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Relation between Beam Driven Seed-Current and Rotation in Steady State FRC

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Abstract

We consider an FRC configuration whose current is maintained by a steady state beam. Without quadrupole fields, back current can be inhibited by the Ohkawa effect if $Z_b < Z_{\text{eff}}$, where Z_b and Z_{eff} are the beam charge number and effective charge number of background ions. However, the resulting rotation of the plasma often leads to instability. For systems with a large bootstrap effect, the rotation can be moderate, but it is then difficult to contain fusion products. An additional problem is that the Ohkawa effect due to alpha particles tends to disassemble the equilibrium. It has previously been shown that the presence of a quadrupole field inhibits back current. Here we show that a steady state flux can be maintained with moderate input power in both reactors and present day experiments with the resulting rotation slow enough to fulfill stability conditions. However, experimental means must be devised to supply a continual source of particles and additional energy.

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1. Introduction

Steady state operation is one of the highly desirable properties for a fusion reactor. For a field steady state reversed configuration (FRC), it is essential to maintain the toroidal current which creates the reversal. A steady state beam-driven FRC has been considered by Baldwin and Rensink [1], Pearlstein et al. [2], Hammer and Berk [3], Hirano [4], and Okamoto [5]. Neutral beam injection is relatively straightforward if back current against beam current can be avoided. Ohkawa current [6] has been suggested as a means to maintain the toroidal current [1,2,4,5]. However, in axisymmetric systems, Ohkawa current created by unidirectional beam injection can induce rotation of the plasma which may lead to rotational instability. In an ignited plasma, it is difficult to obtain large $\bar{s}_f$ (the ratio of plasma radius to the Larmor radius of fusion products) to retain fusion products. In such a case fusion products create a large amount of beam current. This problem has been considered in detail by Berk, Momota and Tajima [7]. However, in a $D-T$ ignited plasma a difficult problem arises in that the Ohkawa current generated by energetic alpha particles flows in the opposite direction of the plasma current (if $Z_\alpha > Z_{\text{eff}}$, where $Z_\alpha = 2$ and Z_{eff} is the effective charge number of the background ions). Such a situation tends to disassemble the equilibrium.

On the other hand, several solutions may be offered in the configuration examined by Hammer and Berk [3] where the FRC is embedded in a multipole magnetic field. Their theory predicts that in a neutral beam driven system multipole fields induce annihilation of back current due to magnetic pumping mechanism [8], and field lines do not open appreciably. In their theory, a unidirectional beam current is a seed, but the rotation rate remains moderate in the favorable curvature of the multipole field, thereby avoiding the rotational instability. There is no tendency to disassemble the equilibrium of a burning plasma, since the beam generated by fusion products has negligible back current.

The most dangerous $n = 2$ rotational instability of FRC plasma has been experimentally controlled by the application of quadrupole fields [9,10] or helical fields [11]. The theoretical basis for multipole field stabilization of the rotational instability has also been analyzed by Ishimura [12].

Recently, Okamoto [5] considered the relationship between a seed current (Ohkawa current) necessary to sustain an FRC and the transport. To avoid plasma rotation, Okamoto analyzed a balanced beam configuration first suggested by Ohkawa [6] and the steady state configuration that can be achieved with a given seed current and particle input rate was determined. It was found that the amount of seed current depends strongly on τ_B/τ_N, where τ_B and τ_N are the magnetic diffusion time and the particle confinement time, respectively, and the importance of bootstrap current effect was emphasized.

In this paper, we consider how much rotation is induced in a beam driven steady state FRC where an input beam gives rise to a seed current (or equivalently an effective voltage) due to either the Ohkawa effect, or due to magnetic pumping of an FRC imbedded in a quadrupole field. We show that in a standard FRC, significant rotation rates develop that, in large system, violate rotational stability criteria. In contrast, with quadrupole fields, the rotation rates are moderate; low enough to avoid rotational instability and large enough to produce a skin effect that prevents open field lines ~~to penetrate~~ from penetrating into the plasma.

In Sec. 2, basic equations for an FRC without a multipole field are derived in the limit of infinitely axially elongated FRC. In Sec. 3, the magnitude of plasma rotation is evaluated based on the equation for the rotation derived in Sec. 2. Plasma rotations in an FRC with multipole fields are considered in Sec. 4 and the results are compared with that in Sec. 3. Section 5 is devoted to conclusions and discussions.

2. Basic Equations for an Axisymmetric FRC Plasma

In this section, we consider a plasma in an axisymmetric FRC without a multipole field. The equations of motion for electrons and plasma ions are given by

$$n_j m_j \frac{\partial \mathbf{v}_j}{\partial t} + n_j m_j (\mathbf{v}_j \cdot \nabla)\mathbf{v}_j = -\nabla p_j + eZ_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) + \mathbf{R}_j + \boldsymbol{\xi}_j, \tag{2.1}$$

with

$$\boldsymbol{\xi}_j = \mathbf{M}_j - \nabla \cdot \overleftrightarrow{\pi}_j, \tag{2.2}$$

where j stands for electrons or plasma ions, $n_j, m_j, \mathbf{v}_j$, and p_j are density, mass, velocity and pressure, respectively. $\mathbf{E}$ and $\mathbf{B}$ are electric field and magnetic field and Z_j is the charge number of jth species ($Z_e = -1,\ e > 0$). $\mathbf{R}_e(\mathbf{R}_i)$ is the rate of electron (ion) momentum density due to collisions with ions (electrons) and $\mathbf{R}_e + \mathbf{R}_i = 0$. M_j is the input of momentum density to the jth species (electrons or plasma ions) from the externally driven beam and $\overleftrightarrow{\pi}_j$ is the stress tensor of jth species.

We assume that the FRC is infinitely elongated in the axial direction (z-direction) to simplify the analysis. The set of basic equations will be reduced to a one-dimensional problem in the radial direction (r-direction). All the variables are independent of axial z-direction and azimuthal θ-direction. In order to solve Eq. (2.1) with (2.2), we make an assumption that collision frequencies are small parameters. If $\nu \sim O(\epsilon)$, where ν is a collision frequency and ϵ is a small parameter, we can order so that $v_{j\theta}^2 \sim O(1),\ v_{jr} \sim O(\epsilon),\ R_{j\theta} \sim O(\epsilon),\ M_{j\theta} \sim O(\epsilon), R_{jr} \sim O(\epsilon^2),\ M_{jr} \sim O(\epsilon^2)$ and $\partial/\partial t \sim O(\epsilon)$. The lowest order of Eq. (2.1) gives

$$-n_j m_j \frac{v_{j\theta}^2}{r} = -\frac{\partial p_j}{\partial r} + eZ_j n_j (E_r + v_{j\theta} B_z), \tag{2.3}$$

and the first order equation becomes

$$n_j m_j \frac{\partial v_{j\theta}}{\partial t} + n_j m_j v_{jr} \frac{1}{r}\frac{\partial}{\partial r}(r v_{j\theta}) = eZ_j n_j (E_\theta - v_{jr} B_z) + R_{j\theta} + \xi_{j\theta}. \tag{2.4}$$

Electrons are treated as massless particles ($m_e \to 0$), but the inertia of ions are taken into account. From Eqs. (2.3) and (2.4), we obtain

$$v_{e\theta} = -\frac{E_r}{B_z} - \frac{1}{en_eB_z}\frac{\partial p_e}{\partial r}, \tag{2.5}$$

$$v_{i\theta} = -\frac{r}{2}\Omega_i\left\{1-\sqrt{1+\frac{4}{r}\frac{1}{\Omega_i}\left(-\frac{E_r}{B_z}+\frac{1}{eZ_in_iB_z}\frac{\partial p_i}{\partial r}\right)}\right\}, \tag{2.6}$$

$$v_{er} = \frac{E_\theta}{B_z} - \frac{R_{e\theta}+M_{e\theta}-\left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta}{en_eB_z}, \tag{2.7}$$

$$v_{ir} = \left(\frac{E_\theta}{B_z}+\frac{R_{i\theta}+M_{i\theta}-\left(\nabla\cdot \overleftrightarrow{\pi}_i\right)_\theta}{eZ_in_iB_z}-\frac{1}{\Omega_i}\frac{\partial v_{i\theta}}{\partial t}\right)\frac{1}{1+(1/\Omega_i)(1/r)(\partial/\partial r)(rv_{i\theta})}, \tag{2.8}$$

where $\Omega_i = eZ_iB_z/m_i$ is the cyclotron frequency of plasma ions. The changes in momentum density are given by

$$\left.\begin{aligned}
R_{e\theta} &= R_{ei} &&= -\frac{1}{\tau_{ei}}n_em_e(v_{e\theta}-v_{i\theta}),\\
R_{i\theta} &= R_{ie} &&= -R_{e\theta},\\
M_{e\theta} &= M_{eb} &&= -M_{be} = -\frac{1}{\tau_{eb}}n_em_e(v_{e\theta}-v_{b\theta}),\\
M_{i\theta} &= M_{ib} &&= -M_{bi} = -\frac{1}{\tau_{ib}}n_im_i(v_{i\theta}-v_{b\theta}),
\end{aligned}\right\} \tag{2.9}$$

where τ_{ei}, τ_{eb} and τ_{ib} are the momentum relaxation times between electrons and plasma ions, electrons and beam ions, and plasma ions and beam ions, respectively. The explicit forms of these functions will be given later.

