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Abstract

A model that incorporates both the effects of temperature gradients and magnetic shear on the drift wave monopole solutions is analyzed. In the case where the former effect is treated improperly and the latter is neglected, it was shown in Ref. 1 that there exist exact monopole solutions, which can further be shown [Ref. 4] to be equivalent to the existence of a point spectrum for a nonlinear eigenvalue problem. When both the effects are included, this spectrum becomes a banded continuous spectrum. An eigenvalue of this spectrum is associated with a localized vortex structure that undulates in space about a fixed level, eventually matching to a radiative ion acoustic tail. A novel separatrix crossing technique is used to investigate this problem.

In 1977 Petviashvili analyzed[1] nonlinear drift waves with the inclusion of a temperature gradient. He was able to show that the model equation he used admits localized monopole solutions, which come in two types: cyclones with negative vorticity and anticyclones with positive vorticity. Recently, authors[2,3] have come to realize that the model equation used by Petviashili is incomplete; for a consistent ordering an explicit x-dependent term arising from the nonconstancy of the temperature must be included. The inclusion of this term precludes the existence of the exact monopole solutions. If one adds the local effect of magnetic shear, then an additional explicit x-dependence is obtained, and thus further blocking the possibility of exact monopole solutions. The purpose of this paper is to present a technique for analyzing equations that have, in some sense, near monopole solutions.

In a previous work[4] we analyzed Petviashvili's original model, and described the existence of monopole solutions in terms of a nonlinear eigenvalue problem, where the amplitude of the monopole at $r = 0$, r being the cylindrical coordinate, corresponds to the eigenvalue. The spectrum in the case of Petviashvili's model is composed of a single point, assuming $k^2 = 1 - v_d/u$ is fixed. Here u is the monopole speed and v_d is either the drift wave speed or the coriolis parameter β for rotating neutral fluids. In this work, we see that for a model problem the presence of the explicit x-dependence changes the one-point nonlinear eigenvalue spectrum for the monopole into a continuous "banded" spectrum. An element of the banded spectrum corresponds to a localized vortex structure with a nonmonotonic profile. In fact these bands occur upon imposing radiative tail boundary conditions. Qualitatively one can understand the banded spectrum effect as arising from the variation of the Sagdeev potential for a "soliton"; i.e. pure monopole [see Fig. 1(a)], to that of an ion-acoustic wave [see Fig. 1(b)]. This variation arises because of the explicit x-dependence.

The presence of magnetic shear, forces a coupling of the drift wave, where $\omega = k_y v_d/(1 + k^2)$ to ion acoustic waves where $\omega^2 = k_\parallel^2 c_s^2$. This happens because the magnetic field twists over the scale length L_s. Including both the sheared field and temperature gradient effects, the perpendicular and parallel momentum equations are

$$\left(\frac{1}{\tau(x)} - \nabla^2\right) \frac{\partial \varphi}{\partial t} \;+\; v_d \frac{\partial \varphi}{\partial y} + \frac{\tau'(x)}{\tau^2(x)} \varphi \frac{\partial \varphi}{\partial y} - [\varphi, \nabla^2 \varphi] + Sx \frac{\partial v_\parallel^i}{\partial y} = 0 \tag{1}$$

$$\frac{\partial v_\parallel^i}{\partial t} \;+\; [\varphi, v_\parallel^i] + Sx \frac{\partial \varphi}{\partial y} = 0 \tag{2}$$

where the normalized electron temperature $\tau(x) \equiv T_e(x)/T_0$, the spatial coordinates are scaled by $\rho_{i0} \equiv (T_0/m_i \omega_{ci}^2)^{1/2}$ with $\omega_{ci} \equiv eB_0/cm_i$, time is scaled by ω_{ci} and as usual φ is made dimensionless by a factor e/T_0. The parallel velocity is scaled by $\rho_{i0}\,\omega_{ci} = c_{s0} = \sqrt{T_0/m_i}$, and the drift velocity is also scaled by c_{s0}. The quantity $S \equiv \rho_{i0}/L_s$ is a measure of the magnetic shear.

