

Pressure Gradient-Driven Modes in Finite Beta Toroidal Plasmas

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Abstract

When the ion temperature gradient is finite, the ideal kinetic theory plasma destabilizes the MHD mode below the critical β of MHD theory. For the temperature gradient $\eta_i > 2/3$ and inverse aspect ratio $\epsilon_{Ti} < 0.35$ the electrostatic toroidal η_i -mode is unstable. The coupling between the two modes is investigated by solving the fourth-order system describing the shear Alfvén-drift wave and the ion acoustic wave in the finite β tokamak. The ion kinetic velocity integral including the gyroradius effects and the ion magnetic drift resonances are used to obtain $\gamma_k(k, q^2\beta, \eta_i, s, \epsilon_n, q, \tau)$ and $\gamma_k/\langle k_x^2 \rangle$ for the modes. The study emphasizes the β and q -dependence of the transport associated with $\gamma_k/\langle k_x^2 \rangle$.

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I. Introduction

Drift wave instabilities associated with the ion temperature gradient are used to interpret the anomalous ion thermal conductivity in tokamak confinement studies. While the magnitude of the anomalous transport agrees with the experiments the observed dependence of the confinement on plasma current, which occurs in the dimensionless variables through $q = rB/RB_\theta$, is inadequately given by theory. This shortcoming of previous formulas for the ion thermal conductivity χ_i suggests that a more detailed understanding of the ion pressure gradient-driven mode with an emphasis on the q and β dependence is required. Here we investigate the dependence of the toroidal η_i -mode on q , β and other system parameters due to the toroidal coupling of the η_i -mode to the stable, kinetically modified MHD modes.

In slab geometry, the finite β effects on the stability of the η_i -mode are reported by several authors. Using the kinetic-local dispersion relation, Pu and Migliuolo¹ report that the finite β stabilization is ineffective and the high values of β , above the MHD ballooning mode threshold, are necessary for stabilization. This conclusion was confirmed by Migliuolo² later through analytic, nonlocal mode analysis. In contrast, Dong et al.³ report that finite β of several percent could have a significant stabilizing effect on the slab η_i -modes where they use an integral equation formulation for the sheared slab model.

The η_i -mode in toroidal geometry has been studied with electrostatic (E.S.) and electromagnetic (E.M.) theory both in the fluid description and the kinetic description with various restrictions.⁴⁻⁶ These studies⁴⁻⁶ indicate that the toroidal η_i -mode is not finite β stabilized, but the studies do not adequately describe the gyroradius at $k_\perp \rho_i \approx 1$ and the full curvature $-\nabla B$ drift resonance effects. In Ref. 6 the full kinetic integrals were used but the analysis was local to the outside of the torus which misses the toroidally induced mode coupling. In the present study we investigate the toroidal η_i -mode with the full gyroradius effects includ-

ing $k_{\perp}\rho_i \gg 1$ and the ion magnetic drift resonance $\omega = \omega_{Di}$ effects and compare the results with the previous works.

The model equations investigated here contain the MHD ballooning mode in the small $k_{\perp}\rho_i$, ω_D/ω hydrodynamic expansion. In the MHD expansion the stability is given by $\alpha = q^2\beta_i[1 + \eta_i + \tau(1 + \eta_e)]/\varepsilon_n$ and $s = rq'/q$ with the MHD threshold occurring at $\alpha \cong 0.8s^{1/2}$. The kinetic description in the Hastie and Hesketh⁷ study and our recent work⁸ show shows that the kinetic theory with finite η_i and $\omega = \omega_{Di}$ resonance effects destabilize the MHD ballooning mode below the MHD β critical. In this subcritical unstable range the kinetic MHD mode has a relatively small growth rate that is comparable with the toroidal η_i -mode growth rate. The frequency of the kinetic MHD mode in the subcritical range is $\omega_k \approx \omega_{*p_i}$ which is greater than the frequency of the toroidal η_i -mode, $\omega \sim \omega_{Di}$ which is also unstable in this regime. The kinetic MHD mode has finite but small E_{\parallel} represented here by $qRE_{\parallel} = -\partial_{\theta}(\phi - \psi)$, whereas the η_i -mode has $|\psi/\phi| \sim \beta \ll 1$.

Equations describing the coexistence of the two modes have been investigated earlier by Tang et al.⁹ They do not focus on the β and q dependence of γ_k and $\gamma_k/\langle k_x^2 \rangle$ for the η_i -mode, but consider principally the soft onset of the kinetic MHD mode. The Tang et al.⁹ study includes collisional electron dissipation, and the study also emphasizes the kinetic theory reduction of the maximum growth rate below the MHD growth rate.

Another kinetic stability study that contains the physics of both the η_i -mode and MHD mode is the work of Cheng.¹⁰ With an approximate treatment of the ion kinetic velocity integral, Cheng¹⁰ reports the coexistence of the two branches of eigenmodes. The Cheng study also emphasizes the kinetic modification of the MHD branch.

Dominguez and Moore¹¹ also investigate the two branches with fluid theory and show that the toroidal η_i -mode tends to be stabilized by increasing β . The coexistence of the two modes and the β stabilization are shown clearly in their Figs. 9 and 10. In Fig. 9b they report a curious stabilizing effect of increasing η_i on the unstable kinetic MHD branch. Recently

Jarmen et al.¹² considered the same modes with improved fluid equations and reports that the electrostatic toroidal η_i -mode persists both in the limit of small and large β . But these studies^{10,11,12} neglect the ion dynamics associated with the compressional ion acoustic waves and solve second order differential equation for the eigenmodes containing only the Alfvén waves. The parallel ion dynamics is important for the η_i -mode. In slab geometry the parallel ion dynamics destabilizes the η_i -mode, but in toroidal regime it gives a weak stabilizing effect.

The general formulations of eigenmode equations from kinetic theory in ballooning mode formalism are given by Antonsen and Lane,¹³ Tang et al.¹⁴ and ours.⁶ In the present work we solve the fourth-order system of equations for the coupled electrostatic potential and parallel vector potential. The perpendicular vector potential A_\perp is neglected based on our earlier local, kinetic stability analysis work showing negligible effects from A_\perp .⁶ We find that the correct analytic properties of the ion kinetic response function $P_k(\omega)$ [Ref. 10, $Q_k(\omega)$] require the separate v_\parallel and v_\perp velocity integrals for the ∇B -curvature drift resonances. We neglect the resonance at $k_\parallel v_\parallel = \omega - \omega_{Di}$ and expand in $k_\parallel v_i / |\omega - \omega_{Di}|$ to obtain the second differential operator description of the ion acoustic waves. We monitor the value of $\langle k_\parallel^2 \rangle / |\omega|^2$, to determine the limits of validity of this expansion a posteriori. Recent investigations by Romanelli¹⁵ include the $k_\parallel v_\parallel = \omega - \omega_{Di}$ resonance and show that it gives a small stabilizing effect. The dissipative trapped electron effects are also neglected since they obscure the dominant physics of the η_i -modes.

The organization of the work is as follows. In Sec. II we derive the fourth-order differential equations describing the shear Alfvén-drift wave and ion acoustic mode coupling. The solutions of these equations are presented in Sec. III. We discuss the finite β stabilization of the toroidal η_i -mode and the parametric dependence of the measure of the ion thermal conductivity given by $\gamma_k / \langle k_x^2 \rangle$ obtained from the eigenvalues and the eigenfunctions. Conclusions are given in Sec. IV.