We suppose that the beam is injected into the plasma in the azimuthal θ-direction to generate a net seed current. We assume that energetic beam ions are born between the null point r_0 and the separatrix $r_s (r_0 \simeq r_s/\sqrt{2})$, and slow down to be thermalized near r_0. The Larmor radius of beam ion is comparable to the major radius ($\simeq r_0$). To model their dynamics we assume that they are injected in concentric orbits. To the extent that only drag is taken into account in the collisions they remain in concentric orbits, and we can use a simple orbit equilibrium to describe their dynamics. A single orbit of a beam energetic ion is described by

$$\left.\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= \frac{eZ_b}{m_b}E_r + \Omega_b r\dot{\theta} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= -\Omega_b\dot{r} + f_{b\theta} + \frac{Z_b e E_\theta}{m_b}\end{aligned}\right\}, \tag{2.10}$$

where Z_b is the charge number of beam ions and $\Omega_b = eZ_bB_z/m_b$ with m_b the beam ion mass. E_r is the radial electric field, E_θ the toroidal electric field, and $f_{b\theta}$ is the drag force due to collisions with electrons and plasma ions. We solve Eq. (2.10) assuming the nearly concentric orbit ($\ddot{r} \simeq 0$) and $f_{b\theta}$ is approximated by

$$f_{b\theta} = -\frac{1}{n_b m_b}(M_{e\theta} + M_{i\theta}), \tag{2.11}$$

where n_b is the beam density. We obtain

$$v_{br} \;=\; \dot{r} \;= -g\frac{E_\theta}{B_z} + \frac{g}{n_b m_b \Omega_b}(M_{e\theta} + M_{i\theta}), \tag{2.12}$$

$$v_{b\theta} \;=\; r\dot{\theta} = -\frac{1}{2}r\Omega_b\left\{1 + \sqrt{1 - \frac{4}{\Omega_b}\frac{E_r}{rB_z}}\right\}. \tag{2.13}$$

Here, g is given by

$$g = \frac{\Omega_b^2}{\frac{1}{2}(\Omega_\beta^2 - \Omega_b^2)\left\{1 + \frac{1}{\sqrt{1-\frac{4}{\Omega_b}\frac{E_r}{rB_z}}}\left(1 - \frac{2}{\Omega_b}\frac{E_r}{rB_z}\right)\right\} + \Omega_b^2\sqrt{1 - \frac{4}{\Omega_b}\frac{E_r}{rB_z}} - \frac{r\Omega_b}{\sqrt{1-\frac{4}{\Omega_b}\frac{E_r}{rB_z}}}\frac{\partial}{\partial r}\left(\frac{E_r}{rB_z}\right)}, \tag{2.14}$$

where $\Omega_\beta^2 = \Omega_b\partial(r\Omega_b)/\partial r$ is the square of the betatron frequency (in absence of E_r). Since E_r/B_z is comparable to the plasma rotation, $E_r/(\Omega_b r B_z)$ is much less than unity, except near the null point ($B_z \simeq 0$). In the limit of small $E_r/(\Omega_b r B_z)$,

$$g \simeq \frac{\Omega_b^2}{\Omega_\beta^2}. \tag{2.15}$$

Continuity equations for electrons, plasma ions, and beam ions are given, respectively, by

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{v}_e) \;=\; Z_b\frac{S_b}{r}\delta(r - r_b) + S_e^{\text{extra}}, \tag{2.16}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = \frac{Z_b}{Z_i}\frac{S_b}{r}\delta(r - r_0) + S_i^{\text{extra}}, \tag{2.17}$$

$$\frac{\partial n_b}{\partial t} + \nabla \cdot (n_b \mathbf{v}_b) = \frac{S_b}{r}\delta(r - r_b) - \frac{S_b}{r}\delta(r - r_0). \tag{2.18}$$

We have assumed that energetic beam ions are born at $r_b(r_0 < r_b < r_s)$ and slow down to be thermalized at r_0. Extra particle sources, such as by pellet injection, are expressed as S_e^{extra} and S_i^{extra} ($S_e^{\text{extra}} = Z_i S_i^{\text{extra}}$). From Eqs. (2.16) to (2.18) and the charge neutrality condition, the r-component of total current density, J_r should vanish;

$$J_r = -en_e v_{er} + eZ_i n_i v_{ir} + eZ_b n_b v_{br} = 0. \tag{2.19}$$

This equation gives, together with Eqs. (2.7), (2.8) and (2.12),

$$\begin{aligned} n_i m_i \frac{\partial v_{i\theta}}{\partial t} &= -eE_\theta \left\{ Z_b n_b (1+g) + (n_e + gZ_b n_b)\frac{1}{\Omega_i}\frac{1}{r}\frac{\partial}{\partial r}(r v_{i\theta}) \right\} \\ &\quad + (M_{e\theta} + M_{i\theta})\left\{ 1 + g\left[1 + \frac{1}{\Omega_i}\frac{1}{r}\frac{\partial}{\partial r}(r v_{i\theta}) \right] \right\} \\ &\quad + (R_{e\theta} + M_{e\theta})\frac{1}{\Omega_i}\frac{1}{r}\frac{\partial}{\partial r}(r v_{i\theta}) - \left(\nabla\cdot \overleftrightarrow{\pi}_i\right)_\theta - \left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta \left[1 + \frac{1}{\Omega_i}\frac{1}{r}\frac{\partial}{\partial r}(r v_{i\theta}) \right] \end{aligned} \tag{2.20}$$

This equation can be written as

$$\begin{aligned} n_i m_i \frac{\partial v_{i\theta}}{\partial t} &= (M_{e\theta} + M_{i\theta})(1+g) - eZ_b n_b (1+g) E_\theta - \left(\nabla\cdot \overleftrightarrow{\pi}_i\right)_\theta - \left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta \\ &\quad + \left[-(en_e + egZ_b n_b)E_\theta + (M_{e\theta} + M_{i\theta})g + R_{e\theta} + M_{e\theta} - \left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta \right] \\ &\quad \times \frac{1}{\Omega_i}\frac{1}{r}\frac{\partial}{\partial r}(r v_{i\theta}). \end{aligned} \tag{2.21}$$

If we use Eqs. (2.7) and (2.12) to replace the terms $(M_{e\theta} + M_{i\theta})g$ and $R_{e\theta} + M_{e\theta} - \left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta$ in the bracket on the right hand side of Eq. (2.21) by n_{br} and v_{er}, respectively, we obtain

$$\begin{aligned} & n_i \frac{\partial}{\partial t}(r m_i v_{i\theta}) + n_i v_{ir}\frac{\partial}{\partial r}(r m_i v_{i\theta}) = \\ & \qquad r\left[(M_{e\theta} + M_{i\theta})(1+g) - eZ_b n_b (1+g)E_\theta - \left(\nabla\cdot \overleftrightarrow{\pi}_i\right)_\theta - \left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta \right]. \end{aligned} \tag{2.22}$$

Here, we have used the charge neutrality condition given by Eq. (2.19) to obtain the second term on the left hand side. This term represents the convection of angular momentum

$rm_i v_{i\theta}$ by the radial transport of plasma ions. The first term on the right hand side stands for the momentum input from the beam to the plasma corrected by the betatron effect of the beam ions. This term drives the plasma rotation in the direction of the injected beam. The second term is the effect of the inductive electric field. Using that $v_{b\theta}$ is negative, the inductive field term tends to cancel the first term when B_z increases and adds to the first term when B_z decreases. Neglecting the inductive electric field and viscosity terms, we observe that Eq. (2.22) implies that momentum transfer of the beam to electrons and ions cause increased spinning of the ions as well as radial convection of the spin. Even the momentum input from the beam (the g-term) adds to the spin as a result of the electric fields arising from the quasi-neutrality constraint. The ion viscosity term is given by

$$\begin{aligned} \left(\nabla\cdot \overleftrightarrow{\pi}_i\right)_\theta &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\pi^i_{r\theta}), \\ \pi^i_{r\theta} &= -\eta_2 r\frac{\partial}{\partial r}\left(\frac{v_{i\theta}}{r}\right) + \eta_3 r\frac{\partial}{\partial r}\left(\frac{v_{ir}}{r}\right), \end{aligned} \tag{2.23}$$

where η_2 and η_3 are coefficients of shear viscosity and gyro-viscosity, respectively [13]; $\eta_2 = n_iT_i/(\Omega_i^2\tau_i)$, $\eta_3 = n_iT_i/(2\Omega_i)$. In Eq. (2.23), shear viscosity damps the rotation, however, the collisionless gyro-viscosity term damps or enhances the rotation; according to the details of plasma ion radial transport. The electron viscosity term in Eq. (2.22) is negligibly small when compared to ion viscosity term.

If $|E_r/B_z| \gg |(1/eZ_in_iB_z)\partial p_i/\partial r|$ and $|E_r/(\Omega_i rB_z)| \ll 1$ (this is valid except in the vicinity of field null), it follows from Eq. (2.6) that $\partial v_{i\theta}/\partial t \simeq -\partial(E_r/B_z)/\partial t$. It is clear as mentioned above that Eq. (2.22) determines the plasma rotation. The rotation equation (balance of angular momentum) which corresponds to Eq. (2.22) in the absence of externally driven beam was first derived by Hamada and Tomiyama [14].