For small parallel speed, we take the ion acoustic dynamics as linear. Equation (2) then becomes

$$\frac{\partial v_\parallel^i}{\partial t} = -Sx \frac{\partial \varphi}{\partial y} \,. \tag{3}$$

Combining Eqs. (1) and (3), we get the following model equation:

$$\left(\frac{1}{\tau} - \nabla^2\right) \frac{\partial \varphi}{\partial t} + v_d \frac{\partial \varphi}{\partial y} + \frac{\tau'}{\tau^2} \varphi \frac{\partial \varphi}{\partial y} - [\varphi, \nabla^2 \varphi] - S^2 x^2 \int_0^t \frac{\partial^2 \varphi}{\partial y^2}\, dt = 0 \,. \tag{4}$$

In the limit of $S \to 0$ and in the inconsistent limit $\dfrac{1}{\tau} \to 1$ while $\tau'/\tau^2 \to$ constant we obtain the incomplete Petviashvili model. In this "limit" the monopole solution is obtained upon substituting $\varphi = \varphi_0(r)$ where $r = [x^2 + (y - ut)^2]^{1/2}$ into (4). One obtains the following equation:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\varphi_0}{dr}\right) - 4k^2 \varphi_0 + \frac{\alpha}{2u} \varphi_0^2 = 0 \tag{5}$$

where $k^2 = \dfrac{1}{4}\left(1 - \dfrac{v_d}{u}\right)$ and $\alpha = \tau'/\tau^2$. Note that the presence of shear and temperature gradient removes the radial symmetry of this equation. Assuming $\varphi(x, y - ut)$ Eq. (4) reduces exactly to

$$\nabla^2 \varphi - \frac{1}{\tau(x)} \varphi - \ell n\, n_0(x) + \frac{S^2 x^3}{3u} = F(\varphi - ux) \tag{6}$$

where, usually $\ell n\, n_0(x) \simeq -v_d\, x$, $v_d = -n_0'/n_0$ is a constant and F is an arbitrary function. Choosing $F = -\frac{v_d}{u}(\varphi - ux) - \frac{S^2}{3u^4}(\varphi - ux)^3$, $\tau(x) = (1-\alpha x)^{-1}$, and defining $X = x + \frac{\alpha u^2}{2S^2}$, Eq. (6) becomes

$$\nabla^2\varphi \;+\; \varphi\left(\frac{v_d}{u} - 1 - \frac{\alpha^2 u^2}{2S^2}\right) + \frac{S^2 X^2}{u^2}\varphi + \frac{\alpha}{2u}\varphi^2$$

$$+ \frac{S^2}{3u^4}\varphi^3 - \frac{S^2}{u^3} X \varphi^2 = 0 \; . \tag{7}$$

Equation (7) is a difficult equation to solve, and so for the purposes of this paper we modify it in two ways. For tractibility of mathematics we replace ∇^2 by d^2/dx^2. This semi-tractable simplified mathematical model is of course not rigorously correct; in particular interesting asymmetrical behavior has been eliminated. The model does in some sense capture the behavior in the x-direction and it exhibits the main physical feature of coupling between the integrable solitary wave potential and the ion acoustic wave potential at large x. We thus obtain a one-dimensional nonautonomous problem. Our technique can handle general problems of this kind. For simplicity our second modification is to drop the last two terms of Eq. (7). Elsewhere we will present results including these terms. The qualitative behavior of banded spectra obtained here exists also in the "correct" model.

Taking into account the above assumptions and introducing the new variables $t = \widehat{k}\left(x + \frac{\alpha u^2}{2s^2}\right)$, where $\widehat{k}^2 = \frac{1}{4}\left(1 - \frac{v_d}{u} + \frac{\alpha^2 u^2}{S^2}\right)$, $\varphi = \varphi_m \psi(t)$ and $s = \frac{S}{\widehat{k}u}$, Eq. (7) becomes

$$\frac{d^2\psi}{dt^2} - 4\psi + 6\gamma\psi^2 + s^2 t^2 \psi = 0 \;, \tag{8}$$

where we have set

$$\varphi_m = \frac{12\gamma u \widehat{k}^2}{\alpha}$$

and γ is a constant that remains to be determined. The boundary condition $\psi(t \to \infty) \to 0$ and the initial conditions

$$\psi(t=0) = 1 \qquad \text{and} \qquad \frac{d\psi(t=0)}{dt} = 0 \;, \tag{9}$$

together with Eq. (8), define a nonlinear eigenvalue problem for γ. In the limit $s \to 0$ the solution of the model Eq. (7) is $\gamma(s=0) \equiv \gamma_0 = 1$ with the homoclinic orbit $\varphi_0 = \mathrm{sech}^2 t$ where v_d, u and α/S are restricted to $\widehat{k}^2 > 0$.