II. Formulations

Following the calculations of the currents and charge densities in Ref. 6, we obtain the condition of quasi-neutrality

$$a\phi + b\psi = 0 \quad (1)$$

and the parallel component of Ampere's law

$$b\phi + d\psi = 0 \quad (2)$$

where

$$a(k, \omega, \theta) = -1 + \tau(P - 1) - \frac{c_s^2}{\omega^2 q^2 R^2} \frac{\partial}{\partial \theta} P_3 \frac{\partial}{\partial \theta}$$

$$b(k, \omega, \theta) = 1 - \frac{\omega_{*e}}{\omega} + \frac{c_s^2}{\omega^2 q^2 R^2} \frac{\partial}{\partial \theta} P_2 \frac{\partial}{\partial \theta}$$

and

$$d(k, \omega, \theta) = \frac{\rho^2 v_A^2}{\omega^2 q^2 R^2} \frac{\partial}{\partial \theta} \nabla_\perp^2 \frac{\partial}{\partial \theta} - \left(1 - \frac{\omega_{*e}}{\omega}\right) + \left(1 - \frac{\omega_{*pe}}{\omega}\right) \frac{\omega_{De}}{\omega} - \frac{c_s^2}{\omega^2 q^2 R^2} \frac{\partial}{\partial \theta} P_1 \frac{\partial}{\partial \theta}.$$

Here we used the ballooning mode representation and assumed the mode frequencies are in between the transit frequencies of the ions and electrons which will be justified later. Similar differential eigenmode equations in ballooning mode formalism are found in Refs. 13 and 14 from kinetic equations, and two fluid equations in Ref. 16. The ion kinetic response functions, P and P_j ($j = 1, 2, 3$) are given in Ref. 6 and we use the fluid limit in P_j ($j = 1, 2, 3$) functions

$$P_1^f = P_2^f = P_3^f \cong \left(1 - \frac{\omega_{*i}}{\omega}\right) \Gamma_0(b) - \frac{\omega_{*i}}{\omega} \eta_i [\Gamma_0(b) + b(\Gamma_1(b) - \Gamma_0(b))]$$

with $\Gamma_j(b) = I_j(b)e^{-b}$ and $b = k_\perp^2 \rho_i^2$. All frequencies are measured in units of c_s/r_n and the wavenumber k_θ in units of $\rho_s = c(m_i T_e)^{1/2}/eB$. The dimensionless complex frequency

$\omega[c_s/r_n]$ is a function of the seven dimensionless parameters $k, \beta_e, q, \varepsilon_n, s, \eta_i$ and $\tau = T_e/T_i$.

With the dimensionless variables, we write Eqs. (1) and (2) as

$$\left[1 - \tau(P - 1) + P_3^f \frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2}\right] \phi = \left[1 - \frac{\omega_{*e}}{\omega} + P_2^f \frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2}\right] \psi \quad (3)$$

$$\begin{aligned} \left[\frac{\varepsilon_n^2}{q^2 \omega^2} \frac{2}{\beta_e} \frac{\partial}{\partial \theta} k_\perp^2 \frac{\partial}{\partial \theta} + \left(1 - \frac{\omega_{*e}}{\omega}\right) \left(1 - \frac{\omega_{De}}{\omega}\right) + P_1^f \frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2}\right] \psi \\ = \left[1 - \frac{\omega_{*e}}{\omega} + P_2^f \frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2}\right] \phi. \end{aligned} \quad (4)$$

First we consider the limit which allows ion acoustic coupling terms to be zero with $\omega_A = \frac{\varepsilon_n}{q} \left(\frac{2}{\beta_e}\right)^{1/2}$ fixed. If $q \rightarrow \infty$ but $q^2 \beta_e$ finite, Eqs. (3) and (4) reduce to the 2nd order differential equation

$$[1 - \tau(P - 1)] \left\{ \frac{\omega_A^2}{\omega^2} \frac{\partial^2}{\partial \theta^2} k_\perp^2 \frac{\partial}{\partial \theta} + \left(1 - \frac{\omega_{*e}}{\omega}\right) \left(1 - \frac{\omega_{De}}{\omega}\right) \right\} \psi - \left(1 - \frac{\omega_{*e}}{\omega}\right)^2 \psi = 0, \quad (5)$$

where we used

$$\frac{\psi}{\phi} = \frac{1 + \tau(1 - P)}{1 - \frac{\omega_{*e}}{\omega}}$$

from Eq. (3) and note that $E_{||} = ik_{||} \phi(1 - \psi/\phi)$. The eigenmodes of Eq. (5) have been analyzed in earlier works.⁷⁻¹¹ With the full ion kinetic velocity space integral P we investigated the kinetic effects on MHD ballooning mode.⁶ Equation (5) also governs the toroidal η_i -mode in the low beta limit $\omega_A^2 \rightarrow \infty$ by $1 \cong \tau(P - 1)$ with the mode characteristics ($|\phi| \gg |\psi|$), the high beta limit and the MHD ballooning mode ($|\phi| \sim |\psi|$).

In $\beta \rightarrow 0$ limit Eq. (5) reduce to

$$[1 - \tau(P - 1)] \frac{\omega_A^2}{\omega^2} \frac{\partial}{\partial \theta} k_\perp^2 \frac{\partial}{\partial \theta} \psi = 0. \quad (6)$$

Assuming $[1 - \tau(P - 1)] \neq 0$, we obtain the solution of Eq. (6) as $\psi \sim \tan^{-1} \theta$, which is an unphysical solution having $\int d\theta \psi^2 \rightarrow \infty$. To have a solution which tends to zero for large θ , we must have

$$[1 - \tau(P - 1)] = 0. \quad (7)$$

Equation (7) is the local dispersion relation of toroidal η_i -mode and was studied in previous works.⁶

Dispersion relation (7) gives unstable η_i -mode when

$$\eta_i > \eta_c \sim \frac{2}{3} \quad \text{and} \quad \epsilon_{T_i} = \frac{r_{T_i}}{R} < 0.35.$$

Above the threshold the mode has $\omega_k \simeq \omega_{Di} = -2k\epsilon_n$ and $\gamma_k \sim v_i/(Rr_{T_i})^{1/2}$. Recent H -mode discharge experiments show inverted gradient profiles with $\eta_i < 0$ and $\epsilon_n < 0$. For the dissipative drift wave and the trapped electron mode the inverted profile showed a substantial gain in stability for $\nu_{*e} < 0.3$ regime. For fixed or local value of θ the condition $\text{Im } P = 0$ yields the marginal stability frequency

$$\omega_m \simeq \frac{1 + \frac{3}{2}|\eta_i|}{1 - |\eta_i|/|\epsilon_n|} \omega_{*i}$$

and

$$\text{Re } P(\omega_m) \simeq \left| \frac{\eta_i}{\epsilon_n} \right| \cdot \frac{1}{\Gamma_0(b)}.$$

This leads to the instability condition of

$$\epsilon_{T_i} < \frac{\tau \Gamma_0(b)}{1 + \tau}$$

for inverted gradient profiles. In terms of the electrostatic dispersion relation $D = 1 + \tau(1 - P)$, the high frequency collisionless dissipation becomes anomalous in sign for modes rotating in the ion direction when $|\eta_i| > |\epsilon_n|$. The anomalous high frequency sign of the dissipation gives the Nyquist diagram of $D = D(\omega)$ encircling the origin for ω in the upper half plane under the condition $\eta_i > \frac{2}{3}$ and $\epsilon_{T_i} < 0.35$, or $\epsilon_{T_i} < \frac{\tau \Gamma_0(b)}{1 + \tau}$ for inverted gradient profiles.

In this work, we analyze both the MHD ballooning and toroidal η_i -mode by solving Eqs. (3) and (4).

III. Stability Analysis

A. Electrostatic Limit

In the limit of $\beta = 0$, Eqs. (3) and (4) reduce to

$$\left(\left(1 - \frac{\omega_{*i}}{\omega} \right) \Gamma_0 - \frac{\omega_{*i}}{\omega} \eta_i [\Gamma_0 + b(\Gamma_1 - \Gamma_0)] \right) \frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2} \phi + [1 - \tau(P - 1)] \phi = 0. \quad (8)$$

We solve Eq. (8) with the full P -function and compare with the solutions from fluid or local approximations. As $\theta \rightarrow \infty$, $P \rightarrow 1/k_\perp^2 \rightarrow 0$ due to $J_0^2(k_\perp v_\perp / \omega_{ci})$ and the ω_{Di} -functions, we obtain the boundary condition as $\phi \rightarrow 0$ by requiring that ϕ must be spatially decaying. The analysis of Eq. (8) shows that $|\omega_r| \sim \gamma$ in contrast to the solution from fluid equations ($|\omega_r| \ll \gamma$).

