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F. Y. Gang, B. D. Scott
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

P. H. Diamond
Department of Physics, B-019
University of California at San Diego
La Jolla, California 92093
and
General Atomics, Inc.
San Diego, California 92138

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F. Y. Gang and B. D. Scott†
Institute for Fusion Studies
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P. H. Diamond
Department of Physics, B-019
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Abstract

The absolute equilibrium statistical mechanics of a two-field model of drift wave turbulence is investigated with emphasis on predicting the direction of spectral transfer, and on exploration of the role of desity-potential correlation in constraining the nonlinear transfer process. The results indicate that departure from the adiabatic relation $\tilde{n}/n_0 = e\tilde{\phi}/T_e$ allows transfer of total energy to small scales in fully developed turbulence. This prediction is in distinct contrast to intuition based on the one-field Hasegawa-Mima model. The results are verified by direct numerical simulation of the basic mode coupling equations.

† *Present address: Centro de Electrodinãmica de Universidade Tẽcnica de Lisboa, 1096 Lisboa, Portugal.*

The question of the direction of energy transfer in drift wave turbulence is a subject of considerable interest and one which is poorly understood. Previous studies[1−4]of this subject were mostly in the context of one-field models with adiabatic electron response ($\tilde{n}/n_0 = e\tilde{\phi}/T_e$). They show that for low frequency fluctuations, the total energy tends to flow to large scales (the so-called inverse cascade). In obtaining this result, the adiabatic constraint is crucial. However, in realistic systems, the adiabatic relation does not hold absolutely, so the density evolution dynamics must be considered.

In this paper, we present a study of the statistical mechanics of a two-field model of drift wave turbulence first derived by Wakatani and Hasegawa.[5] The adiabatic restriction in this model is relaxed due to the introduction of a small but finite resistivity, so that the decoupling between $\tilde{n}$ and $\tilde{\phi}$ can be taken into account. In order to facilitate the analysis, we will proceed in $\vec{\mathbf{k}}$-space. The fourier transformed version of the mode coupling equations (see equations (6) and (7) in ref.(5)) are:

$$\frac{\partial}{\partial t}(k^2\tilde{\phi}_{\vec{k}}) + \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{z}) \cdot \vec{k}_2 \tilde{\phi}_{\vec{k}_1}(k_2^2\tilde{\phi}_{\vec{k}_2}) = \chi_e k_{\parallel}^2(\tilde{\phi}_{\vec{k}} - \tilde{n}_{\vec{k}}) - \mu k^4 \tilde{\phi}_{\vec{k}} \tag{1}$$

$$\frac{\partial}{\partial t}\tilde{n}_{\vec{k}} + \sum_{\vec{k}_1+\vec{k}_2=\vec{k}} (\vec{k}_1 \times \vec{z}) \cdot \vec{k}_2 \tilde{\phi}_{\vec{k}_1}\tilde{n}_{\vec{k}_2} = \chi_e k_{\parallel}^2(\tilde{\phi}_{\vec{k}} - \tilde{n}_{\vec{k}}) - i\omega_{*e}\tilde{\phi}_{\vec{k}} \tag{2}$$

where $\tilde{\phi}_{\vec{k}}$ and $\tilde{n}_{\vec{k}}$ are the fourier modes of potential fluctuation and density fluctuation respectively, $\chi_e = v_{te}^2/\nu_{ei}\omega_{ci}$ is electron parallel thermalconductivity, μ is ion kinetic viscosity, and $\omega_{*e} = -k_y d(\ln n_0)/dx$ is electron drift frequency. The $\vec{\mathbf{k}}$-space is truncated so that $k_{min} \leq k \leq k_{max}$.

A simple analytically tractable approach to the problem of determining the direction of energy flow is that of equilibrium statistical mechanics.[6] The idea of this approach is that the direction of energy transfer is determined by the properties of the nonlinear mode coupling terms, and the constraints due to nonlinearly conserved quantities. In order to study the properties of these terms, we consider an isolated system with no linear dissipation and drive. In this case, the nonlinear mode couplings will drive the system to an absolute equilibrium state, thus the equilibrium spectrum indicates the direction of energy flow. Although the use of equilibrium statistical mechanics approach restricts us to consideration of two limiting cases (i.e. adiabatic electrons with $\chi_e = \infty$, and

hydrodynamic electrons with $\chi_e = 0$), the results are sufficient to indicate how the trend of energy flow changes when the system departs from the adiabatic electron assumption. In particular, the results presented in this paper will be a good approximation to the case where $\chi_e k_\parallel^2 \simeq 0$, i.e. the strong hydrodynamic regime.