Equation (8) can be written in the form of Hamilton's equation for an effective particle with coordinate $q = \psi$, time t, momentum $p = \dfrac{d\psi}{dt}$ and effective potential $V(\psi, t) = -2\psi^2 + 2\gamma\psi^3 + \dfrac{s^2t^2}{2}\psi^2$. The Hamiltonian is

$$H(\psi, p, t) = \frac{1}{2}p^2 + 2\gamma\psi^3 + 2\psi^2\left(\frac{s^2t^2}{4} - 1\right) , \qquad (10)$$

and the dynamical equations are

$$\begin{aligned} \dot{p} &= -\frac{\partial}{\partial\psi}\left(-2\psi^2 + 2\gamma\psi^3 + \frac{s^2t^2}{2}\psi^2\right) = 4\psi - 6\gamma\psi^2 - s^2\,t^2\phi \\ \dot{\psi} &= p\,. \end{aligned} \qquad (11)$$

As noted above, the effect of shear is to couple the vortex solution to the ion wave by changing the potential energy $V(\psi, t)$ with time. The critical time for particles passing from the solitary wave potential into ion acoustic wave potential is $t_0 = \dfrac{2}{s}$. At the critical time, the potential can no longer contain trapped particles (see dashed line in Fig. 1(b)). For $t > t_0$ trapped particle orbits exist in the nieghborhood of the origin; i.e. in the ion acoustic potential.

From numerical integration of the model Eqs. (8) and (9), we obtain the spectrum of eigenvalues $\gamma_n(s)$. This is done by choosing a zero momentum initial condition and integrating beyond t_c to determine if there is trapping for all time in the ion acoustic potential. We are interested in the critical value of the amplitude for such trapping. Physically the trapping implies the radiative tailing at large t-values. See. Fig. 2. A detailed study of the numerical spectrum yielded the $\gamma_n(s)$ curve shown in Fig. 3. The curve shows the amplitude at $t = 0$ (γ) for eigenfunctions $\psi(t)$ that are bounded as $t \to \infty$, versus the shear parameters, s. Observe that the upper and lower boundaries of the "bounded region" oscillate as $s \to 0$

$s \to 0$ (c.f. Fig. 1(b)) with an eventual limit at $\gamma = 2/3$. Each of thse cycles corresponds to a class of eigenfunctions that oscillate a *given* number of times about the minimum of the drift wave part of the potential before becoming trapped in the ion acoustic feature. As one approaches smaller s the effective potential bounces more times before the transition in mode at t_c. Empirically, we have discovered the following relations:

$$\begin{aligned} \gamma_n(s) &= \gamma_{n+1}(s) + \frac{1}{n^2} \quad \text{for even } n \text{ and } \quad \gamma > \frac{2}{3} \\ \gamma_n(s) &= \gamma_{n+1}(s) - \frac{1}{n^2} \quad \text{for odd } n \text{ and } \quad \gamma < \frac{2}{3} \\ \gamma_{n\to\infty} &= \frac{2}{3} \end{aligned} \tag{12}$$

where n is the number of effective particle oscillations that occurs for $0 < t < t_0 = 2/n$. The numerical results show the relation (12) works very well when $n > 4$.

We give a qualitative explanation of Fig. 3 and Eqs. (12) by appealing to the analogy of a dropped ball in the gravitational potential

$$V(\psi, t) = -2\psi^2 + 2\psi^3 + \frac{s^2t^2}{2}\psi^2 \tag{13}$$

as shown in Fig. 1 (setting $\gamma = 1$). From Eq. (13) we find that the depth of the potential at the local minimum is

$$V_m = -\frac{8}{27}\left(1 - \frac{s^2t^2}{4}\right)^3 \quad \text{at} \quad \psi_m = \frac{2}{3}\left(1 - \frac{s^2t^2}{4}\right) .$$

We consider the cases without ($s = 0$) and with ($s \neq 0$) shear.