Study of the spectrum of unstable toroidal η_i -mode eigenvalues $\omega(k) + i\gamma(k)$ as a function of $k_\theta \rho_s = k$ for $\eta_i > \eta_c$ and $\varepsilon_{Ti} < 0.35$ shows the growth rate increasing linearly with k reaching a maximum around $k = 0.3$ with $\gamma_m \lesssim 0.1$ where $\omega_m \cong -0.1$ to -0.2 . The frequency varies approximately as $\omega_k \cong -2\varepsilon_n k$ demonstrating the importance of the $\omega = \omega_{Di}$ resonance for the thermal ions.

Now we fix the wavenumber k in the region of maximum growth rate and vary the other parameters. Varying η_i we find that the growth rate increases as $\gamma \cong (\eta_i - \eta_c)^{1/2}$ and $\omega_k \cong -2k\varepsilon_n \left(1 + \frac{\eta_i}{2} \right)$. The reference parameters are $\varepsilon_n = 0.25$, $s = 1$, $\tau = 1$, $q = 2$. The critical η_i occurs near $\eta_c \simeq 1.5$ for both $k = 0.3$ and 0.5 , which is considerably higher than the $\eta_c \sim 2/3$ from the local kinetic theory which requires $q \rightarrow \infty$. Thus we find that critical $\eta_i = \eta_c(q)$ is the function of q shown in Fig. 1.

Now we vary the toroidicity ε_n and the shear parameter $s = rq'/q$. Solutions of the ballooning mode equation show that the growth rate increase for small ε_n and has a maximum at $\varepsilon_n \sim 0.1$ for $k = 0.5$, and at $\varepsilon_n \sim 0.5$ for $k = 0.3$, then $\gamma(\varepsilon_n)$ decreases as ε_n increases. This is because ε_n measures not only the strength of magnetic drift but the strength of the magnetic shear as well $L_s/L_n = q/\varepsilon_n s$. The frequency varies again as $\omega_k = -2\varepsilon_n k$. For

small s the growth rate is independent of shear. The growth rate decreases as s increases for $s > 1/2$ when $k = 0.5$. For $k = 0.3$, the growth has a broad maximum in the shear weakly decreasing for $s > 1.0$.

In our study, we find that there are several unstable eigenmodes for a given parameter set. For the modes not emphasized here the wavefunctions are very broad $\langle \theta^2 \rangle \gg 1$ and oscillatory. We believe these modes are related to the sheared slab branches, but their study requires the integral formulation to treat the $\omega = k_{\parallel} v_{ti}$ resonance without expansion which is beyond the context of this study. In view of transport, as measured by $\gamma_k / \langle k_x^2 \rangle$, these secondary modes appear less important than the θ -localized mode which we identify as the toroidal η_i -mode.

Finally, we consider the current or q dependence of the growth rate and eigenfunctions. As noted before the local theory is recovered for $q \rightarrow \infty$ and we find that $\gamma(q) < \gamma(q \rightarrow \infty)$ for the reference parameters. For $q \simeq 1$ we find that the system is essentially stable due to the strong coupling to the ion-acoustic waves and the associated parallel compression in the ion motion. With respect to the eigenfunction we find that $\langle k_x^2 \rangle = k^2 s^2 \langle \theta^2 \rangle$ decreases with q as given in Eq. (11) of Ref. 4. Thus, we find a strong q dependence of the $\gamma_k / \langle k_x^2 \rangle$ measure for anomalous transport shown in Fig. 2. With the fluid expansion to the kinetic response function P , the growth rate of the mode changes very little with q as also reported by Romanelli.¹⁵

B. Finite β Plasmas

For finite β there are two eigenmodes with different eigenvalues and different mode characteristics. One is a finite β modified η_i -mode with $|\phi| \gg |\psi|$ and the other is an MHD ballooning mode with $|\phi| \sim |\psi|$.

We prove that for $\eta_i = 0$ the ion acoustic coupling does not affect the marginal stability of the MHD ballooning mode,⁸ from the ideal MHD and kinetic equation. The ideal MHD

equations with the parallel ion acoustic dynamics, Eqs. (A21) and (A22) which are derived in the Appendix at $\omega = 0$ reduce to the ideal MHD equation at marginal stability

$$\frac{2}{\beta_e} \frac{\varepsilon_n^2}{q^2} \frac{\partial}{\partial \theta} k_\perp^2 \frac{\partial}{\partial \theta} \psi + \omega_{De} \omega_{*e} \left(1 + \frac{1 + \eta_i}{\tau}\right) \psi = 0. \quad (9)$$

The parallel ion dynamics does not change the marginal stability but reduces the growth rate. For $\omega = \omega_{*i}$ the kinetic Eqs. (1) and (2) also reduce to the ideal MHD Eq. (9) at marginal stability, which proves the kinetic theory threshold is the MHD threshold β_c^{MHD} for $\eta_i = 0$.

The origin of the instability below β_c^{MHD} is found to be from the ion magnetic drift resonance effect when $\eta_i \neq 0$. We explain this using Eq. (5) which neglects the ion acoustic coupling.

For $\eta_i \neq 0$, the kinetic response function does not vanish at the MHD marginal stability frequency. For $\omega \cong \omega_{*i}$, the η_i response adds to the $V_{\text{eff}}(z)$ defined in Ref. 7 through the function

$$\begin{aligned} \Delta P(\omega_{*i}, k, z) = & -\eta_i \left\langle \frac{(v^2/v_i^2 - 3/2) J_0^2(k_\perp v_\perp / \omega_{ci})}{1 - \omega_{Di}(v_\perp, v_\parallel, z) / \omega_{*i}} \right\rangle \\ & + i\eta_i \sqrt{2\pi} \hat{\omega}_z H(\hat{\omega}_z) \int_0^{\hat{\omega}_z^{1/2}} dv_\parallel \exp\left(\frac{1}{2}v_\parallel^2 - \hat{\omega}_z\right) J_0^2(k_\perp \sqrt{\hat{\omega}_z - 2v_\parallel^2}) \\ & \times \left(\frac{3}{2} + \frac{1}{2}v_\parallel^2 - \hat{\omega}_z\right), \end{aligned} \quad (10)$$

where $\hat{\omega}_z = 2\omega_{*i}/\omega_{Di}(z, s)$ and $H(x)$ is the heaviside step function. The second term in the right hand side of Eq. (10) is from the ion magnetic resonance contribution. In the neighborhood of small η_i we expand Eq. (5) about the MHD marginal stability with $\omega = \omega^{(0)} + \omega^{(1)}$, and we obtain

$$\psi''(z) + \left\{ \frac{\alpha \left[\cos(z/s) + z \sin(z/s) + \frac{\frac{\omega^{(1)}}{\omega_{*i}} (1 - \Gamma_0(k_\perp)) - \Delta P}{1 + \tau} \right]}{1 + z^2} - \frac{1}{(1 + z^2)^2} \right\} \psi = 0.$$

Perturbation theory for the complex frequency shift $\omega^{(1)}$ due to η_i gives

$$\omega^{(1)} = \omega_{*i} \frac{\int_{-\infty}^{\infty} dz \bar{\psi}_0^2 \frac{\Delta P}{1+z^2}}{\int_{-\infty}^{\infty} dz \bar{\psi}_0^2 \frac{1}{1+z^2} (1 - \Gamma_0(k_{\perp} \rho_i))}. \quad (11)$$

where the eigenfunction $\bar{\psi}_0(z)$ is real and given by Eq. (9). Most contributions of $\text{Im } \Delta P$ of Eq. (10) comes from $z \leq 1$ and $\text{Im}(\Delta P) < 0$ for $z \leq 1$ makes $\text{Im } \omega^{(1)}$ positive-unstable.