As usual, we introduce a phase space in which the coordinates are the real and imaginary parts of $\tilde{n}_{\vec{k}}$ and $\tilde{\phi}_{\vec{k}}$. The phase space density P(Y) is defined in such a way that P(Y)dY is the probability of finding the system in the phase space region dY around phase point Y, where phase point $Y = (Re\tilde{n}_{\vec{k}}, Im\tilde{n}_{\vec{k}}, Re\tilde{\phi}_{\vec{k}}, Im\tilde{\phi}_{\vec{k}})$. Using equations (1) and (2), we can show that in the absence of dissipation and drive, the flow of phase space points is incompressible, i.e. $\partial\dot{Y}_{\vec{k}}/\partial Y_{\vec{k}} = 0$. Then Liouville equation follows directly as: $\partial P(Y)/\partial t + \sum_{\vec{k}} \dot{Y}_{\vec{k}} \partial P(Y)/\partial Y_{\vec{k}} = 0$. Based on this Liouville's equation, we can introduce a H-theorem for the drift wave system by direct analogy with the statistical mechanics of many particle system,[7] namely $dH(\hat{p})/dt \leq 0$, where the entropy functional is defined as $H(\hat{p}) = -\int dY \hat{p}(Y) \ln \hat{p}(Y)$, with $\hat{p}(Y)$ being the coarse grained phase space density.

For an isolated system with several invariants of motion, such as the total energy $E = E_0$, total enstrophy $\Omega = \Omega_0$ etc., the phase space density is a microcanonical distribution given by $\hat{P}(Y) = C\delta(E - E_0)\delta(\Omega - \Omega_0)\cdots$. The single mode distribution function which will be used in the caculation of equilibrium spectra can be easily obtained by integrating out the other variables. The result is a Gaussian distribution[8]: $\hat{p}(Y) = Cexp(-\alpha\hat{E}_{\vec{k}} - \beta\hat{\Omega}_{\vec{k}} - \cdots)$, where α, β, etc., are inverse "temperatures" related to the invariants E_0, Ω_0, etc.

In the following, we will discuss the equilibrium state of the drift wave system in two limiting cases, namely, the adiabatic case: $\chi_e k_\parallel^2 = \infty$, and the hydrodynamic case: $\chi_e k_\parallel^2 = 0$.

In the first case, the model equations reduce to the Hasegawa-Mima (one-field) equation, and the system admits two invariants of motion:total energy $E = \int d^3x((\tilde{\phi})^2 + (\vec{\nabla}\tilde{\phi})^2)$, and total enstrophy $\Omega = \int d^3x(\tilde{\phi} - \nabla^2\tilde{\phi})^2$. The equilibrium spectrum is obtained to be: $< \tilde{\phi}_{\vec{k}}^2 >= 1/[(1+k^2)(\alpha + \beta(1+k^2))]$, and the total energy and enstrophy spectra are:

$$E(k) = \frac{L^2}{2\pi} \frac{k}{\alpha + \beta(1+k^2)}, \tag{3}$$

$$\Omega(k) = \frac{L^2}{2\pi} \frac{k(1+k^2)}{\alpha + \beta(1+k^2)}. \tag{4}$$

These spectra, which have been noted by Hasegawa and Mima, show that in the equilibrium state E(k) tends to accumulate at small k (large scale), while $\Omega(k)$ tends to accumulate at large k (small scale). Thus, it was predicted that the total energy cascades to large scale while the total enstrophy cascades to small scales.