Without shear, the potential depth $V_m = -\frac{8}{27}$, at $\psi_m = \frac{2}{3}$ are constant in time. If we drop the ball within $0 < \psi < 1$ at $t = 0$, it would oscillate around the bottom forever (with $\psi(t = 0) = \frac{2}{3}$, the ball would stay motionless at the bottom); if $\psi(t = 0) = 1$, the ball takes infinite time to get to the $\psi = 0$ point, which is the homoclinic orbit for the solitary wave solution studied in Ref. 1. If $\psi(t = 0) < 0$ or $\psi(t = 0) > 1$, the ball goes over the potential hill at $\psi = 0$ off to negative infinite as $t \to \infty$. Here, the expansion of the

Boltzmann distribution $n_e = N(x)\exp(e\Phi/T_e) = N(1+\varphi)$ used in deriving Eq. (4) breaks down. The potential has depleted the electron density to the unphysical point where n_e becomes negative.

With $s \neq 0$, however, the potential V, depth V_m of the potential well, and position ψ_m of its bottom change with time. The potential well becomes shallower and shallower with increasing t until $V_m = 0$ and $\psi_m = 0$ when $t = t_0(s) = \frac{2}{s}$ as shown by dashed line in Fig. 1(b). When $t > \frac{2}{s}$, the potential well changes its shape into that shown by the solid line in Fig. 1(b). Therefore, the question of whether the ball becomes eventually trapped in the well shown by the solid line in Fig. 1(b), or it goes into the "Hell" as $t \to \infty$, is determined not only by the initial potential energy (the initial position ψ of the ball) but also by the magnitude of s. The shear parameter s serves as an inverse characteristic time for the change of the well. It is also obvious that the number of oscillations that the ball performs around the bottom of the potential well within $t < \frac{2}{s}$ is determined by value $\frac{2}{s}$. The first formula (for $\gamma > \frac{2}{3}$) of Eq. (12) or the upper branch of Fig. 3 corresponds to dropping the ball from the right side of the bottom of the potential well at $t = 0$, while the second one (for $\gamma < \frac{2}{3}$) of Eq. (12), or the lower branch of Fig. 3, from the left side of the bottom.

Although Eqs. (12) have not been proven, we understand that coupling of the waves at large t to the vortex at $t = 0$ results in a spectrum of vortices with an increasing number of oscillations of ψ in the nonlinear trapping region. Thus, the inhomogeneity acts to split up the vortex point spectrum into continuous bands given by $\gamma_n(s)$. The shear inhomogeneity is a defocusing effect.

By introducing the action-angle variables

$$J = \frac{1}{2\pi}\oint pdq = \frac{1}{2\pi}\oint \sqrt{2(H-V)}\,d\varphi$$

and separatrix crossing theory we can relate the action at $t = 0$ for the homoclinic orbit

$$J(t=0,\gamma) = \frac{1}{2\pi}\oint \sqrt{2H(t=0) - 4(\gamma\psi^3 - \psi^2)}\,d\psi \tag{14}$$

to the orbits for $t = t_c > t_0$

$$J_s(t_c, \gamma) = \frac{1}{2\pi} \oint \sqrt{2H - 4(\gamma\psi^3 - \left(1 - \frac{s^2 t_c^2}{4}\right) \psi^2 \, d\psi} \ . \tag{15}$$

The particles will be trapped if their velocities are positive (along positive ψ direction) when

$$J(t=0, \gamma) = J_s(t_c, \gamma) \ . \tag{16}$$

From Eq. (16), we can find $t_c = t_c(s, \gamma)$. On the other hand we can approximately calculate the period T of oscillation of particles by

$$T(\gamma) = \oint dt = \oint \frac{d\psi}{p} = \oint \frac{d\psi}{[2[H(t=0) - V(t=0)]]^{1/2}} \ . \tag{17}$$

Equation (16) will be true if the particles have completed more than a half cycle. We therefore can write

$$t_c(s, \gamma) \simeq \frac{nT(\gamma)}{2} \tag{18}$$

where $n = 1, 2, 3, \cdots$ is the number of oscillations that the particles makes in the monopole well (Fig. 1) before passing $t = \dfrac{2}{s}$ into the ion acoustic spectrum. We then can get the critical $\gamma(s)$ from Eq. (18).

In Fig. 4 we compare the analytic and numerical results for $\gamma(s)$. We see from the figure that they agree quantitatively. The agreement is about as good as one would expect, considering the crudeness of the theory. Keeping higher order adiabatic invariants and folding in the result of the phase shift at the crossings would improve the comparison.