In Figs. 3 and 4 we show the eigenvalues of the two modes and the transport measure $\gamma / \langle k_x^2 \rangle$ versus $q^2 \beta$. The parameters are $\varepsilon_n = 0.25$, $\eta_i = 2$, $\tau = 1$, $k = 0.5$. As $q \rightarrow \infty$, the ion acoustic terms vanish and we recover the local solution of the coupled $\phi - \psi$ equations studied in previous works. When $q = 2$, we find the finite β stabilization of the toroidal η_i -mode is strong and the growth rate is zero for $\beta > \beta_c^{\eta_i} \sim 0.01$.

We find that for the MHD ballooning mode, the parallel compressibility effects are weak. The growth rate is larger than that of the toroidal η_i -mode, but the transport measure $\gamma / \langle k_x^2 \rangle$ is relatively smaller as shown in Fig. 4.

IV. Conclusions

New results are presented for the behavior of the pressure gradient driven modes in tokamaks by solving the fourth-order system describing the shear Alfvén-drift wave and ion acoustic wave in the finite β plasma. The toroidal η_i -mode is unstable for $\eta_i > \eta_c(q)$ with $\eta_c(q = 1) \approx 2.0$ and $\eta_c(q = 10) \sim 2/3$. The toroidal η_i -mode is characterized by $|\phi| \gg |\psi|$, and the real frequency varies as $\omega \cong -2\varepsilon_n k$ which means that the $\omega = \omega_{Di}$ resonance is essential. The growth rate γ is smaller than the real frequency, which is in contrast to the usual fluid result in which the solution with $|\gamma| > |\omega_k|$ is obtained.

The MHD ballooning mode is characterized by $|\phi| \sim |\psi|$ and the real frequency varies as $\omega_k \cong \omega_{*pi}$, and the kinetic theory growth rate is much smaller than the fluid theory growth rate. For the kinetic MHD ballooning mode, instability below β_c^{MHD} is shown to arise directly from the finite η_i effect. For $\eta_i = 0$, we prove that the kinetic equation gives the same β_c as

the ideal MHD equation. We give that the kinetic instability growth rate below the β_e^{MHD} is due to the ion magnetic drift resonance when $\eta_i \neq 0$ by perturbation theory.

In the electrostatic limit the growth rate is smaller than the local kinetic theory which neglects the ion acoustic coupling ($q \rightarrow \infty$ with $q^2 \beta_e$ finite). The finite β effects on the toroidal η_i -mode is a strong stabilization for smaller values of q . The experimentally observed dependence of plasma transport on plasma current, which appears in the dimensionless parameters through $q = rB/RB_\theta$, may be related this low q , fixed β , stabilizing effect shown in Fig. 2. In contrast, for the MHD ballooning mode, the ion acoustic coupling gives weak effect.

As a measure of transport we have studied the parametric variation of $\gamma_k / \langle k_x^2 \rangle$ for k near the maximum of γ_k . While mode coupling in the turbulence may change some aspects of the transport formula it may be useful to compare such theories or to compare experiments with the parametric dependence given by linear ballooning mode theory. We find that for $\tau = 1$ the ballooning eigenmodes give a diffusivity of the form

$$\chi_i \simeq \frac{q(\eta_i - \eta_c)^{1/2}}{s + s_1} \left(\frac{\rho_s}{r_n} \frac{cT_e}{eB} \right) \quad (12)$$

where $\eta_c(q)$ is given in Fig. 1, and

$$s_1 = \frac{q}{\sqrt{2}} \frac{\left(\frac{\eta_i - \eta_c}{\tau} \right)^{3/4}}{(2\epsilon_n)^{1/4}} \frac{\rho_s}{r_n}.$$

For sufficiently small s , profile effects¹⁷ on the broad radial eigenmodes are important. For $s \gg s_1$ and $\eta_i \gg \eta_c$ the formula is the same form as given by Eq. (36) in Ref. 4. For $r_n \rightarrow \infty$ the effective scale length becomes

$$\chi_i(\eta_i \rightarrow \infty) \simeq \frac{q\rho_s}{s(r_{Ti}R)^{1/2}} \left(\frac{cT_e}{eB} \right). \quad (13)$$

Formula (12) for the anomalous ion thermal conductivity is given as a model recognizing that inverse cascade or other turbulence effects may modify the scaling with some parameters.

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Appendix

Here we derive the set of reduced equations from the two-fluid equations which govern the pressure gradient driven modes in finite β plasmas.

The ion continuity equation is given as

$$\frac{\partial}{\partial t} n_i + \mathbf{v}_{\perp i} \cdot \nabla n_i + n_i \nabla \cdot \mathbf{v}_{\perp i} + n_i \nabla \cdot \mathbf{v}_{pi} + n_i \nabla_{\parallel} v_{\parallel i} = 0 \quad (\text{A1})$$

where the hydrodynamic velocities are given by

$$\mathbf{v}_{\perp i} = \frac{c \hat{\mathbf{b}} \times \nabla \Phi}{B} + \frac{c \hat{\mathbf{b}} \times \nabla P_i}{en_i B}, \quad (\text{A2})$$

$$\mathbf{v}_{pi} = \frac{m_i c^2}{e B^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_{di} \cdot \nabla \right) \nabla \Phi \quad (\text{A3})$$

and

$$\frac{dv_{\parallel i}}{dt} = -en_i \left(\hat{\mathbf{b}} \cdot \nabla \Phi + \frac{1}{c} \frac{\partial}{\partial t} A_{\parallel} \right) - \hat{\mathbf{b}} \cdot \nabla P_e. \quad (\text{A4})$$

We used $\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial}{\partial t} A_{\parallel} \hat{\mathbf{b}}$ and $\mathbf{B} = \nabla A_{\parallel} \times \hat{\mathbf{b}}$ for the electromagnetic fields. The thermal ion balance equation is

$$\frac{3}{2} \frac{\partial}{\partial t} (n_i T_i) + \nabla \cdot \left(\frac{3}{2} n_i T_i \mathbf{v}_{\perp i} + \mathbf{q}_i \right) + n_i T_i \nabla \cdot \mathbf{v}_{\perp i} = 0 \quad (\text{A5})$$

where

$$\mathbf{q}_i = \frac{5}{2} \frac{P_i \hat{\mathbf{b}} \times \nabla T_i}{m_i \omega_{ci}}.$$

For electrons, we write the electron continuity equation as

$$\frac{\partial}{\partial t} n_e + \mathbf{v}_{\perp e} \cdot \nabla n_e + n_e \nabla \cdot \mathbf{v}_{\perp e} + n_e \nabla_{\parallel} v_{\parallel e} = 0 \quad (\text{A6})$$

where we neglected the electron polarization drift velocity term and $\mathbf{v}_{\perp e}$ is given by

$$\mathbf{v}_{\perp e} = \frac{c \hat{\mathbf{b}} \times \nabla \Phi}{B} - \frac{c \hat{\mathbf{b}} \times \nabla P_e}{en_e B}. \quad (\text{A7})$$

For the electron temperature equation, we use the adiabatic relation

$$\hat{b} \cdot (\nabla P_e + en_e \mathbf{E}) = 0 \quad (\text{A8})$$

with the electron temperature constant along the magnetic field

$$\hat{b} \cdot \nabla T_e = 0. \quad (\text{A9})$$

We consider the ballooning type modes in the usual circular cross-section tokamak with $B_T = B_0/(1 + \epsilon \cos \theta)$ and $B_\theta = \frac{\epsilon}{q} B_T$ where $\epsilon = \frac{r}{R}$ and $q(r) = r B_T / R B_\theta$. The fluctuations are taken to vary as

$$f = f(r) + \delta f(\theta) e^{i\ell(q(r)\theta - \phi) - i\omega t} + \text{c.c.}.$$

This mode gives rise to $k_\perp^2 = k_\theta^2(1 + s^2\theta^2)$ with $k_\theta = \ell q/r$ and the parallel derivative $ik_\parallel = \frac{1}{qR} \frac{\partial}{\partial \theta}$.