To complete the set of basic equations, we require Maxwell equations;

$$\frac{\partial B_z}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta), \tag{2.24}$$

$$\mu_0 J_\theta \;=\; -\frac{\partial B_z}{\partial r}, \tag{2.25}$$

with

$$J_\theta = -en_e v_{e\theta} + eZ_i n_i v_{i\theta} + eZ_b n_b v_{b\theta}. \tag{2.26}$$

We restrict ourselves only to the case when the electron and plasma ion temperature, T_e and T_i are assumed to be spatially uniform without solving energy balance equations. Equations (2.5) to (2.8), (2.12), (2.13), (2.16) to (2.18), (2.22), (2.24) and (2.25) form the set of basic equations.

Equation (2.7) can be rewritten as

$$\frac{1}{en_e}\left(\nabla\cdot \overleftrightarrow{\pi}_e\right)_\theta - v_{er}B_z = \eta(J_\theta - J_\theta^*), \tag{2.27}$$

where η is the Spitzer resistivity, $\eta = m_e/(e^2 n_e \tau_{ei})$, and the seed current density J_θ^* is given by

$$J_\theta^* = eZ_b n_b v_{b\theta}\left(1 - \frac{n_e Z_b}{n_i Z_i^2}\right) - eZ_b n_b\left(v_{i\theta} - \frac{n_e Z_b}{n_i Z_i^2} v_{e\theta}\right). \tag{2.28}$$

If $n_b \ll n_e \simeq Z_i n_i$ and $v_{i\theta}$ and $v_{e\theta}$ are much smaller than $v_{b\theta}$ (small rotation case), Eq. (2.28) is reduced to the so-called "Ohkawa current" [6];

$$J_\theta^* \simeq eZ_b n_b v_{b\theta}\left(1 - \frac{Z_b}{Z_{\text{eff}}}\right). \tag{2.29}$$

Here Z_{eff} is the effective charge number of the background plasma ions. In the strong rotation case, plasma ions and electrons are accelerated in the beam direction ($v_{i\theta} \sim v_{e\theta} \sim -E_r/B_z$), resulting in reduction of the net seed current.

3. Rotation of a Steady-State Plasma

We now study steady state rotation in an FRC when current is maintained by a steady state beam. In this section we consider a plasma in an FRC without a multipole field and solve Eq. (2.22) to estimate the rotation. In Eq. (2.23) the term $\eta_2 \partial/\partial r\,(v_{i\theta}/r)$ forces a relaxation to rigid motion where $v_{i\theta}/r = \omega$ is independent of r. Thus we conclude that the plasma is stationary and rotates rigidly; Further, in the stationary case $E_\theta = 0$, we neglect the electron viscosity, which is smaller than that of ions.

Integration of Eq. (2.22) multiplied by r over $r = 0$ to $r = r_s$ (r_s the separatrix radius) yields

$$\omega = \frac{v_{i\theta}}{r} = -\frac{\int_0^{r_s} M_b(1+g)r^2dr + r_s^2\pi^i_{r\theta}(r_s)}{2\int_0^{r_s} n_i m_i v_{ir} r^2 dr}, \tag{3.1}$$

with

$$M_b = -(M_{e\theta} + M_{i\theta}) = -\nu_b n_b m_b v_{b\theta}, \tag{3.2}$$

where $\nu_b = n_e/\tau_{eb}n_b + n_i/\tau_{ib}n_b$, is the momentum relation rate of beam ions and M_b is the momentum loss rate of beam ions. Let $B_z > 0$ for $r > r_0$ and $B_z < 0$ for $r < r_0$, where r_0 is the field null point ($r_0 = r_s/\sqrt{2}$). Then $J_\theta < 0$ and the beam is injected in the negative direction of $\theta(v_{b\theta} < 0)$. The beam ions slow down from drag forces due to collisions with electrons and plasma ions in the positive direction ($M_b > 0$). The first term in the numerator of Eq. (3.1) can be written as

$$\begin{aligned} A &\equiv \int_0^{r_s} M_b(1+g)r^2dr \\ &= -\int_{r_0}^{r_b} n_b m_b v_{b\theta}\left(\nu_b - \frac{v_{br}}{r}\right) r^2 dr \\ &\simeq -\int_{r_0}^{r_b} (r\nu_b n_b m_b v_{b\theta} + r\Omega_b n_b m_b v_{br}) r\, dr, \end{aligned} \tag{3.3}$$

where we have used Eqs. (3.2) and (2.12) and assumed $v_{b\theta} \simeq -r\Omega_b$ (this is valid, except near the null point). The first term in Eq. (3.3) is the total rate of beam angular momentum, which beam ions lose directly to the background plasma. The second term represents beam angular momentum lost due to beam convection. In Eq. (3.3), A stands for the total angular momentum input which drives the plasma rotation in the beam direction (negative direction of θ). Therefore, one can set A to the direct beam angular momentum input rate as we will subsequently do below. The term

$$\omega B \equiv 2 \int_0^{r_s} n_i m_i v_{ir} r^2 dr\, \omega \tag{3.4}$$

represents the effect of transport of plasma ions. This term appears to represent the convection of angular momentum of the background plasma, although the factor of 2 is difficult to interpret. The second term of numerator of Eq. (3.1) is the effect of viscosity. Under the approximation of a rigid rotor, the collisional shear viscosity term is zero and the remaining term can be expressed as

$$C \equiv r_s^2 \pi^i_{r\theta}(r_s) = r_s^3 \left[\eta_3 \frac{d}{dr} \left(\frac{v_{ir}}{r} \right) \right]_{r=r_s}, \tag{3.5}$$

$$\eta_3 = \frac{n_i T_i}{2\Omega_i}.$$

This term damps or enhances the rotation according to the plasma ion transport.

First we estimate the value of A given by Eq. (3.3). We replace variables in the integration of A by averaged values to obtain

$$\begin{aligned} A &= \int_0^{r_s} (1+g) M_b r^2 dr \\ &\simeq -(1+\bar{g})\bar{\nu}_b \bar{n}_b m_b \bar{v}_{b\theta} \int_{r_0}^{r_b} r^2 dr \\ &= -(1+\bar{g})\bar{\nu}_b \bar{n}_b m_b \bar{v}_{b\theta} \times 1.2(r_b - r_0) r_0^2, \end{aligned} \tag{3.6}$$

where, for simplicity, we have taken $r_b \simeq (r_0 + r_s)/2 = 1.2 r_0$. On the other hand, total angular momentum input from the beam can be given in terms of the initial beam speed v_b^0 and neutral beam equivalent current I_b^0,

$$A = -r_b \frac{I_b^0}{eZ_b} m_b v_b^0 \frac{1}{4\pi L}, \tag{3.7}$$

where L is the half-length of the FRC. Equating Eqs. (3.6) and (3.7) yields

$$\bar{n}_b \bar{v}_{b\theta} = \frac{I_b^0 v_b^0}{(1+\bar{g})\bar{\nu}_b (r_b - r_0) r_0 e Z_b 4\pi L}. \tag{3.8}$$

We assume that the rotation speed $|v_{i\theta}|$ is small enough to express the seed current density by Eq. (2.29);

$$J_\theta^* = eZ_b\bar{n}_b\bar{v}_{b\theta}\left(1-\frac{Z_b}{Z_{\text{eff}}}\right) = \frac{I_b^0 v_b^0}{(1+\bar{g})\bar{\nu}_b(r_b-r_0)r_0\cdot 4\pi L}\left(1-\frac{Z_b}{Z_{\text{eff}}}\right), \tag{3.9}$$

where we have used Eq. (3.8). The total seed current I_* is given by

$$I_* = 2L(r_b-r_0)J_* = \frac{I_b^0 v_b^0}{2\pi r_0\bar{\nu}_b(1+\bar{g})}\left(1-\frac{Z_b}{Z_{\text{eff}}}\right). \tag{3.10}$$

Eliminating $I_b^0 v_b^0$ from Eq. (3.7) using Eq. (3.10), we obtain

$$A = -\frac{0.605\frac{m_b r_0^2}{eZ_b L}\bar{\nu}_b(1+\bar{g})I_*}{\left(1-\frac{Z_b}{Z_{\text{eff}}}\right)}. \tag{3.11}$$

The total plasma current inside the separatrix I_t is obtained by integrating Eq. (2.25) to yield

$$I_t = -\frac{2L}{\mu_0}\int_0^{r_s}\frac{dB_z}{dr}\,dr = -\frac{4L}{\mu_0}B_s, \tag{3.12}$$

where $B_s = \sqrt{1-\beta_s}B_\omega$ (β_s is the beta value at the separatrix and B_ω is the magnetic field at the wall). This relation comes from the fact that the pressure balance $d(p + B_z^2/2\mu_0) = 0$ holds since there are no beam ions, and inertia effects are assumed small at the separatrix. Finally we obtain

$$A = \frac{\frac{2.42 m_b r_0^2 B_\omega}{eZ_b\mu_0}\sqrt{1-\beta_s}\bar{\nu}_b(1+\bar{g})}{\left(1-\frac{Z_b}{Z_{\text{eff}}}\right)}\frac{I_*}{I_t}. \tag{3.13}$$

Note that if $1 - Z_b/Z_{\text{eff}} > 0,\ I_* < 0,\ I_t < 0$ and $A > 0$.