The effects of magnetic shear and temperature gradients on a model of the monopole drift wave vortex has been investigated by analytical and numerical methods. The results show that even small shear causes non-perturbative changes from the solitary to the radiative solution, due to the sensitivity of the homoclinic orbit to the defocusing inhomogeneity. The perturbation leads to the banded spectrum $\gamma_n(s)$ with radiative tails coupling to the exterior ion acoustic waves. The numerical results show that the number n of oscillation of φ within

$t < \frac{2}{s}$ satisfies $n = \frac{1}{s}$ and when s is small ($s < .25$), the eigenvalue $\gamma_n(s)$ satisfy the recursion relations of Eq. (12).

Even though the present formulation is an oversimplified model, it exhibits the main physical features of coupling the integrable solitary wave to the ion acoustic wave due to the presence of the magnetic shear. The study of a more comprehensive model which involves the two fields, φ and v_z, will be presented elsewhere.

Acknowledgments

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Figure Captions

1. Effective potential for nonlinear drift waves.

   (a) Behavior of effective potential near the center of the drift wave structure: $t = 0$.

   (b) Behavior of effective potential far from the center of the drift wave structure: $t > 2/s$.

2. (a) Nonlinear eigenfunction for $s = \frac{1}{8}$ showing monopole vortex and wave solutions for $\gamma > \frac{2}{3}$ (upper bound).

   (b) Nonlinear eigenfunction for $s = \frac{1}{7}$ showing monopole vortex and wave solutions for $\gamma < \frac{2}{3}$ (lower bound).

3. Spectrum of critical amplitude γ versus shear s showing vortex branches $\gamma_n(s)$ and nonlinear wave solutions.

4. Comparison of $\gamma(s)$ from separatrix crossing theory with that from numerical integration rsults of Eq. (7). The solid lines represent the analytical results and the dashed lines, numerical results.