Using this model, the linearized version of Eqs. (A1)-(A9) can be written as

$$\frac{\partial}{\partial t} n_i = -i\omega_{*e}\phi + i\omega_{De}\phi + i\omega_{De}p_i + \frac{\partial}{\partial t} \nabla_\perp^2 \phi + i\omega_{*pi} \nabla_\perp^2 \phi - ik_\parallel v_{\parallel i} \quad (\text{A10})$$

$$\frac{\partial}{\partial t} v_{\parallel i} = -ik_\parallel \phi - \frac{\partial}{\partial t} A - ik_\parallel p_i - i\omega_{*pi} A \quad (\text{A11})$$

$$\frac{\partial}{\partial t} p_i = i\omega_{*pi}\phi - i\frac{5}{3}\omega_{Di}p_i + i\frac{5}{3\tau}\omega_{Di}n_i + \frac{5}{3\tau} \left(\frac{\partial}{\partial t} n_i + i\omega_{*e}\phi \right) \quad (\text{A12})$$

$$\frac{\partial}{\partial t} A = -ik_\parallel \phi + ik_\parallel n_e - i\omega_{*e} A \quad (\text{A13})$$

$$\frac{\partial}{\partial t} k_\perp^2 \phi = -i\omega_{*pi} k_\perp^2 \phi + i\omega_{De} p_i + i\omega_{De} n_e - ik_\parallel (v_{\parallel i} - v_{\parallel e}) \quad (\text{A14})$$

$$k_\perp^2 A = \frac{\beta_e}{2} (v_{\parallel i} - v_{\parallel e}) \quad (\text{A15})$$

where we defined the dimensionless space-time variable by

$$x, y \rightarrow \rho_s(x, y) \quad , \quad z \rightarrow r_n z \quad , \quad t \rightarrow r_n t / c_s$$

and scaled the amplitude of the fields as follows:

$$\frac{\tilde{P}_i}{P_e} = \frac{\rho_s}{r_n} P_i, \quad \frac{\tilde{v}_{||i}}{c_s} = \frac{\rho_s}{r_n} v_{||i}, \quad \frac{e}{T_e} \Phi = \frac{\rho_s}{r_n} \phi \quad \text{and} \quad \frac{c_s}{c} \frac{e}{T_e} A_{||} = \frac{\rho_s}{r_n} A.$$

These equations are combined to give 2 coupled 2nd order differential equations with $ik_{||}\psi = -\frac{\partial}{\partial t}A$:

$$\frac{2}{\beta_e} \frac{\varepsilon_n^2}{q^2} \frac{\partial}{\partial \theta} k_{\perp}^2 \frac{\partial}{\partial \theta} \psi + \omega(\omega - \omega_{*pi}) k_{\perp}^2 \psi + f(\phi - \psi) \quad (\text{A16})$$

$$+ \frac{\omega \left[\omega_{De} \omega_{*e} \left(1 + \frac{1+\eta_i}{\tau} \right) + \frac{\omega_{De}^2 \omega_{*e}}{\omega} \frac{5}{3\tau} \left(1 + \frac{1}{\tau} \right) \right]}{\omega - \frac{5}{3} \omega_{Di}} \psi = 0 \quad (\text{A17})$$

$$\begin{aligned} \frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2} g + \frac{\omega_{De}}{\omega} g + \left(1 - \frac{\omega_{*pi}}{\omega} \right) k_{\perp}^2 \phi + \left(1 - \frac{\omega_{*e}}{\omega} \right) (\phi - \psi) \\ = -\frac{\omega_{De}}{\omega} \left(1 - \frac{\omega_{*pi}}{\omega} \right) \psi \end{aligned} \quad (\text{A18})$$

with $f(\theta)$ and $g(\theta, \phi, \psi)$ are defined as

$$f(\theta) = k_{\perp}^2 \omega(\omega - \omega_{*pi}) - \frac{\omega \left[\frac{\omega_{De} \omega_{*e}}{\tau} \left(\frac{2}{3} - \eta_i \right) - \left(1 + \frac{5}{3\tau} \right) \omega_{De} \omega - \frac{5}{3\tau} \left(1 + \frac{1}{\tau} \right) \omega_{De}^2 \right]}{\omega - \frac{5}{3} \omega_{Di}} \quad (\text{A19})$$

$$\begin{aligned} g(\theta, \phi, \psi) = \phi - \left(1 - \frac{\omega_{*pi}}{\omega} \right) \psi - \frac{5}{3\tau} (-\omega + \omega_{Di}) \left(\phi - \frac{\omega - \omega_{*e}}{\omega} \psi \right) \\ + \frac{\frac{\omega_{De}}{\tau} \left(\frac{2}{3} - \eta_i \right) \phi}{\omega - \frac{5}{3} \omega_{Di}}. \end{aligned} \quad (\text{A20})$$

In the limit of $\omega \gg \omega_{*i}$, the Eqs. (A16) and (A17) reduce to the ideal MHD equations

$$\frac{2}{\beta_e} \frac{\varepsilon^2}{q^2} \frac{\partial}{\partial \theta} k_{\perp}^2 \frac{\partial}{\partial \theta} \psi + \omega^2 k_{\perp}^2 \psi + \omega_{De} \omega_{*e} \left(1 + \frac{1+\eta_i}{\tau} \right) \psi = -\omega_{De} \zeta \quad (\text{A21})$$

$$\left[\frac{\varepsilon_n^2}{q^2 \omega^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{1 + \frac{5}{3\tau}} \right] \zeta = -\omega_{De} \psi. \quad (\text{A22})$$

The growth rate from 4th order differential equation is smaller than the one from 2nd order differential equation (Eq. (A20) with $\zeta = 0$), but the critical β is not changed.

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Figure Captions

1. The threshold value of η_i versus the safety factor q . The parameter values are $\varepsilon_n = 0.25$, $s = 1$, $\tau = 1$, $k_\theta \rho_s = 0.5$, and $\beta_e = 0$.
2. The measure of the anomalous transport, $\gamma_k / \langle k_x^2 \rangle$ for η_i mode. Same parameter values except $\eta_i = 2$.
3. The eigenvalues of both η_i mode and kinetic MHD-like ballooning mode. The parameter values are $\varepsilon_n = 0.25$, $\eta_i = 2$, $\tau = 1$, and $k = 0.5$.
4. The measure of the anomalous transport, $\gamma_k / \langle k_x^2 \rangle$ for both η_i mode and kinetic MHD-like ballooning mode. Same parameter values as in Fig. 3.