In order to express B given by Eq. (3.4) in terms of particle input rate Q, we combine Eqs. (2.16) to (2.18) to get

$$\nabla\cdot(\rho\mathbf{v}) = Q, \tag{3.14}$$

$$\begin{cases} \rho &= n_e m_e + n_i m_i + n_b m_b \simeq n_i m_i, \\ \mathbf{v} &= m_e n_e \mathbf{v}_e + m_i n_i \mathbf{v}_i + m_b n_b \mathbf{v}_b \\ &\simeq m_i n_i \mathbf{v}_i + m_b n_b \mathbf{v}_b, \end{cases}$$

where

$$Q = m_b \frac{S_b}{r} \delta(r - r_b) + m_i S_i^{\text{extra}}. \tag{3.15}$$

The first term is the beam particle input rate and the second is the extra one such as that from a continual pellet injection. The ion continuity equation (2.17) becomes

$$\frac{1}{r}\frac{d}{dr}(r m_i n_i v_{ir}) = Q + m_b \frac{S_b}{r}\left\{\delta(r - r_0) - \delta(r - r_b)\right\}. \tag{3.16}$$

The parallel flow term $\partial(n_i m_i v_{iz})/\partial z$ can be approximated by $n_i m_i w/L$ ($v_{iz} = zw/L$) in the present one-dimensional treatment [15], but we have neglected this term in Eq. (3.16), assuming $|v_{ir}/r| \gg |w/L|$. Then

$$\begin{aligned} B &= 2\int_0^{r_s} r\,dr \left[\int_0^r Q(r')r'dr' + m_b S_b \int_0^r \left\{\delta(r' - r_0) - \delta(r' - r_b)\right\} dr'\right] \\ &= 2\int_0^{r_s} r\,dr \int_0^r Q(r')r'\,dr' + m_b S_b (r_b^2 - r_0^2). \end{aligned} \tag{3.17}$$

The first term corresponds to v_{er} and the second is attributed to the presence of beam ions. It is noted that the second beam term produces the difference of $n_e v_{er}$ and $Z_i n_i v_{ir}$ and enhances the ion particle diffusion. It is then clear that the presence of beam ions tends to reduce the rotation in the denominator of Eq. (3.1), whereas it enhances the momentum input to the plasma as we have seen in Eq. (3.3).

The first term of Eq. (3.17) can be expressed in terms of the particle input rate. We define, as in Ref. [5],

$$\sum = \frac{2\mu_0^2 r_s^2}{\eta_0 B_\omega^2} Q\Theta, \quad \Theta = \frac{Z_i T_e + T_i}{m_i}, \tag{3.18}$$

where the resistivity η is decomposed as: $\eta = \eta_0 \eta_R(y)$ with η_0 constant and $\eta_R(y) \approx O(1)$, but spatially varying. Though our analysis is classical, we will consider the possibility that η be anomalous. The first term becomes

$$\begin{aligned} B_1 &\equiv 2\int_0^{r_s} r\,dr \int_0^r Q(r')r'\,dr' = \frac{r_s^4}{2}\int_0^1 dy \int_0^y Q(y')dy' \\ &= \frac{r_s^2 \eta_0 B_\omega^2}{4\mu_0^2 \Theta} \int_0^1 dy \int_0^y \sum(y')\,dy'. \end{aligned} \tag{3.19}$$

As in Ref. [5], we define

$$S(y) = \frac{1}{\eta_R}\int_{\frac{1}{2}}^{y} \sum(y')\,dy' = \frac{S_0}{\eta_R}\int_{\frac{1}{2}}^{y} \frac{Q(y')\,dy'}{\langle Q\rangle}, \tag{3.20}$$

with

$$S_0 = \frac{2\mu_0^2 r_s^2}{\eta_0 B_\omega^2}\Theta\,\langle Q\rangle\,, \tag{3.21}$$

where $\langle Q\rangle = \int_0^{r_s} Q 2\pi r\,dr/(\pi r_s^2) = \int_0^1 Q\,dy$. Equation (3.19) becomes

$$B_1 = \frac{r_s^2\eta_0 B_\omega^2}{4\mu_0^2\Theta}\int_0^1 dy\,[-S(y=0)\eta_R(y=0) + S(y)\eta_R(y)]\,. \tag{3.22}$$

It is well-known that the p = const. surface shifts from the flux surface when inertia is taken into account. However, we neglect this shift in our rough estimation assuming that the inertia effect is small. In such a circumstance, we choose $Q(y)$ and $\sum(y)$ to be an even function around $y = 1/2$ (null point) as indicated in Ref. [5]. From the definition of $S(y)$ given by Eq. (3.20), $\eta_R(y)S(y)$ is an odd function and $\int_0^1 \eta_R(y)S(y)\,dy = 0$ in Eq. (3.22). Equation (3.20) gives the relation $\eta_R(y=0)S(y=0) = -S_0/2$ and B_1 becomes

$$B_1 = \frac{r_s^2\eta_0 B_\omega^2}{8\mu_0^2\Theta}S_0. \tag{3.23}$$

The second term of Eq. (3.17) can be expressed in terms of A given by Eq. (3.13). The relation of S_b and the equivalent beam current I_b^0 is

$$\int_V \frac{S_b}{r}\delta(r-r_b)\,dV = \frac{I_b^0}{eZ_b}, \tag{3.24}$$

where $dV = 4\pi Lr\,dr$ is the volume element. This relation gives

$$S_b = -\frac{B_w}{\mu_0 I_t}\frac{I_b^0}{\pi e Z_b}. \tag{3.25}$$

The second term of B given by Eq. (3.17) becomes

$$B_2 \equiv m_b S_b(r_b^2 - r_0^2)$$

$$= -m_b \frac{B_s I_b^0}{\mu_0 \pi e Z_b I_t}(r_b^2 - r_0^2). \tag{3.26}$$

Using Eq. (3.10) we obtain

$$B_2 = -\frac{\frac{2m_b B_s (r_b^2 - r_0^2) r_0}{\mu_0 e Z_b I_t v_b^0} \bar{\nu}_b (1+\bar{g}) I_*}{\left(1 - \frac{Z_b}{Z_{\text{eff}}}\right)}$$

$$= -0.826 \frac{r_b^2 - r_0^2}{r_0} \frac{1}{v_b^0} A, \tag{3.27}$$

where we have used Eq. (3.13). The Larmor radius of initial beam ions is $r_b = -v_b^0/\Omega_b^0$, with $\Omega_b^0 = eZ_b B_z(r = r_b)/m_b$. Equation (3.27) becomes

$$B_2 = 0.826 \frac{r_b^2 - r_0^2}{r_b r_0} \frac{A}{\Omega_b^0}. \tag{3.28}$$

We have obtained the expression for B; $B = B_1 + B_2$, where B_1 and B_2 are given by Eqs. (3.23) and (3.28), respectively.

Finally we estimate C given by Eq. (3.5). C is evaluated at the separatrix where beam ions and particle source are assumed to be absent. Therefore, from Eq. (3.16) we obtain

$$v_{irs} = \frac{1}{r_s n_{is} m_i} \int_0^{r_s} Q(r') r' dr', \tag{3.29}$$

and

$$\left.\frac{dv_{ir}}{dr}\right|_{r=r_s} = -\left(1 + \frac{r_s}{n_{is}} \left.\frac{dn_i}{dr}\right|_{r_s}\right) \frac{v_{irs}}{r_s}, \tag{3.30}$$

where $v_{irs} = v_{ir}(r_s)$ and $n_{is} = n_i(r_s)$. Equation (3.5) becomes

$$C = r_s^2 \pi_{r\theta}^i(r_s)$$

$$= -(\eta_3)_{r_s} \left(2 + \frac{r_s}{n_{is}} \left.\frac{dn_i}{dr}\right|_{r_s}\right) v_{irs} r_s. \tag{3.31}$$

It is straightforward to express v_{irs} in terms of S_0 defined by Eq. (3.21) for Q in Eq. (3.29) and transforming the coordinate from r to $y (y = r^2/r_s^2)$, we obtain

$$v_{irs} = \frac{\eta_0 B_\omega^2}{4\mu_0^2 r_s m_i \Theta n_{is}} \left[\frac{S_0}{2} + S(y=1)\eta_R(y=1)\right]. \tag{3.32}$$

We again assume that $\sum(y)$ is even and $S(y)$ is odd around $y = 1/2$. Then $S(y = 1)$ $\eta_R(y = 1) = S_0/2$ and v_{irs} becomes

$$v_{irs} = \frac{\eta_0 B_\omega^2}{4\mu_0^2 r_s m_i \Theta n_{is}} S_0. \tag{3.33}$$