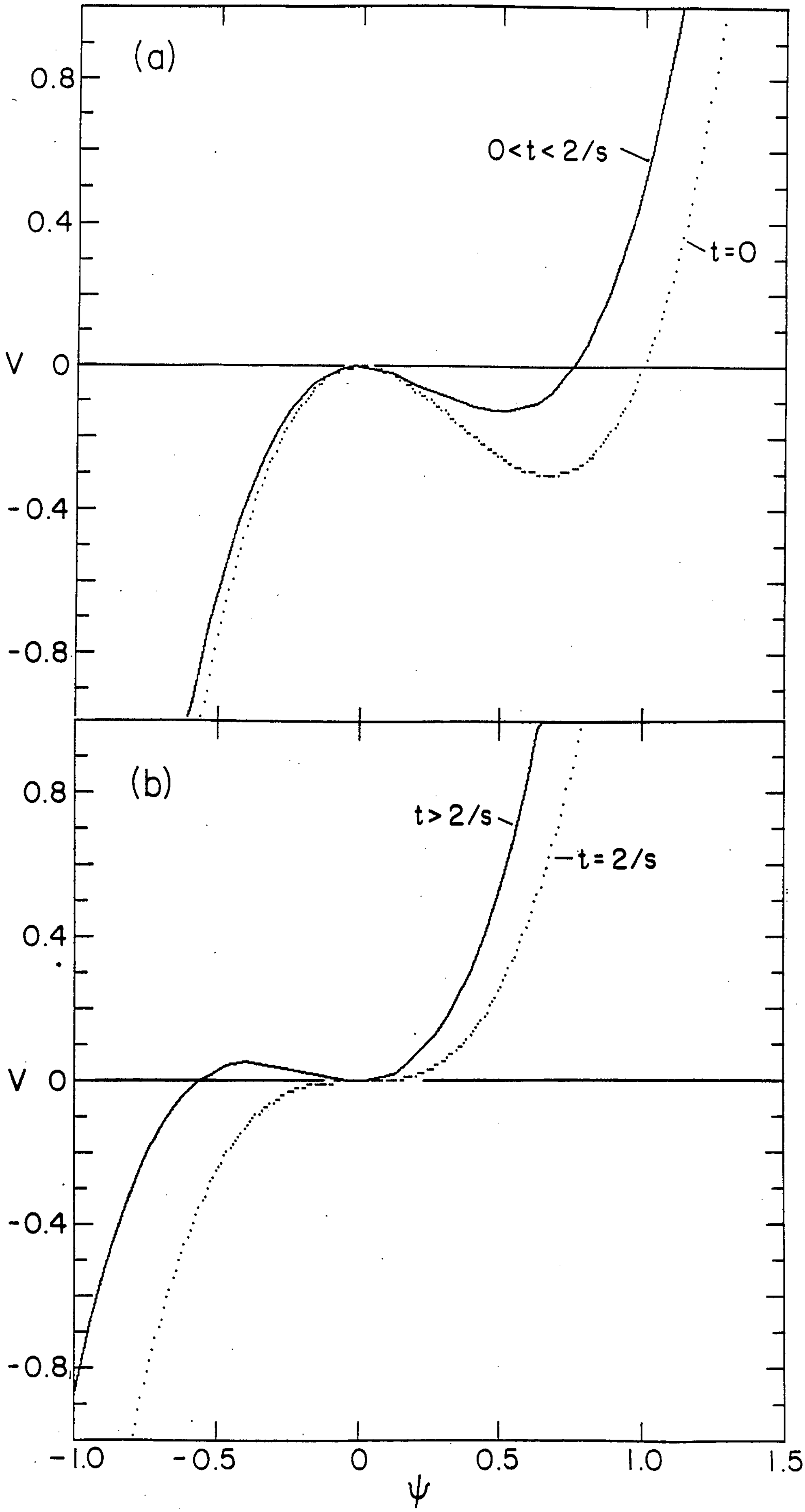

Fig. 1

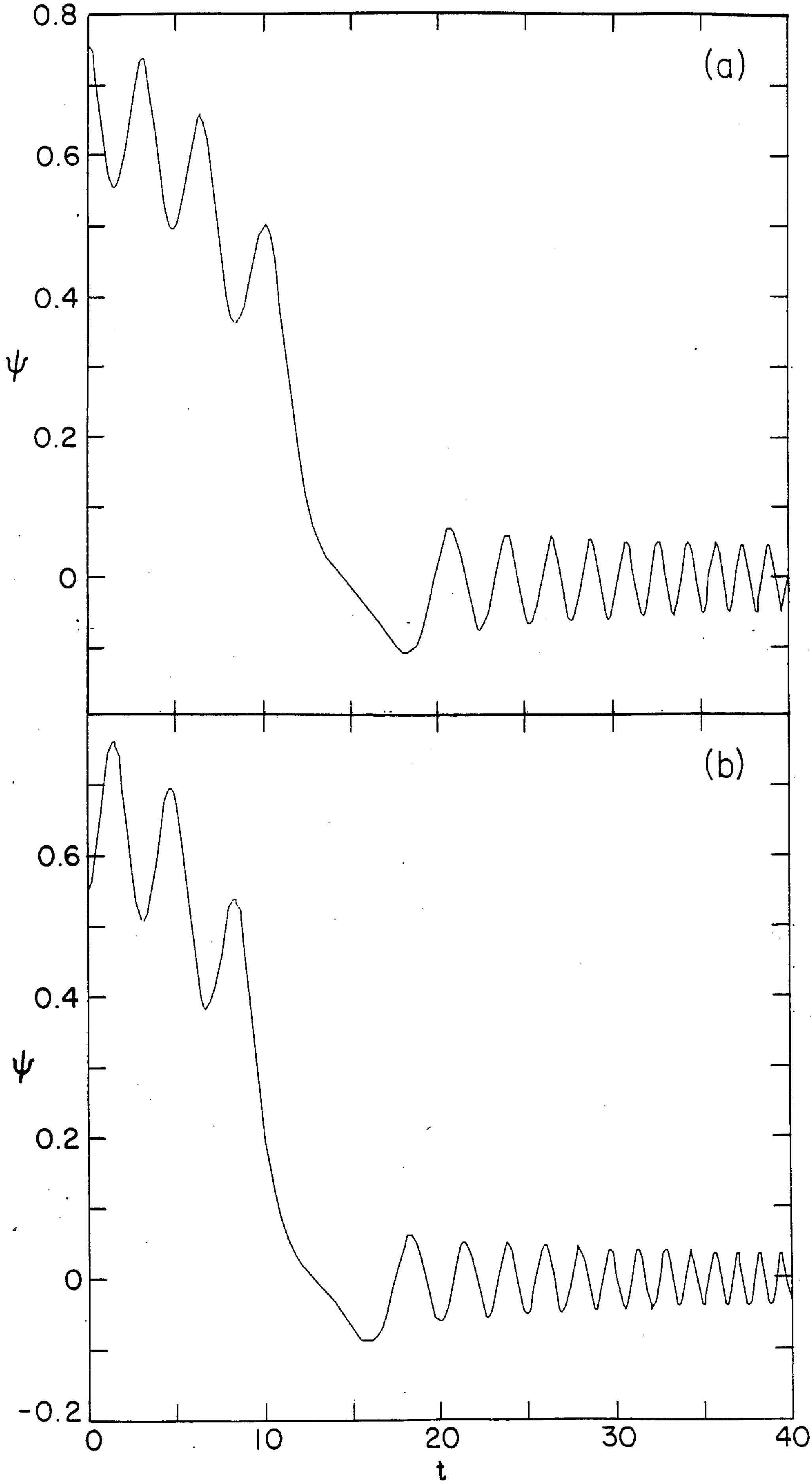

Fig. 2