In the second case, the evolutions of vorticity and density decouple, the model equations then admit four invariants of the motion. These are the fluid kinetic energy $E_\phi = \int d^3x(\vec{\nabla}\tilde{\phi})^2$, fluid internal energy $E_n = \int d^3x(\tilde{n})^2$, fluid enstrophy $\Omega\phi = \int d^3x(\nabla^2\tilde{\phi})^2$, and crosscorrelation $\Gamma = -\int d^3x(\tilde{n}\nabla^2\tilde{\phi})$. We see that the decoupling of the two equations allows not only additive but also independent conservation of E_n and E_ϕ. The equilibrium spectra are thus caculated:

$$< \tilde{n}^2_{\vec{k}} > = \frac{4\left(\beta + \gamma k^2\right)}{4\alpha\left(\beta + \gamma k^2\right) - \eta^2 k^2}, \tag{5}$$

$$< \tilde{\phi}^2_{\vec{k}} > = \frac{4\alpha}{\left(4\alpha\left(\beta + \gamma k^2\right) - \eta^2 k^2\right) k^2}, \tag{6}$$

$$< \tilde{n}_{\vec{k}}\tilde{\phi}_{-\vec{k}} > = \frac{-2\eta}{4\alpha\left(\beta + \gamma k^2\right) - \eta^2 k^2}. \tag{7}$$

The total energy and enstrophy spectra then follow directly:

$$E(k) = \frac{L^2}{2\pi} \frac{k\left(\alpha + \beta + \gamma k^2\right)}{\alpha\beta + \left(\alpha\gamma - \eta^2/4\right) k^2}, \tag{8}$$

$$\Omega(k) = \frac{L^2}{2\pi} \frac{k\left(\beta + \left(\alpha + \gamma - \eta^2\right) k^2\right)}{\alpha\beta + \left(\alpha\gamma - \eta^2/4\right) k^2}. \tag{9}$$

The large k asymptotic behavior of E(k) and $\Omega(k)$ are similar in that they both accumulate at large k or small scale. This strongly suggests that when the adiabatic relation is broken, both total energy and enstrophy tend to flow to small scales. This is not surprising. In the hydrodynamic regime, vorticity and density evolution are governed by the same equations as those that govern 2-D Navier Stokes turbulence. Therefore, by direct analogy, the conceptual and methodological results in 2-D fluid turbulence can be transfered here to explain the result above. Thus, the internal energy E_n and fluid enstrophy Ω_ϕ cascade to small scales due to isodensity and isovorticity contour stretchings by fluid motion which generates small scale density gradient and vorticity gradient perturbations.

The fluid kinetic energy , however, flows to large scales due to the constraint of the conservation of enstrophy. Since the number density of mode increases with increasing k, the transfer of density fluctuation to small scales dominates the total energy cascade process, thus determining the direction of total energy cascade!

Another intersting feature of these spectra is that the presence of finite crosscorrelation (η term in equation (5) and (6)) adds an extra negative term to the denominator which distorts the equilibrium spectra toward high k. This indicates that it will take a much longer time for a system to approach an equilibrium state with finite crosscorrelation than if crosscorrelation vanishes. In other words, the effect of finite crosscorrelation here is to inhibit the nonlinear transfer process.

The results are verified by direct numerical solution of the original model equations. The figure (1) shows evolutions of density and potential contours in the adiabatic regime, while the figure (2) shows corresponding evolutions in the hydrodynamic regime. The initial conditions are the same for all the cases. As shown in figure(1), both density and potential flow to large scales. From figure (2), we see that density cascades to small scales, while potential flows to large scales.

An analysis of the role of the crosscorrelation has been carried out by using an EDQNM closure theory, which will be presented in a future publication. The basic result of this analysis is that the crosscorrelation in drift wave turbulence plays a role analogous to that of crosshelicity $< \vec{v} \cdot \vec{b} >$ in MHD turbulence,[9] namely the presence of crosscorrelation reduces the cascade of internal energy to small scales. Hence, in the strongly adiabatic limit where crosscorrelation is maximal, the cascade of internal energy to small scale is completely inhibited by the cancellation of the nonlinear term due to the perfect density potential correlation, so that the internal energy is carried to large scale by strong linear coupling (i.e.$\tilde{n}/n_0 = e\tilde{\phi}/T_e$). In the hydrodynamic limit, however, the correlation is not perfect (due to the breaking of the adiabatic relation), so that the nonlinear transfer dominates over the linear coupling, leading to internal energy cascade to small scale.