Substituting v_{irs} into Eq. (3.31), we obtain

$$C = -\frac{1}{2\mu_0}\frac{n_{is}T_{is}\eta_0}{\Omega_{is}\beta_s}\left(1 + \frac{\beta_s'}{\beta_s}\right) S_0, \tag{3.34}$$

where T_{is} is the ion temperature at the separatrix, $\beta_s = \beta(r_s) = 2\mu_0 n_{is} m_i \Theta / B_\omega^2$ is the beta value at the separatrix and $\beta_s' = d\beta_s/dy$. We define the particle confinement time by

$$\tau_N = \frac{\int \rho\, dV}{\int Q\, dV} = \frac{\int_0^1 \rho\, dy}{\int_0^1 Q\, dy} = \frac{\langle \rho \rangle}{\langle Q \rangle} = \frac{r_s \langle \rho \rangle}{2\rho_s v_{irs}}, \tag{3.35}$$

where we have used Eq. (3.29), $\rho_s = n_{is} m_i$ and $\langle \rho \rangle$ is the average mass density. Very near the separatrix there are no beam ions and the inertia modification is small. Using the equilibrium $dp/dr = J_\theta B_z$, Ampere's law $\mu_0 J_\theta = -dB_z/dr$ and Ohm's law $-v_{ir} B_z = \eta J_\theta$, we obtain

$$v_{irs} = -\frac{\eta}{B_z^2}\frac{dp}{dr}\bigg|_{r_s} = -\frac{\eta_0 \eta_R}{2\mu_0}\frac{1}{1-\beta_s}\frac{2}{r_s}\beta_s'. \tag{3.36}$$

We have used the relation $B_s^2/B_\omega^2 = 1 - \beta_s$. The particle confinement time is given by

$$\tau_N = -\frac{\mu_0 r_s^2}{2\eta_0}\frac{\langle \beta \rangle (1-\beta_s)}{\eta_R(1)\beta_s \beta_s'}, \tag{3.37}$$

where $\langle \beta \rangle$ is the average beta value. As indicated in Ref. [5], S_0 defined by Eq. (3.21) can be written as $S_0 = \langle \beta \rangle \tau_B / \tau_N$, where $\tau_B = \mu_0 r_s^2 / \eta_0$. We obtain for β_s'/β_s in C

$$\frac{\beta_s'}{\beta_s} = -\frac{S_0}{2}\frac{1-\beta_s}{\eta_R(1)\beta_s^2}. \tag{3.38}$$

Hamada and Tomiyama considered that the intrinsic or inherent rotation of the existing experiments is attributed to the gyro-viscosity effect and explained the rotation speed observed in the experiment from NUCTE-FRC [14]. The results given by Eqs. (3.34) and (3.38) agree with those by Hamada and Tomiyama. Neglecting the beam momentum input for the moment, we observe that the rotation speed due to gyro-viscosity is given by

$$\begin{aligned} \omega &= \frac{v_{i\theta s}}{r_s} = -\frac{C}{B_1} \\ &= v_{this}\rho_{is}\frac{1}{r_s^2}\left(1+\frac{\beta_s'}{\beta_s}\right), \end{aligned} \tag{3.39}$$

where $v_{this} = \sqrt{2T_{is}/m_i}$ and $\rho_{is} = v_{this}/\Omega_{is}$ are the ion thermal speed and ion Larmor radius at the separatrix, respectively. For present experiment and a reactor plasma, $1 \ll |\beta_s'/\beta_s|$, and we have

$$v_{i\theta s} = \frac{\rho_{is}}{r_s}v_{this}\frac{\beta_s'}{\beta_s} = \frac{T_{is}}{m_i\Omega_{is}}\frac{1}{n_{is}}\frac{dn_i}{dr}\bigg|_{r=r_s}, \tag{3.40}$$

which is just the ion diamagnetic flow velocity at the separatrix. This is in the beam direction (negative θ direction) and enhances the rotation. Substituting Eq. (3.38) into Eq. (3.39), we obtain

$$\frac{v_{i\theta s}}{v_{this}} = \frac{\rho_{is}}{r_s}\left(1-\frac{S_0}{2}\frac{1}{\eta_R(1)}\frac{1-\beta_s}{\beta_s^2}\right). \tag{3.41}$$

It is apparent that this value depends strongly on β_s. The value of β_s is determined only when we consider the plasma outside the separatrix. Hamada examined this problem solving transport equation together with equilibrium and obtained the result that $\beta_s \lesssim 0.1$ for a D-T reactor plasma [15]. If we assume that $\beta_s = 0.1$, $T_{is} = 1$ keV, $r_s = 1.2$ m, and $B_\omega = 2T$ $(B_s = B_z(r_s) = \sqrt{1-\beta_s}B_\omega = 1.8T)$, resulting in $v_{i\theta s}/v_{this} = -0.15S_0/\eta_R(1)$. If $\eta_R(r)$ is proportional to the spatial variation of electron temperature, $\eta_R(r) = T_e^{-3/2}(r)/T_e^{-3/2}(0)$, we obtain $\eta_R(1) = 58.1$ for $T_i = 15$ keV. Thus $|v_{i\theta s}/v_{this}| = 0.013 \sim 0.13$ for S_0 of 5 to 50.

However, we are primarily concerned about the effect of the steady beam input in inducing plasma rotation. Since the inherent rotation due to the gyro-viscosity is beyond the scope of the present paper, we will only estimate the rotation speed due to beam injection only.

We define the radius of rotation by

$$r_R^2 = \frac{\int_0^{r_s}\rho r^3 dr}{\int_0^{r_s}\rho r\, dr}. \tag{3.42}$$

This gives that $r_R = r_s/\sqrt{2} = r_0$. Then the average rotation speed normalized by the ion thermal speed $v_{thi} = \sqrt{2T_i/m_i}$ is

$$\left\langle \frac{v_{i\theta}}{v_{thi}} \right\rangle \simeq \frac{-\frac{r_0}{v_{thi}}\frac{A}{B_1}}{\left(1+\frac{B_2}{B_1}\right)}. \tag{3.43}$$

We assume that the constant part of resistivity is given by

$$\eta_0 = \alpha \eta_\perp^{cl} \quad , \quad \eta_\perp^{cl} = \frac{m_e \nu_{ei}}{e^2 n_e}, \tag{3.44}$$

where α is the anomaly factor ($\alpha \geq 1$) and $\eta_\perp^{cl}$ is the Spitzer classical resistivity. We also assume that the dynamics of the resistive anomaly does not effect the beam plasma interaction. We define r_L by

$$r_L = \frac{\sqrt{\Theta}}{\Omega_{i\omega}} \quad , \quad \Omega_{i\omega} = \frac{eZ_iB_w}{m_i}. \tag{3.45}$$

The collision frequency of beam ions with electrons and plasma ions (the reciprocal of beam thermalization time), ν_b is given by

$$\begin{aligned}
\nu_b &= \nu_s(1+\delta), \\
\nu_s &= \frac{Z_b^2}{Z_i}\frac{m_e}{m_b}\nu_{ei}, \\
\delta &= \frac{3\sqrt{\pi}}{4}Z_i\left(1+\frac{m_b}{m_i}\right)\left(\frac{m_b}{m_e}\right)^{1/2}\left(\frac{T_e}{E_b}\right)^{3/2},
\end{aligned} \tag{3.46}$$

where ν_s is the reciprocal of slowing down time, E_b is the beam energy and δ comes from collisions between beam and plasma ions. Using these variables from Eqs. (3.43) to (3.45), we can obtain

$$\begin{aligned}
V_1 &= \frac{r_0}{v_{thi}}\frac{A}{B_1} \\
&= 3.42(1+\bar{g})(1+\delta)\sqrt{\frac{1}{2}+\frac{Z_iT_e}{2T_i}}\sqrt{1-\beta_s}\,\langle\beta\rangle\,\frac{r_s}{r_L}\frac{I_*}{I_t}\frac{1}{\alpha S_0}\frac{\frac{Z_b}{Z_i}}{1-\frac{Z_b}{Z_{\text{eff}}}}
\end{aligned} \tag{3.47}$$

$$\frac{B_2}{B_1} = 0.826\left(\frac{r_b^2}{r_0^2}-1\right)\sqrt{\frac{m_b}{m_i}}\sqrt{\frac{T_i}{E_b}}V_1. \tag{3.48}$$

In order to estimate $\bar{g}$, we assume that g is given by Eq. (2.15); $g = \Omega_b^2/\Omega_\beta^2 = 1/(1 + (r/B_z)dB_z/dr)$ and $B_z \propto r - r_0$. We define $\bar{g} = \int_{r_0}^{r_b} gr^2dr / \int_{r_0}^{r_s} r^2dr$ to obtain $\bar{g} = 0.09 \sim 0.15$ for $r_b = (r_s + r_0)/2 \sim 0.85r_s$.

Calculation parameters chosen for an ignited D-T plasma supported by a steady beam injection are $r_s = 1.2$ m, $L = 6$ m, $B_w = 2T$, $T_e = T_i = 15$ keV, $Z_i = 1$, $A_i = 2.5$ (A_i is the mass number of plasma ions consisting of equal deuterium and tritium), and parameters of a deuterium beam are $A_b = 2$, $Z_b = 1$ and $E_b = 225$ keV. We use the results for S_0, $\langle\beta\rangle$ and I_*/I_t from Ref. [5] assuming that the rotation speed is small enough for these results to be approximately valid.