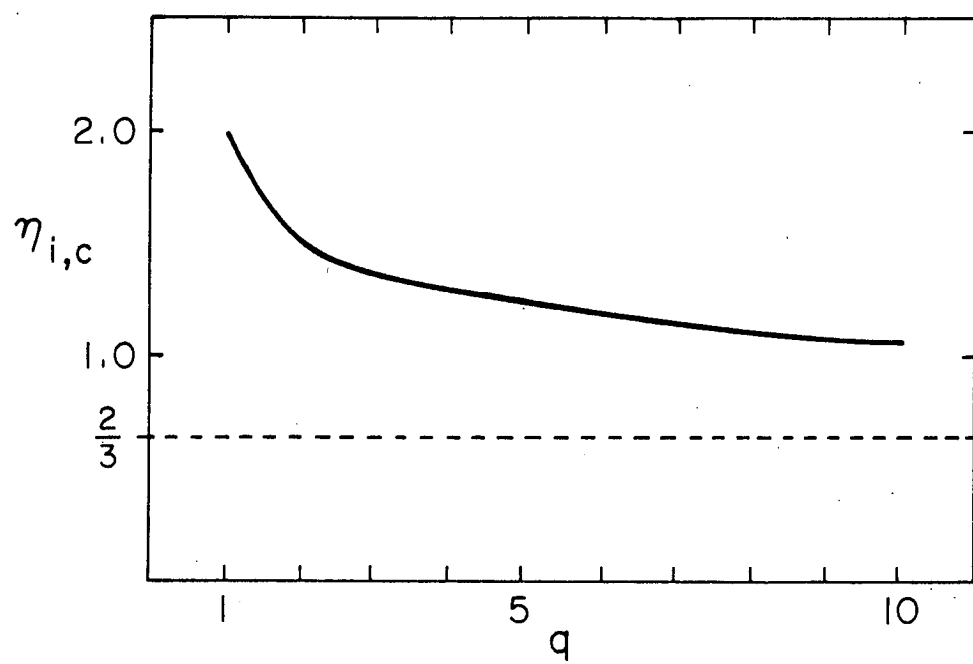


Fig. 1

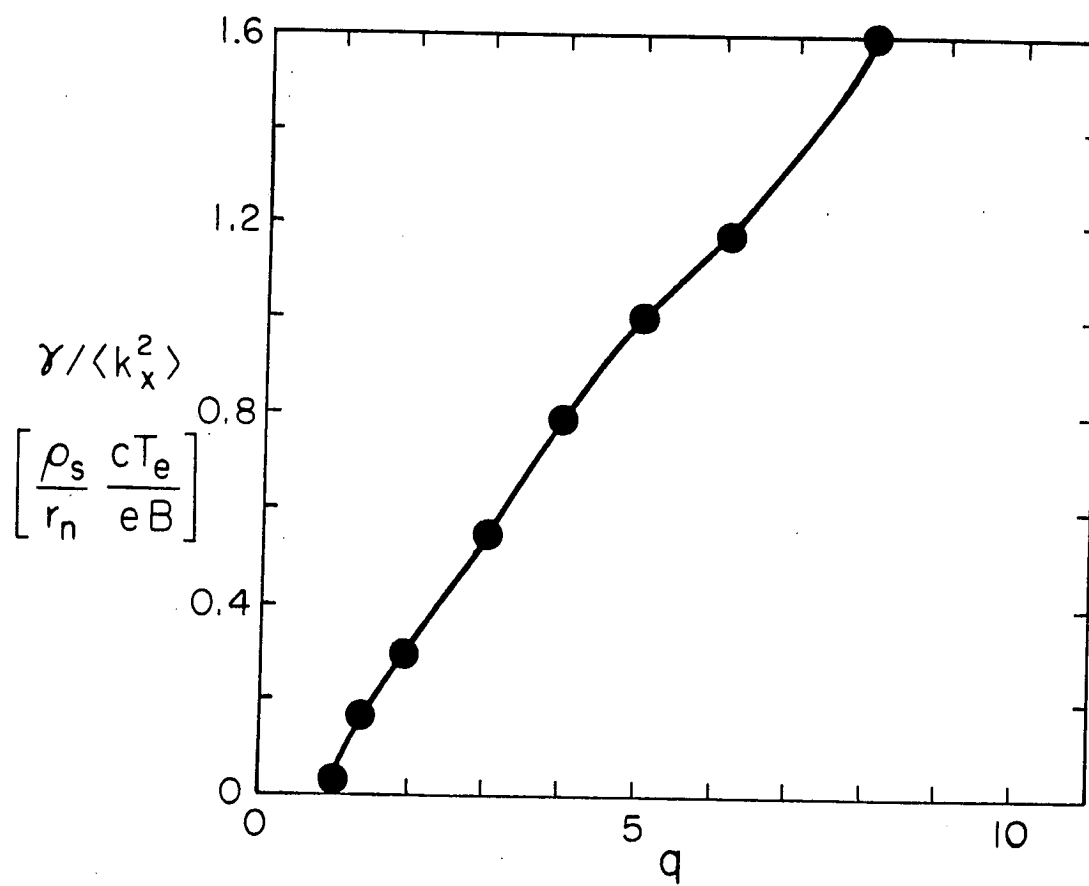


Fig. 2

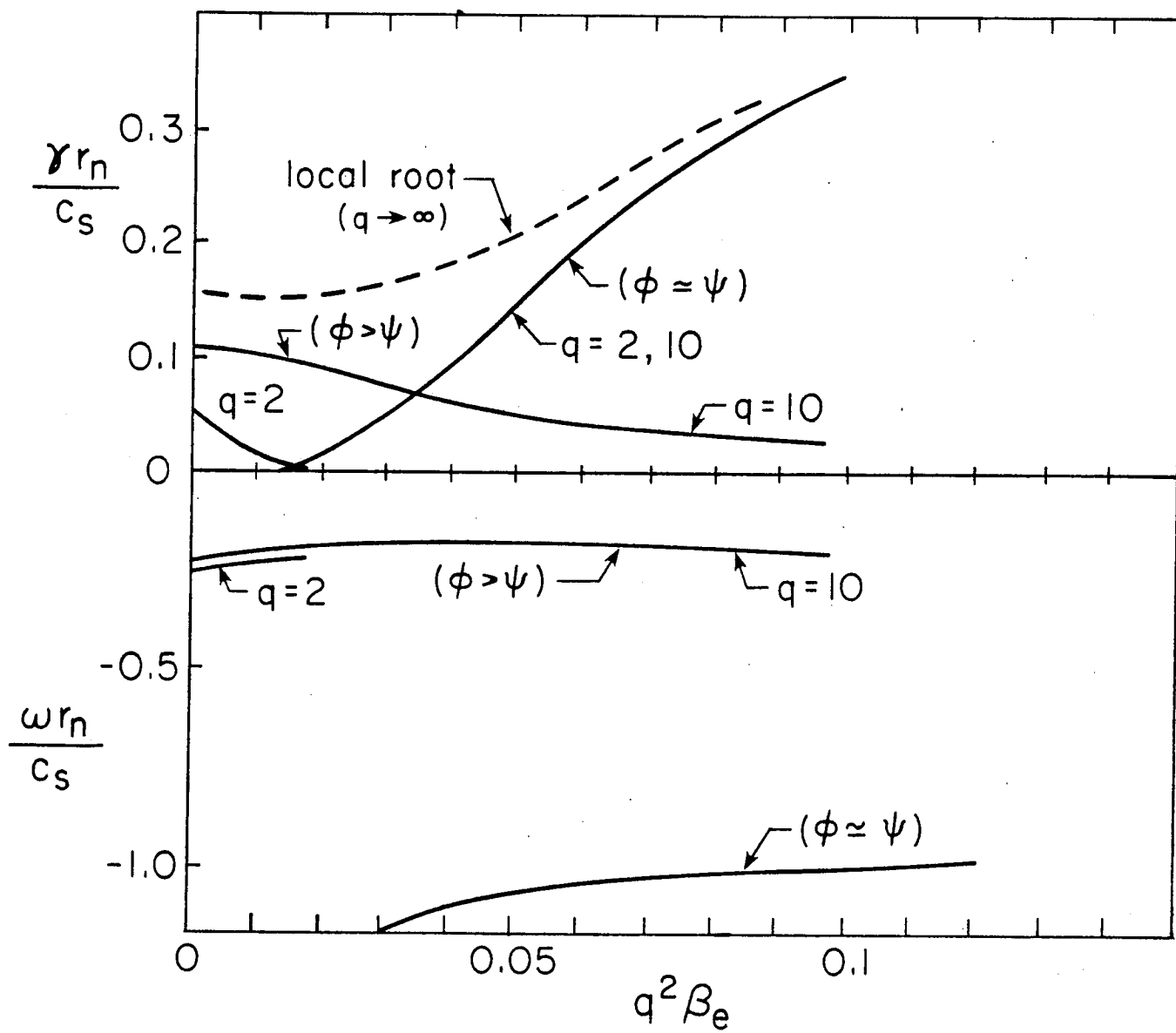


Fig. 3

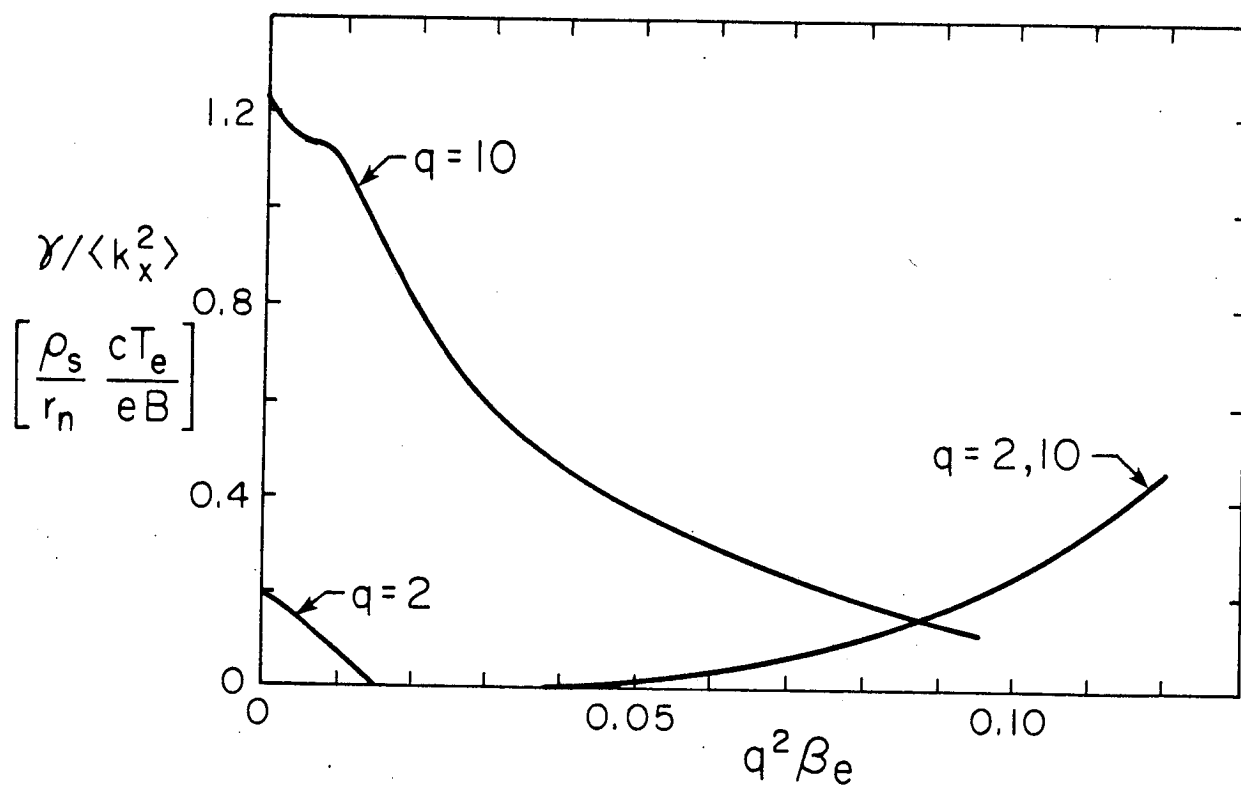
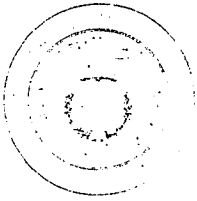


Fig. 4



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October 27, 1988

Dr. Richard J. Palmer, Staff Editor,
Plasma Physics and Controlled Fusion
Techno House
Redcliffe Way
Bristol BS1 6NX ENGLAND

Dear Dr. Palmer:

Please find enclosed the typescript "Pressure Gradient-Driven Mode in Finite Beta Toroidal Plasmas", by Dr. B.G. Hong, D-I. Choi, and myself. In this work we develop new results for the electromagnetic ion temperature gradient instability in toroidal plasmas.

We appreciate your consideration of this new work for publication.

