grid points in the x-direction and 7 Fourier modes ($0 \leq m \leq 6$) in the y-direction were employed, where the mode number m is defined by the relation $\partial/\partial y = k = m\pi/\delta_y$. It was confirmed that these grid points and the number of Fourier modes give sufficient numerical resolution to obtain correct mode saturation in the calculation presented in this section. With these parameters, we obtain the eigenvalue D_c from Eq. (2.17) to be

$$D_c = 0.085 \tag{3.3}$$

and the parameters of Eqs. (2.44) and (2.45) to be

$$\begin{aligned} \frac{\sigma}{(D - D_c)} &= 3.8 \times 10^{-1} \\ l &= 1.2 \times 10^{5}. \end{aligned} \tag{3.4}$$

Because $l \gg \sigma / (D - D_c)$, the mode under consideration is significantly nonlinearly stabilized when $D \gtrsim D_c$.

Figure 1 shows the time evolution of the normalized energy E for the case of $D = 0.13\,(> D_c)$ as calculated from the theory [Eq. (2.46)] and as obtained from numerical simulation of the system Eqs. (2.1)–(2.3), where

$$E = \frac{1}{2\delta_y \Delta} \int_{-\delta_x}^{\delta_x} dx \int_{-\delta_y}^{\delta_y} dy \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right) \tag{3.5}$$

with the typical mode width Δ in the x-direction given by $\Delta = \pi y_0$.[4] The analytical result is in good agreement with the simulation result. The overshooting that appears in the numerical simulation seems to manifest a higher order nonlinear correction, which is not included in Eq. (2.46).

4. Conclusions and Discussion

Using the recently developed general formulation,[1] we have examined nonlinear evolution of the resistive fast interchange mode analytically.[3,4] In the case of the resistive fast interchange mode, the linear operator is nondegenerate at the marginally stable state, i.e., the linear dispersion relation of the mode has a single root of the frequency at the marginally stable state. Therefore, as shown in Ref. 1, the nonlinear amplitude equation of this mode turns out to be of the Landau type. It is found that a new stable equilibrium bifurcates from the initial equilibrium. Comparison between the perturbation theory and numerical simulations of the system (2.1)–(2.3) was made, from which we confirm that the saturation level and saturation time are well estimated from this nonlinear amplitude equation.

Application of the present theory to other resistive modes is also of great interest. We are currently investigating how the nonlinear behavior of the $m = 1$ tearing mode and the $m \geq 2$ tearing modes differ.

Acknowledgments

The authors wish to express their thanks to Professor H. Berk, Professor R. Hazeltine, and Professor W. Horton for their fruitful discussions and encouragement. We are also grateful to Dr. J. Van Dam for suggesting several improvements in the manuscript. This work was supported in part by the U. S. Department of Energy Contract No. DE-FG05-80ET-53088. Part of this work was done while one of the authors (N.N.) was a Visiting Scientist under the auspices of the U.S.-Japan Joint Institute for Fusion Theory.

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Figure Caption

1. Time evolution of the normalized energy E in Eq. (3.5) for $D = 0.13$. Both analytical and numerical results are shown. The unit of the time t is defined in Eq. (2.5).

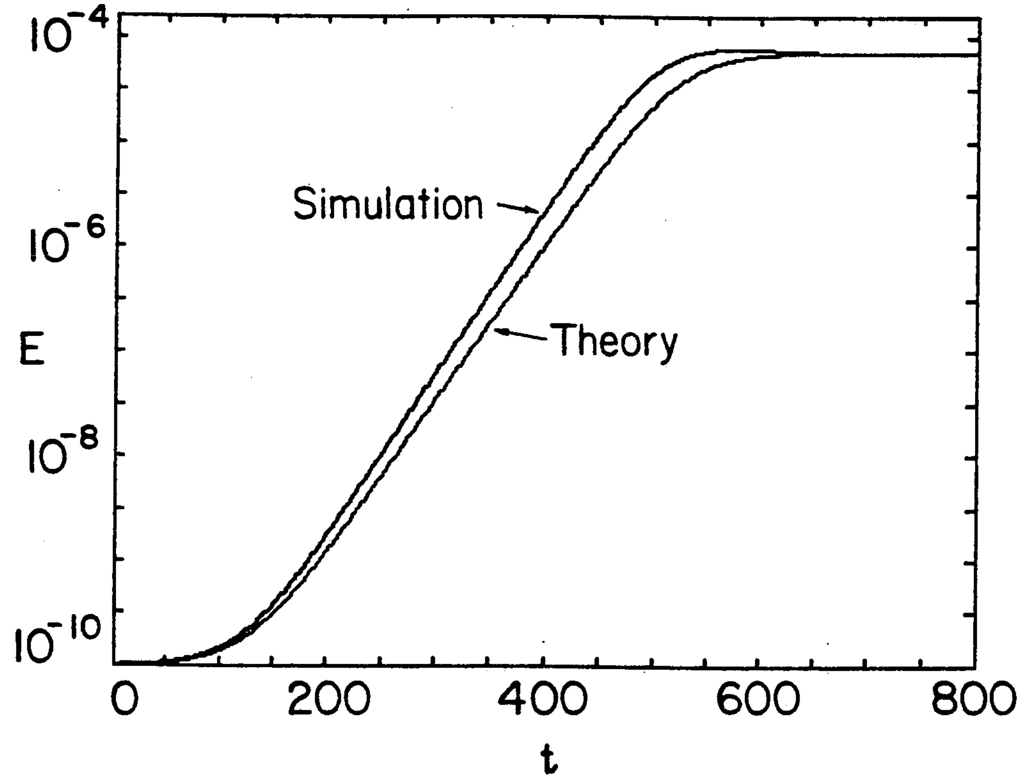

Figure 1