Table I shows the calculation results. The values of S_0, $\langle\beta\rangle$, I_*/I_t, $\bar{s}$, $(\tau_N)_{\min}$ and $\alpha_{\max}$ are employed from Ref. [5] ($\bar{s}$ is the average ratio of r_s to plasma ion Larmor radius). It should be noted that the value of S_0 is uniquely determined only when particle and energy transports are known. A set of values in each row in the Table I is obtained to satisfy the ignition state provided that all the fusion products (α particles) are confined to heat the plasma. The values of $(\tau_N)_{\min}$ and $\alpha_{\max}$ are the minimum of particle confinement time required to achieve the ignition and the permissible maximum enhancement factor α for each S_0. The values of αV_1 are calculated from Eq. (3.47) and $\langle v_{i\theta}/v_{thi}\rangle$ shows the rotation speed for $\alpha = \alpha_{\max}$ calculated based on Eq. (3.43) with Eqs. (3.47) and (3.48). It is seen as expected that the rotation becomes large as I_*/I_t becomes large or as $\bar{s}$ becomes large. The required beam power is given by

$$P_b = \frac{1}{e} I_b^0 E_b^0, \tag{3.49}$$

where I_b^0 and E_b^0 are equivalent beam current [in amperes] and initial beam energy $\left(E_b^0 = \frac{1}{2}m_b(v_b^0)^2\right)$. Substituting Eq. (3.10), we obtain

$$P_b = I_* \frac{\pi r_s \bar{\nu}_b (1+\bar{g})\sqrt{m_b E_b^0}}{e(1 - Z_b/Z_{\text{eff}})}. \tag{3.50}$$

The thermal fusion output power is $P_{\text{out}} = 1GW$ and the values of ratio P_b/P_{out} are shown in Table I. The required circulation powers are moderate even for large rotation case. The last column indicates $\bar{s}\,\langle v_{i\theta}/v_{thi}\rangle$. Since the rough stability condition [16] is $\bar{s}\,\langle v_{i\theta}/v_{thi}\rangle < 1$, the FRC's with $\bar{s}$ larger than 6 are unstable due to large rotation. We also note that the smaller the resistive anomaly factor, the larger the rotation. Thus, a compromise is needed between good overall containment and a slow stable rotation rate. In an FRC with small $\bar{s}$ (say, below 5) or large particle input rate S_0 (equivalently with a large bootstrap effect), the rotation can be moderate. In such the case, however, the intrinsic gyro-viscosity rotation may become dominant. Further, there appears another problem that it is difficult to contain high energy fusion products (alpha particles), which are needed to heat the plasma for small $\bar{s}$. Figure 8 in Ref. [7] shows the containment rate of fusion products as a function of $\tilde{U}_{\max}$. This parameter $\tilde{U}_{\max}$ is just the $\bar{s}$ value of alphas, $\bar{s}_\alpha = (Z_\alpha/Z_i)\sqrt{T_i m_i/E_\alpha m_\alpha}\cdot\bar{s}$, where $Z_\alpha = 2$, $E_\alpha = 3.5$ MeV and m_α is the mass of alphas. If $\bar{s} = 5$, then $\bar{s}_\alpha = 0.1\bar{s} = 0.5$ and it is seen from Fig. 8 in Ref. [7] that very few alphas will be confined to heat the plasma. An additional problem is that if alphas confined in the plasma make a beam as indicated in Ref. [7] to produce a net current, the Ohkawa current flows in the opposite direction of plasma current if $Z_\alpha > Z_{\text{eff}}$. This phenomena tends to disassemble the equilibrium.

4. Multipole Field Effect

It has previously been shown that the presence of quadrupole fields can inhibit back current and a steady flux can be maintained with moderate input beam power in both reactors and present day experiments [3]. In this paper, it is shown that the back current against the input beam current is inhibited by magnetic pumping by transferring electron momentum due to the external coils when the toroidal symmetry is weakly broken by the application of multipole fields.

In this section, we evaluate the magnitude of rotational speed of the plasma in an FRC with multipole fields based on the paper by Hammer and Berk [3]. According to this paper, the seed current is given by

$$J_\theta^* = en_b(v_{b\theta} - v_{i\theta}) = \frac{1}{\eta} f_{mp}, \tag{4.1}$$

$$f_{mp} = \frac{\epsilon^2}{6\pi^2}\left(\frac{r_0}{2m\Delta}\right)^2 \frac{m_i \nu_{ei} v_{i\theta}}{e} \frac{\bar{p}_e}{B_\omega^2} \frac{\overline{M}^2}{\left(1 + \frac{\nu_{ei}^2}{4m^2\omega_0^2}\right)}. \tag{4.2}$$

Here, f_{mp} is the effect of magnetic pumping. In Eq. (4.2), Δ is the radius of the plasma edge ($\Delta \simeq r_s/2$), $\overline{M} = v_{i\theta}/c_s$ is the Mach number with $c_s = \sqrt{2(\bar{p}_e + \bar{p}_i)/m_i\bar{n}}$, $\bar{p}_e$ and $\bar{p}_i$ are electron and ion pressure at the null, $\bar{n}$ is the density at the null ($Z_b = Z_i = 1$ in this section, so that $n = n_e \simeq n_i$), ϵ is the size of symmetry-breaking perturbation and $2m$ is the order of the multipole field. From Eq. (50) in Ref. [3], we obtain

$$\frac{v_{b\theta}}{c_s} = \frac{\bar{n} f_{mp}}{\rho_i n_b \nu_b B_w}. \tag{4.3}$$

$$\left(\rho_i = \frac{m_i c_s}{eB_w} = \sqrt{2} r_L\right)$$

Substituting f_{mp} given by Eq. (4.2) into Eq. (4.3), we obtain

$$\frac{v_{b\theta}}{c_s} = \frac{\bar{n}}{n_b}\frac{\nu_{ei}}{\nu_b}\frac{\epsilon^2}{12\pi}\left(\frac{r_0}{2m\Delta}\right)^2 \beta_e \frac{\overline{M}^3}{\left(1 + \frac{\nu_{ei}^2 r_0^2}{4m^2\overline{M}^2 c_s^2}\right)}, \tag{4.4}$$

where $\beta_e = 2\mu\bar{p}_e/B_\omega^2$. The magnitude of required multipole field is given by Eq. (74) in Ref. [3];

$$\pi^2\left(\frac{\delta B_v}{B_{z0v}}\right)^4 \cong \frac{\epsilon^2\overline{M}^4}{4m^2}, \tag{4.5}$$

where δB_v is the multipole field and B_{z0v} is the vacuum field at the wall without the multipole field ($B_{z0v} \simeq B_\omega$). Eliminating ϵ^2 from Eqs. (4.4) and (4.5) yield

$$\overline{M}\left(1 + \frac{\nu_{ei}^2 r_0^2}{4m^2\overline{M}^2 c_s^2}\right) = M_0 \tag{4.6}$$

$$M_0 = \frac{1}{192}\left(\frac{\delta B_v}{B_{z0v}}\right)^4 \frac{c_s}{v_{b\theta}}\frac{\bar{n}}{n_b}\frac{\nu_{ei}}{\nu_b}\beta_e. \tag{4.7}$$

If the beam ions extend to the size of Δ, we can express $n_b v_{b\theta}$ by I_*/I_t,

$$en_b v_{b\theta}\Delta = B_\omega \frac{I_*}{I_t}, \tag{4.8}$$

where we have used the relation of Eq. (3.12). Eq. (4.7) becomes

$$M_0 = \frac{1}{192\sqrt{2}}\left(\frac{\delta B_v}{B_{z0v}}\right)^4 \frac{I_t}{I_*}\beta_e \frac{\Delta}{r_L}\frac{m_b}{m_e}\frac{1}{1+\delta}, \tag{4.9}$$

where δ is given by Eq. (3.46). From Eq. (4.6), we obtain the Mach number,

$$\overline{M} = \frac{M_0}{2} \pm \sqrt{\frac{M_0^2}{4} - \frac{\nu_{ei}^2 r_0^2}{4m^2 c_s^2}}. \tag{4.10}$$

It is readily shown that the solution given by Eq. (4.10) with the positive sign is unstable and the solution with minus sign is stable. Then, choosing the stable solution, the Mach number is given by

$$\overline{M} = \frac{M_0}{2}\left(1 - \sqrt{1-g^2}\right), \tag{4.11}$$

$$g = \frac{\nu_{ei} r_0}{m c_s M_0}. \tag{4.12}$$

It is clear that solubility condition is $g \leq 1$.