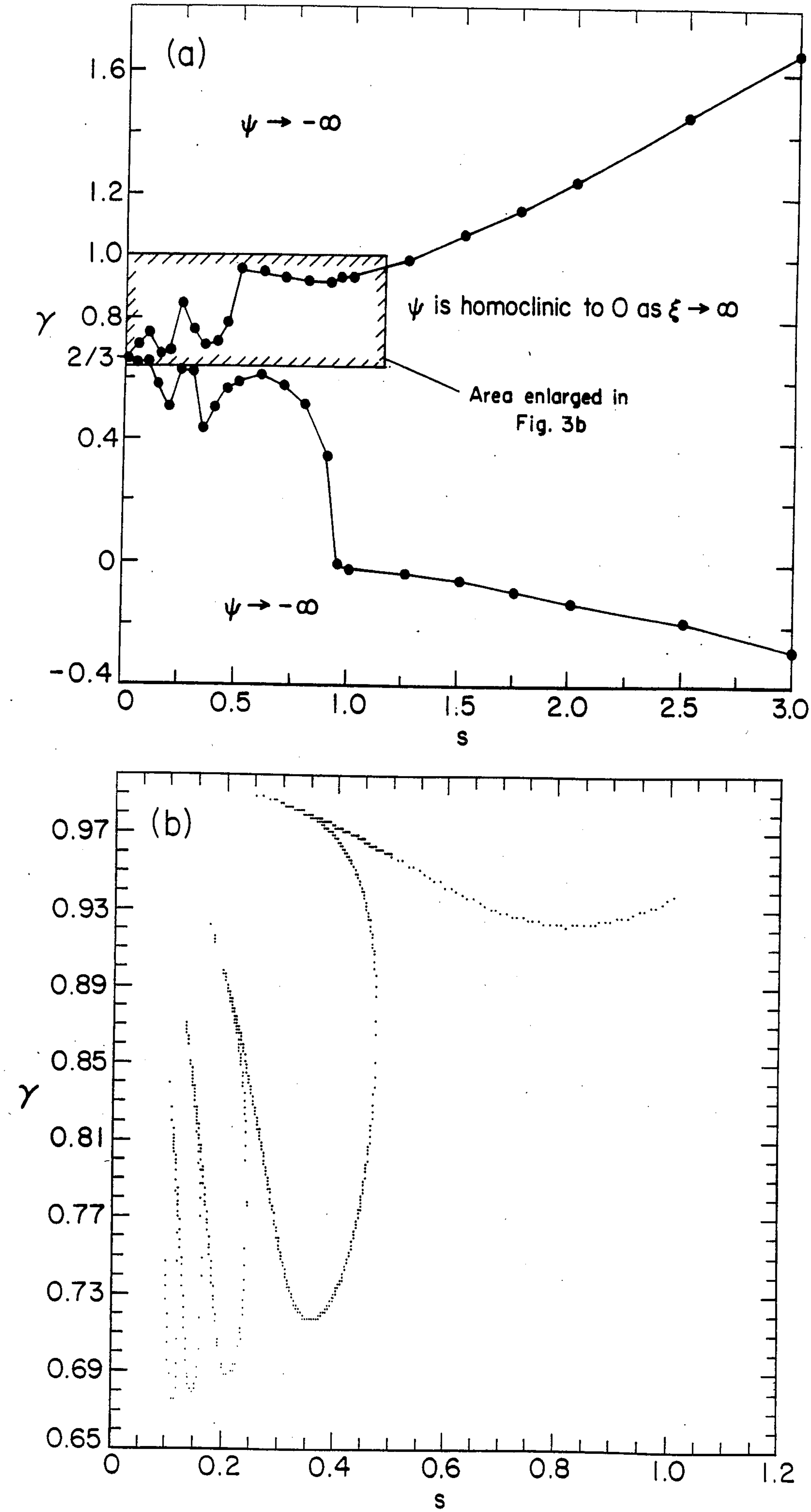

Fig. 3

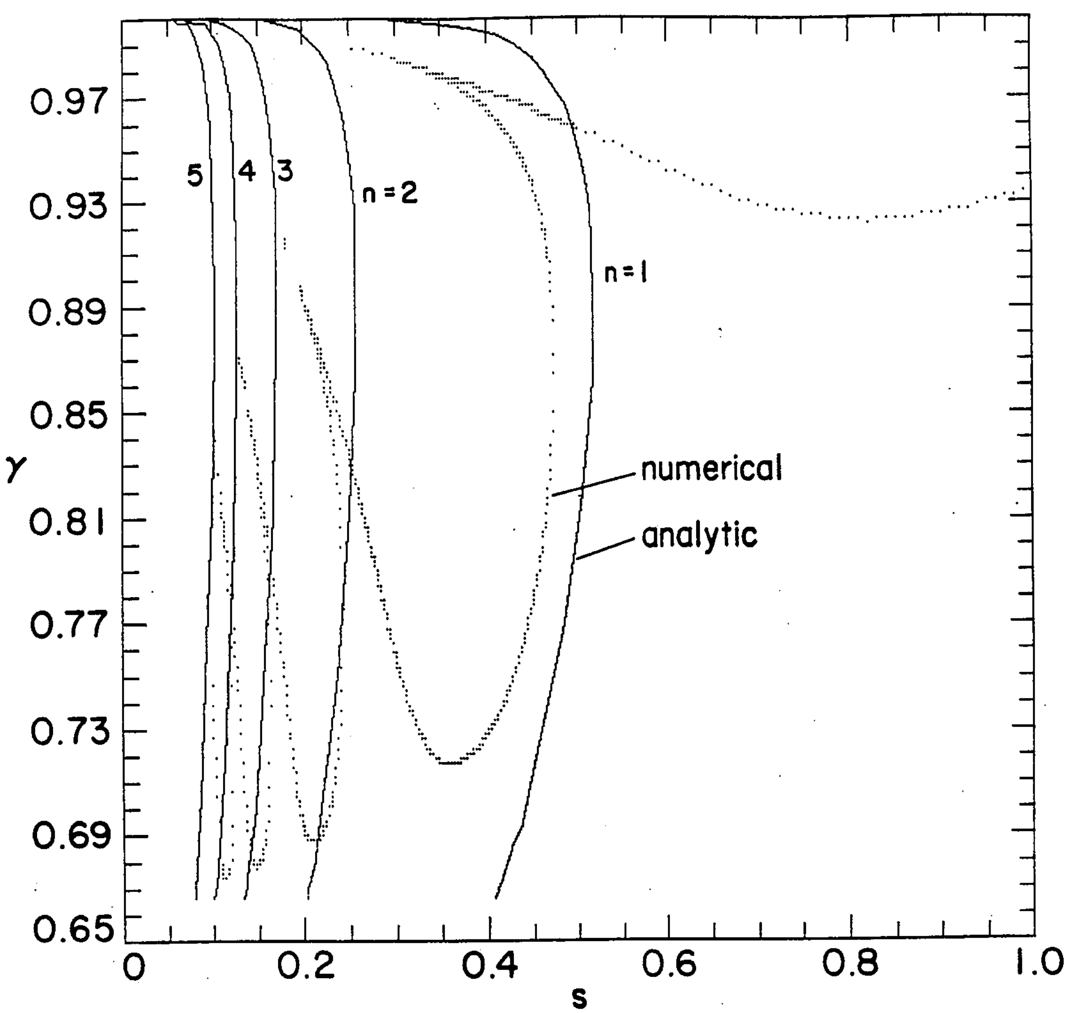
0.97
0.93
0.89
0.85
0.81
0.77
0.73
0.69
0.65
γ
0
0.2
0.4
0.6
0.8
1.0
s
5
4
3
n = 2
n = 1
numerical
analytic

Fig. 4