These results have interesting implications for our understanding of low frequency hydrodynamic microinstabilities in tokamaks. In particular, a collisional drift wave in a sheared magnetic field has the characteristics of both the hydrodynamic and adiabatic

regimes discussed here, depending on whether $\omega\nu_{ei} > \left(k_{\|}'\right)^2 x^2 v_{te}^2$ or $\omega\nu_{ei} < \left(k_{\|}'\right)^2 x^2 v_{te}^2$. Thus, on the basis of the analysis presented here, one might expect generation of small scales to play an important role in the nonliear dynamics of the hydrodynamic layer in collisional drift wave turbulence. Also, an important class of collisionless microinstabilities driven by pressure and curvature (i. e. toroidal ion temperature gradient driven modes and trapped particle modes) are described by nonlinear equations with the same structure as the hydrodynamic regime model discussed here. Hence, energy transfer to small scale and sensitivity to density-potential crosscorrlation can be expected in these systems. Therefore, existing theoretical models based on the assumption of inverse cascade of energy to large scales appear ill founded.

In conclusion,this paper deals with the problem of determining the direction of energy cascade in a two field drift wave turbulence model. Though only two idealized limiting cases are considered here, the results obtained are sufficient to tell how the direction of total energy flow changes when the system passes from adiabatic regime to hydrodynamic regime. More significantly, in this paper we have identified an important and previously ignored issue, namely that decoupling between density and potential fluctuations leads to total energy flow toward small scales, which is in contrast to intuition based on the one-field Hasegawa-Mima model.

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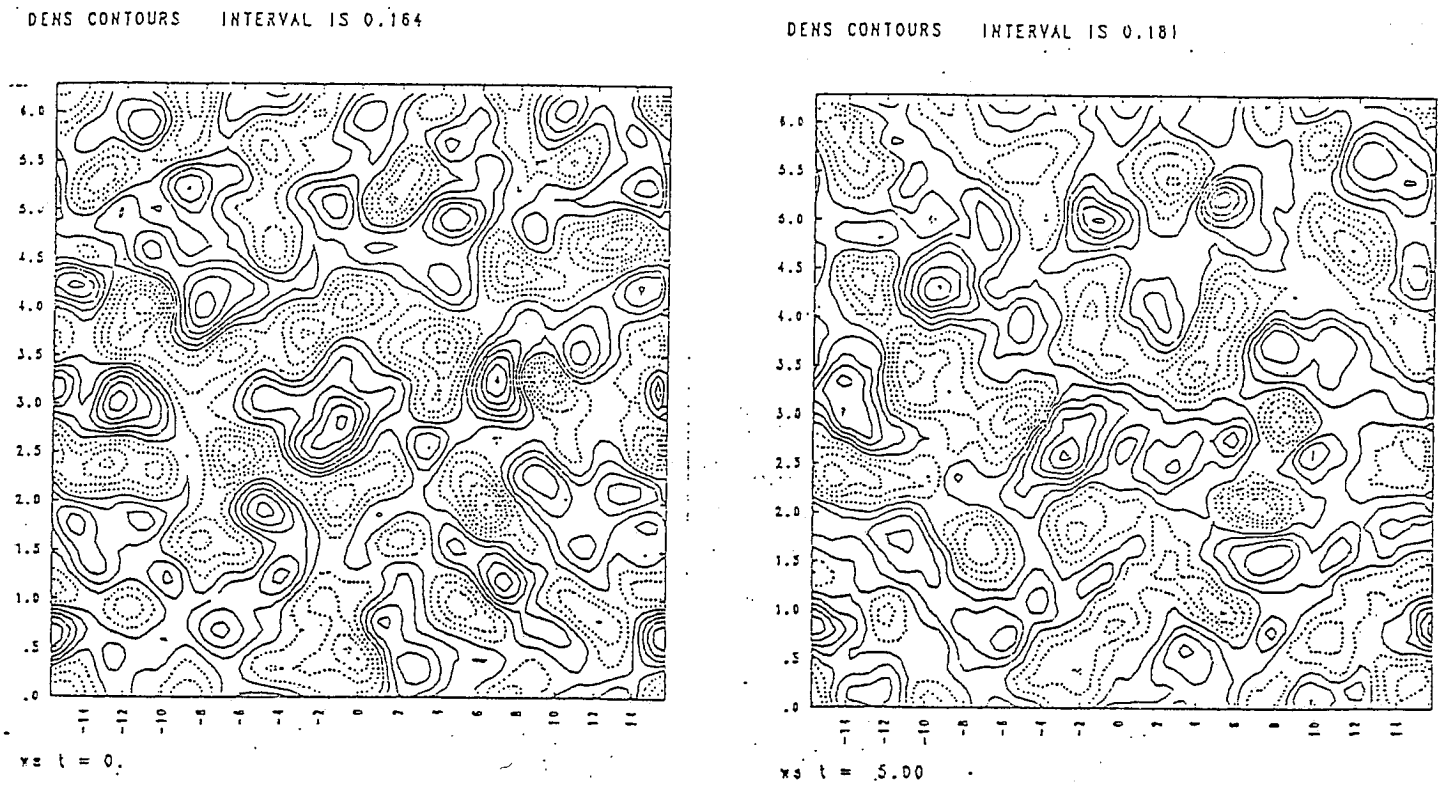
Figure Captions