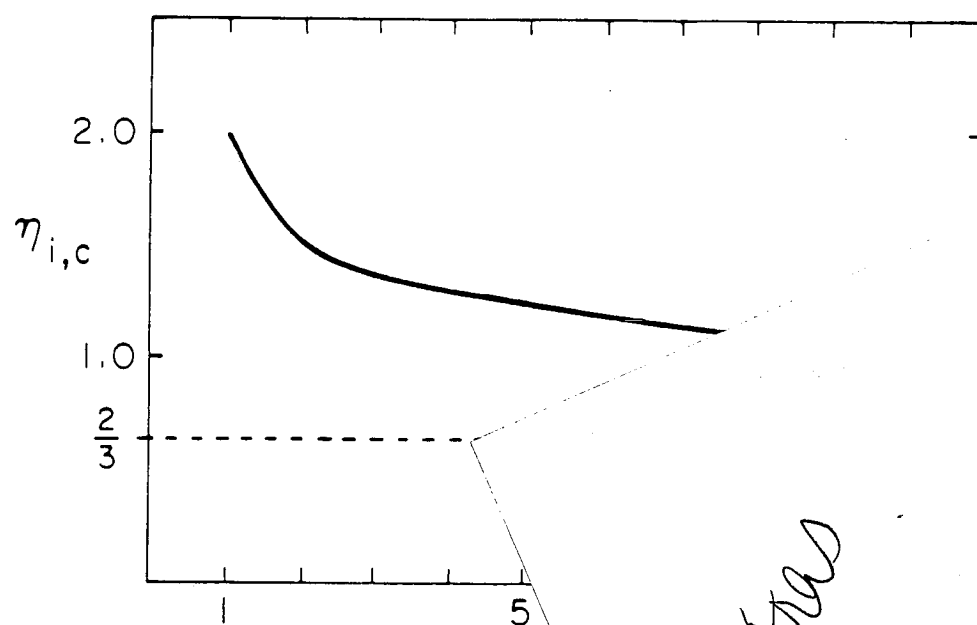
Sincerely,

Wendell Horton

Wendell Horton

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Fig. 1

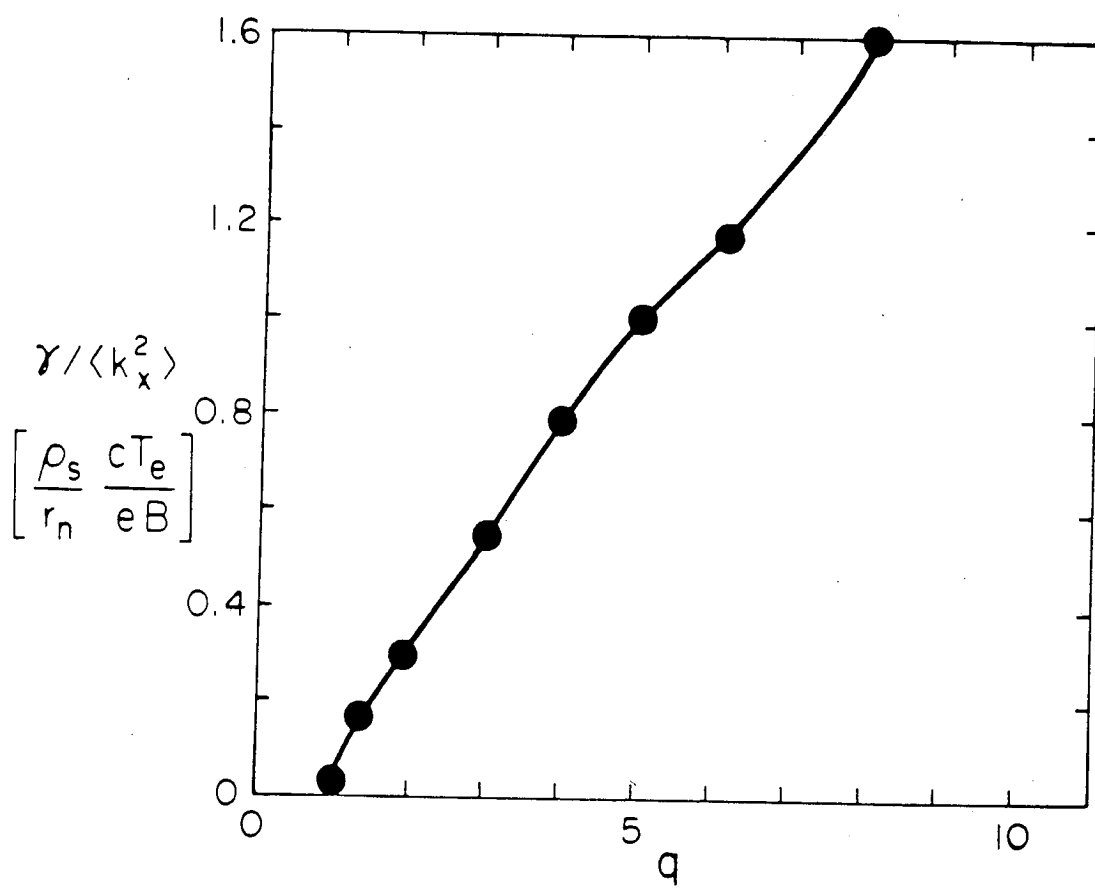


Fig. 2

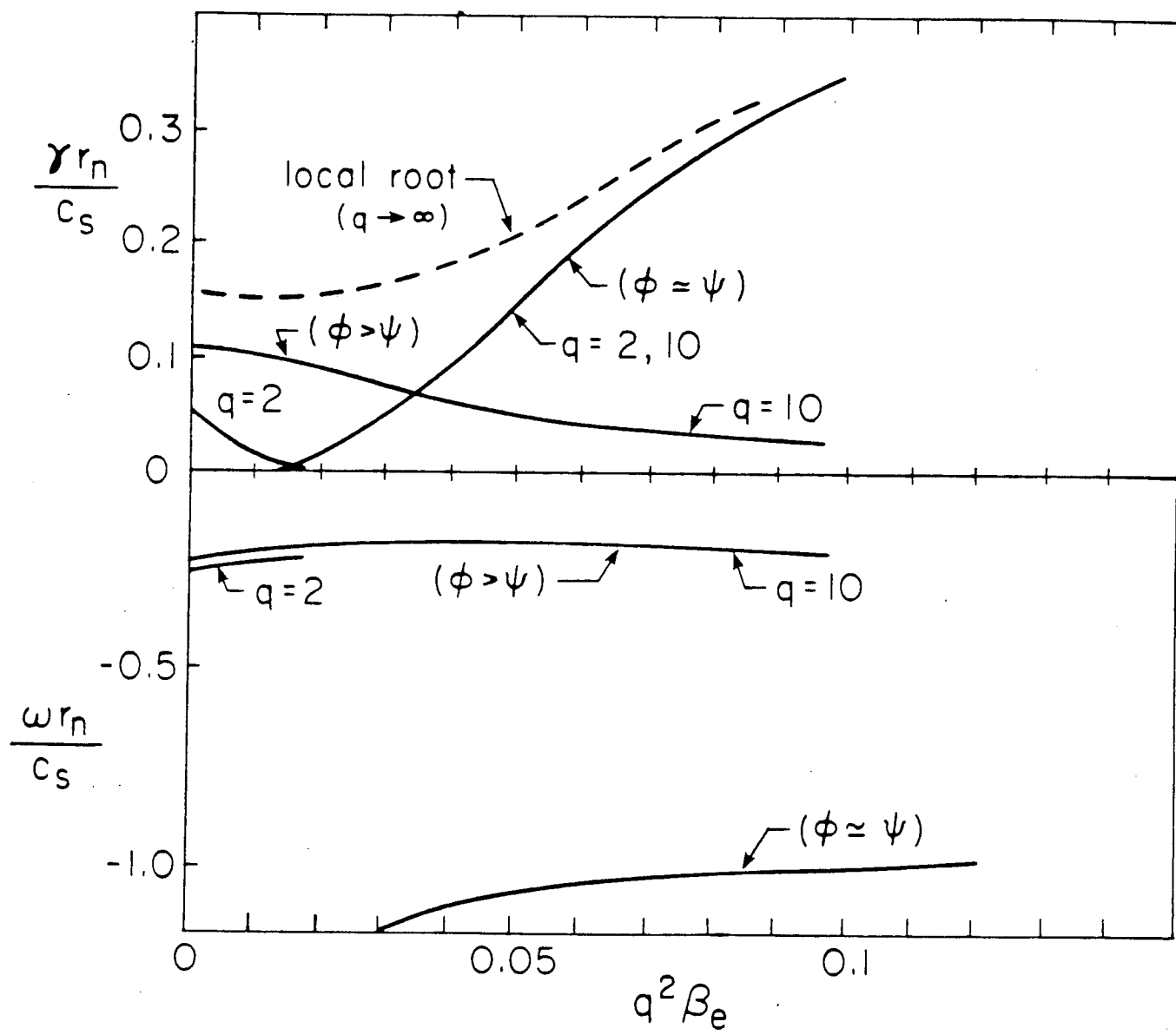


Fig. 3

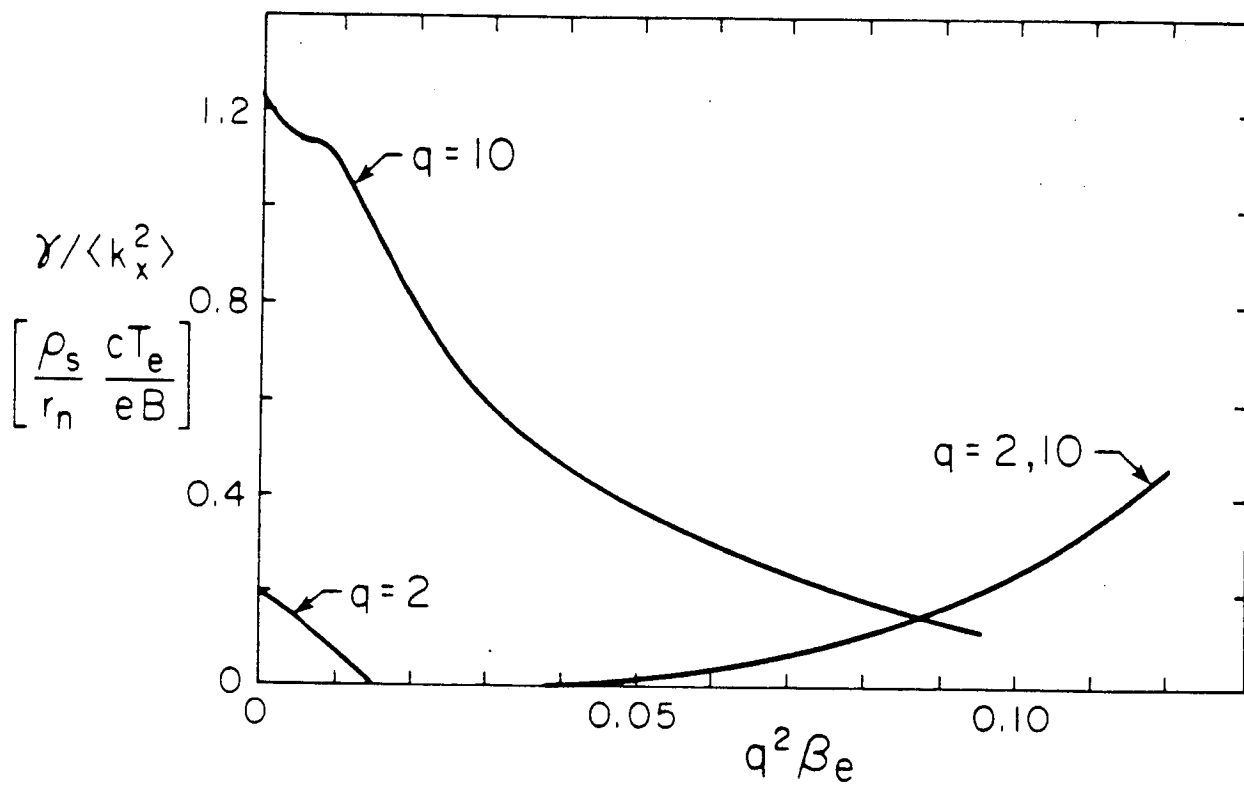


Fig. 4