It is an essential point that the skin depth of multipole fields remain small enough to maintain the reversal. The skin depth is given by

$$\delta_s = \frac{\sqrt{2}c}{(4\pi\omega_0\sigma)^{1/2}}, \tag{4.13}$$

where $\omega_0 = c_s\overline{M}/r_0$ and $4\pi\sigma = \omega_{pe}^2/\nu_{ei}$ (ω_{pe} is the electron plasma frequency). The skin depth becomes

$$\delta_s = \frac{\sqrt{2}c\sqrt{r_0\nu_{ei}}}{(c_s\overline{M})^{1/2}\omega_{pe}}. \tag{4.14}$$

If $g \ll 1$,

$$\frac{\delta_s}{r_L} \simeq 4\sqrt{m}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{1}{\sqrt{g}}. \tag{4.15}$$

The stability condition against the rotational instability given by Ishimura [12] is

$$\delta B_v > \frac{1}{2}(m-1)^{-1/2} r_s \omega_0 (\mu_0 n_i m_i)^{1/2}. \tag{4.16}$$

This condition can be rewritten by

$$\overline{M} < \overline{M}_c = \sqrt{2(m-1)}\frac{\delta B_v}{B_{z0v}}. \tag{4.17}$$

We estimate $\overline{M}$, δ_s and Ishimura's condition for both reactor (D-T ignited plasma) and the present day experiment. We employ the same parameters as in Sec. 3 for a reactor with a quadrupole field ($m = 2$).

Figure 1 shows the results for $\overline{M}$ and δ_s for the reactor case. In Fig. 1, the abscissa is $x = (\delta B_v / B_{z0v})^4 I_t / I_*$ and the measures of $\delta B_v / B_{z0v}$ are drawn in percentage for each value of I_*/I_t. The solubility condition $g < 1$ corresponds to $x < 0.06$. It is remarkable that $\overline{M}$ is very small. The maximum $\overline{M}$ is 1.57×10^{-3} when $g = 1$ and decreases rapidly as δB_v increases. The skin depth increases as the rotation decreases, but it remains small; δ_s becomes comparable to ion Larmor radius r_L when $x = 0.18(\delta B_v / B_{z0v} \simeq 15\%$ for $I_* = 16$ MA and $\delta B_v / B_{z0v} \simeq 5\%$ for $I_* = 0.2$ MA). The stability criterion against the rotational instability is $\overline{M}_c = 1.4 \delta B_v / B_{z0v}$, which is far above $\overline{M}$. The required beam equivalent current is $I_b = 0.21 I_*$ amperes and the beam power is $P_\omega = 4.7 \times 10^4 I_*$ watts. (I_* is in MA units), which is very small. Figures 2 and 3 are the results in the case for FRX-C experimental parameters [17]; $r_s = 1$ m, $2L = 1.3$ m, $B_\omega = 0.8T$, $T_e = 175$ eV, $T_i = 625$ eV, $n \simeq 1.9 \times 10^{21}$ m^{-3}, deuteron beam with $I_b = 50$ A and $E_b = 60$ keV, $I_* = 10$ kA, $I_t = 1.67$ MA, $m = 2$ (quadrupole) for Fig. 2 and for Fig. 3, $r_s = 1$ m, $2L = 1.3$ m, $B_\omega = 0.4T$, $T_e \doteq 200$ eV, $T_i = 630$ eV, $n \simeq 4.5 \times 10^{20}$ m^{-3}, deuteron beam with $I_b = 10$ A and $E_b = 30$ keV, $I_* = 7$ kA, $I_t = 830$ MA, $m = 2$ (quadrupole). In Fig. 2 ($B_\omega = 0.8T$ case), $\overline{M}$ becomes smaller than $\overline{M}_c$ when $\delta B_v / B_{z0v}$ exceeds 18%, whereas in Fig. 3 ($B_\omega = 0.4T$ case) the minimum $\delta B_v / B_{z0v}$ necessary to be stable is 12%. The skin depth is well below the ion Larmor radius at the separatrix for both cases.

5. Conclusion

We have analyzed the steady state rotation established by a neutral beam injected into an FRC, in the case where multipole magnetic fields are either present or not present. In the latter case an equilibrium can be established by the Ohkawa effect if the charge Z_b of the beam is less than the Z_{eff} of the background plasma. If inertial effects are ignored, the configuration would spin-up to the speed of the injected beam. However, here we have included inertial effects of the background plasma, as well as developed a self-consistent model to treat the dynamics of the injected beam. We are able to predict the rotation rate, which is essentially determined by a balance between the input beam angular momentum and the convective outflow of the angular momentum flux of the background plasma. The more rapidly the background plasma can diffuse out of the system, the slower is the rotation. We find that the criterion for stability to rotational instability is readily violated for reactor systems that are large enough to contain alpha particle fusion products. If a resistive anomaly exists to provide a rapid particle loss, the rotation rate can be slow enough in moderate and small sized systems to be compatible with stability theory, but the lack of retention of alpha particles would prevent ignition. Hence, a neutral beam driven FRC in steady state may have difficulty achieving desirable reactor conditions. It is also noted that the Ohkawa effect from alpha particles would tend to drive reverse currents that disassemble the system. Further quantitative study on this aspect is needed.

The difficulty with rotation is considerably ameliorated with the presence of multipole magnetic fields. With multipole fields, one can exploit magnetic pumping to slow the plasma rotation and an injected beam particle becomes a complete seed current, even if $Z_b = Z_{\text{eff}}$. Further, the multipole fields provide for an intrinsic bias to help MHD stabilization as has already been demonstrated experimentally. Hence, we find a steady state equilibrium is possible at a rotation rate compatible with the MHD stability criteria. The study was made for reactor grade and present day experimental parameters.

The main difficulty with multipole fields is the penetration of open field lines into the plasma. With neutral beam injection a remnant rotation rate exists that provides for a skin effect that prevents the penetration of the multipole fields into the bulk of the plasma. If the resistivity determining the skin effect is classical, we find that the penetration depth of the multipole fields at the plasma edge can be less than an ion-Larmor radius at rotation rates that are compatible with MHD stability. Hence, a beam driven FRC in a multipole magnetic field appears to have very desirable steady state and stability features. Improvement of the calculations from the rough methods described in this paper is desirable.

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Table Caption

[1] Rotation Speed and Stability Criterion vs. Particle Input Rate in Ignited States

The definition of symbols are as follows:

S_0 (particle inpu rate), $\langle\beta\rangle$ (average beta), I_*/I_t (seed current/total current), $\bar{s}$ (ratio of r_s to average ion Larmor radius), $(\tau_N)_{\min}$ (minimum particle confinement time in seconds), and $\alpha_{\max}$ (maximum anomaly factor) are employed from Ref. [5]. P_b/P_{out} is the ratio of beam input power to fusion output power ($P_{\text{out}} = 1\text{GW}$), αV_1 is given by Eq. (3.47), $\langle v_{i\theta}/v_{thi}\rangle$ is the average rotation speed divided by ion thermal speed for $\alpha = \alpha_{\max}$ (minimum Mach number for a given S_0), and $\bar{s}\,\langle v_{i\theta}/v_{thi}\rangle$ is the stability criterion against rotational instability.

Figure Captions

[1] Mach number and the skin depth for a reactor case in the presence of quadrupole field.

$\overline{M} = v_{i\theta}/c_s$ is the Mach number of plasma rotation and δ_s/r_L is the skin depth of quadrupole field divided by ion Larmor radius r_L. The abscissa is $x = (\delta B_v/B_{z0v})^4 I_t/I_*$ and field ripple depths at the wall $\delta B_v/B_{z0v}$ are shown for $I_* = 16$ MA and 0.2 MA. There is no solution below $g = 1$. The stability criterion $\overline{M}_c$ is far above $\overline{M}$. The parameters for the reactor are: $B_\omega = 2T$, $r_s = 1.2$ m, $2L = 12$ m, $T_e = T_i = 15$ keV, $\bar{n}_e = 3 \times 10^{20}$ m^{-3}, $I_t = 36$ MA. A D^0-beam is injected with $E_b = 225$ keV and $I_b = 0.21 I_*$ (in MA) Amp.

[2] Mach number and the skin depth for FRX-C parameters in the presence of the quadrupole field.