Figure 1: This figure shows the comparison of evolutions of isodensity and isopotential contours in adiabatic case with $\chi_e = 10\omega_{*e}$; a) isodensity contour evolution, and b) isopotential contour evolution.

Figure 2: This figure shows the comparison of evolutions of isodensity and isopotential contours in the hydrodynamic case with $\chi_e = 0$; a) isodensity contour evolution, and b) isopotential contour evolution.

Figure (1) : Comparison of density and potential contour evolution in the adiabatic regime with $\chi_e = 10\omega_{*e}$.

(a)



(b)

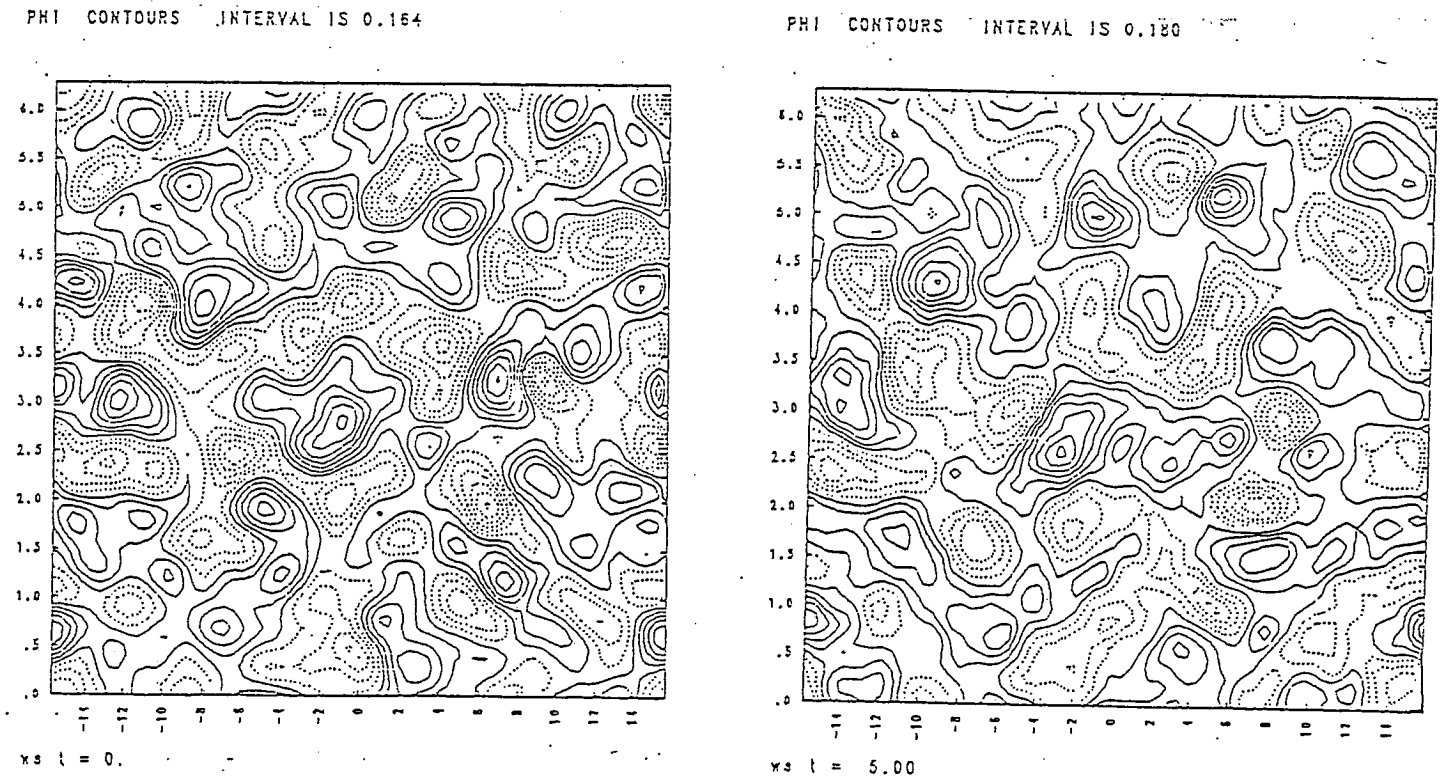
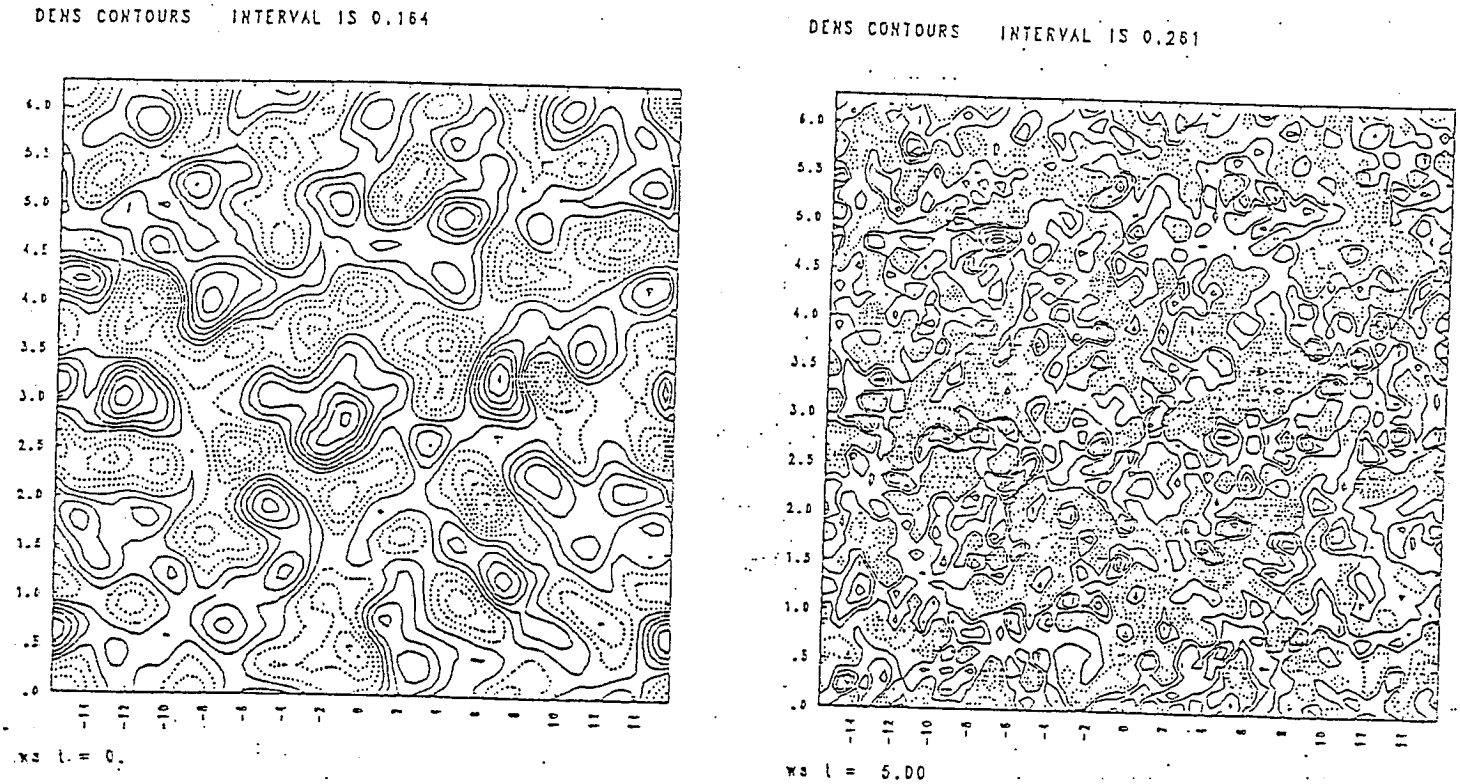


Figure (2) : Comparison of density and potential contour evolution in the hydrodynamic regime with $\chi_e = 0$.

(a)



(b)