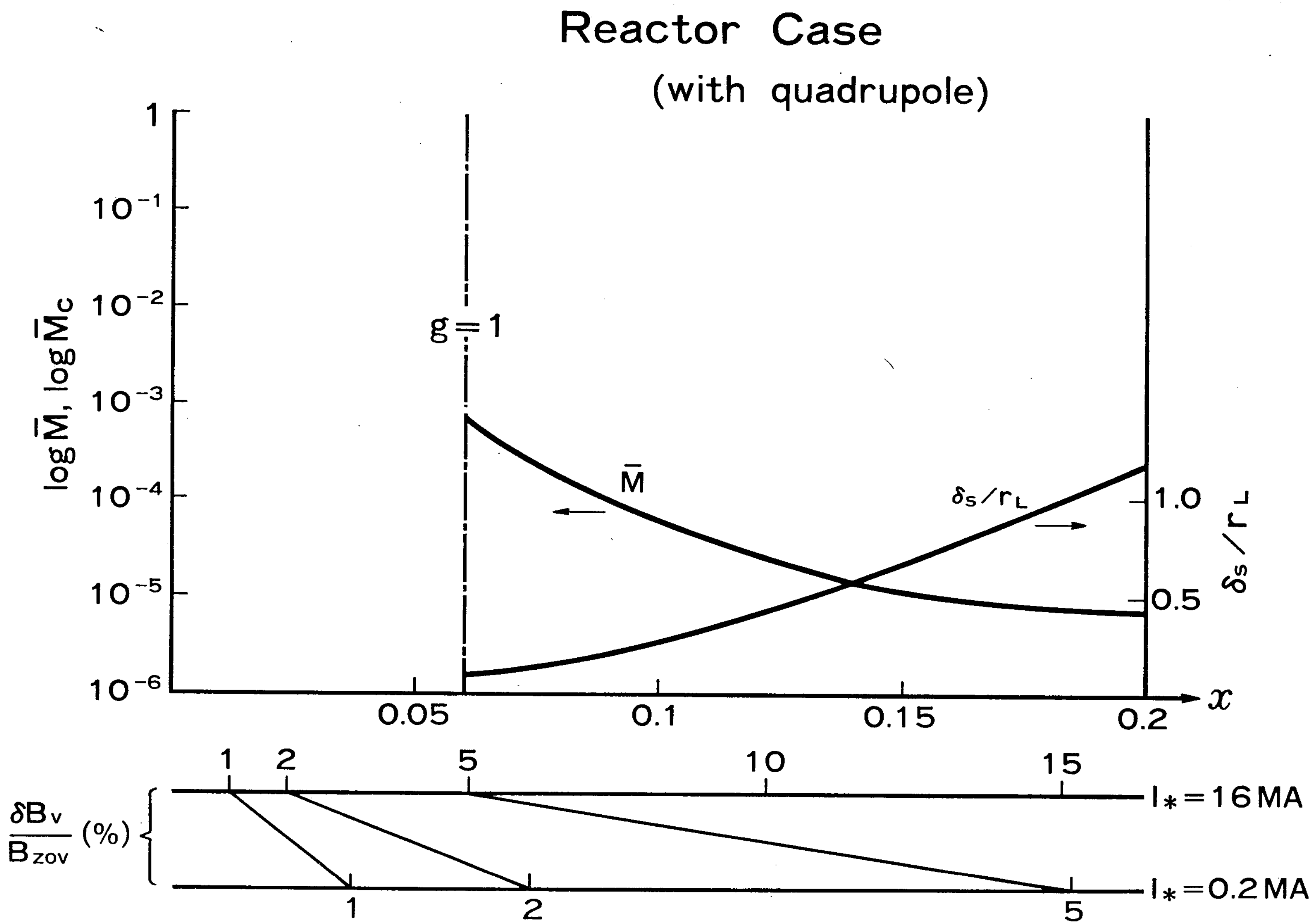
$\overline{M} = v_{i\theta}/c_s$ is the Mach number of plasma rotation, $\overline{M}_c =$ is the Mach number of the stability criterion against rotational instability, and δ_s/r_L is the skin depth of quadrupole field divided by ion Larmore radius r_L. The abscissa is $x = (\delta B_v/B_{z0v})^4 I_t/I_*$ and field ripple depth at the wall $\delta B_v/B_{z0v}$ is shown for $I_* = 10$ kA. There is no solution below $g = 1$. The parameters used for FRX-C 5 mm Torr discharge are: $B_\omega = 0.8T$, $r_s = 0.1$ m, $2L = 1.3$ m, $T_e = 175$ eV, $T_i = 625$ eV, $n_e = 1.9 \times 10^{21}$ m^{-3}, $I_t = 1.67$ MA, and $I_* = 10$ kA. A D^0 beam is injected with $E_b = 60$ keV and $I_b = 50$ Amp.

[3] Mach Number and the skin depth for modified FRX-C parameters in the presence of quadrupole field.

The magnetic field strength is small compared to Fig. 2 to reduce the Mach number of rotation. The density is also reduced to keep the beta value below unity. The parameters are: $B_\omega = 0.4T$, $r_s = 0.1$ m, $2L = 1.3$ m, $T_e = 200$ eV, $T_i = 630$ eV, $n_e = 4.5 \times 10^{20}$ m^{-3}, $I_t = 830$ kA, and $I_* = 7$ kA. A D^0 beam is injected with $E_b = 30$ keV and $I_b = 10$ Amp.

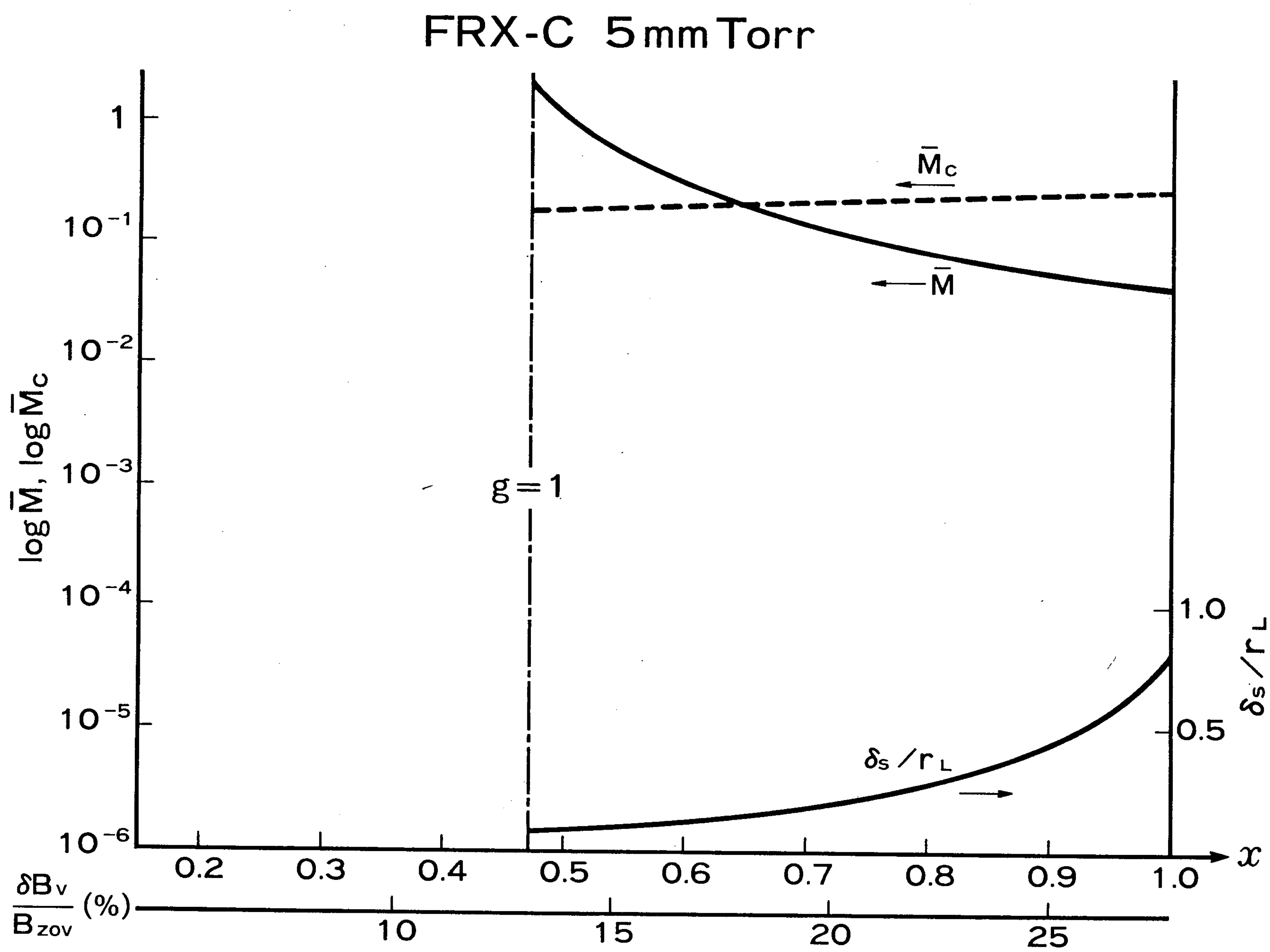
S_0	$\langle\beta\rangle$	I_*/I_t	$\bar{s}$	$(\tau_N)_{min}$ (s)	α_{max}	P_b/P_{out} (%)	αV_1	$\left\langle \frac{v_{i\theta}}{v_{thi}} \right\rangle$ $(\alpha=\alpha_{max})$	$\bar{s}\left\langle \frac{v_{i\theta}}{v_{thi}} \right\rangle$
5	0.71	0.44	10.1	1.37	185	4.75	92.0	0.476	>4.81
10	0.81	0.25	7.9	1.20	121	2.72	29.8	0.241	>1.90
20	0.90	0.09	5.1	1.10	73	1.01	5.96	0.081	>0.41
30	0.93	0.036	3.6	1.06	52	0.39	1.64	0.032	>0.12
40	0.95	0.014	2.7	1.05	40	0.15	0.489	0.012	>0.03
50	0.96	0.0056	2.2	1.03	33	0.06	0.158	0.005	>0.01

Table I



Reactor Case
(with quadrupole)
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
log M̄, log M̄c
g = 1
M̄
δs / rL
1.0
0.5
δs / rL
0.05
0.1
0.15
0.2
x
1
2
5
10
15
I* = 16 MA
δBv / Bzov (%)
1
2
5
I* = 0.2 MA

Fig. 1



FRX-C 5mm Torr
$\log \bar{M}$, $\log \bar{M}_c$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\bar{M}_c$
$\bar{M}$
g = 1
δ_s / r_L
1.0
0.5
x
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{\delta B_v}{B_{zov}}$ (%)
10
15
20
25

Fig. 2

Modified FRX-C

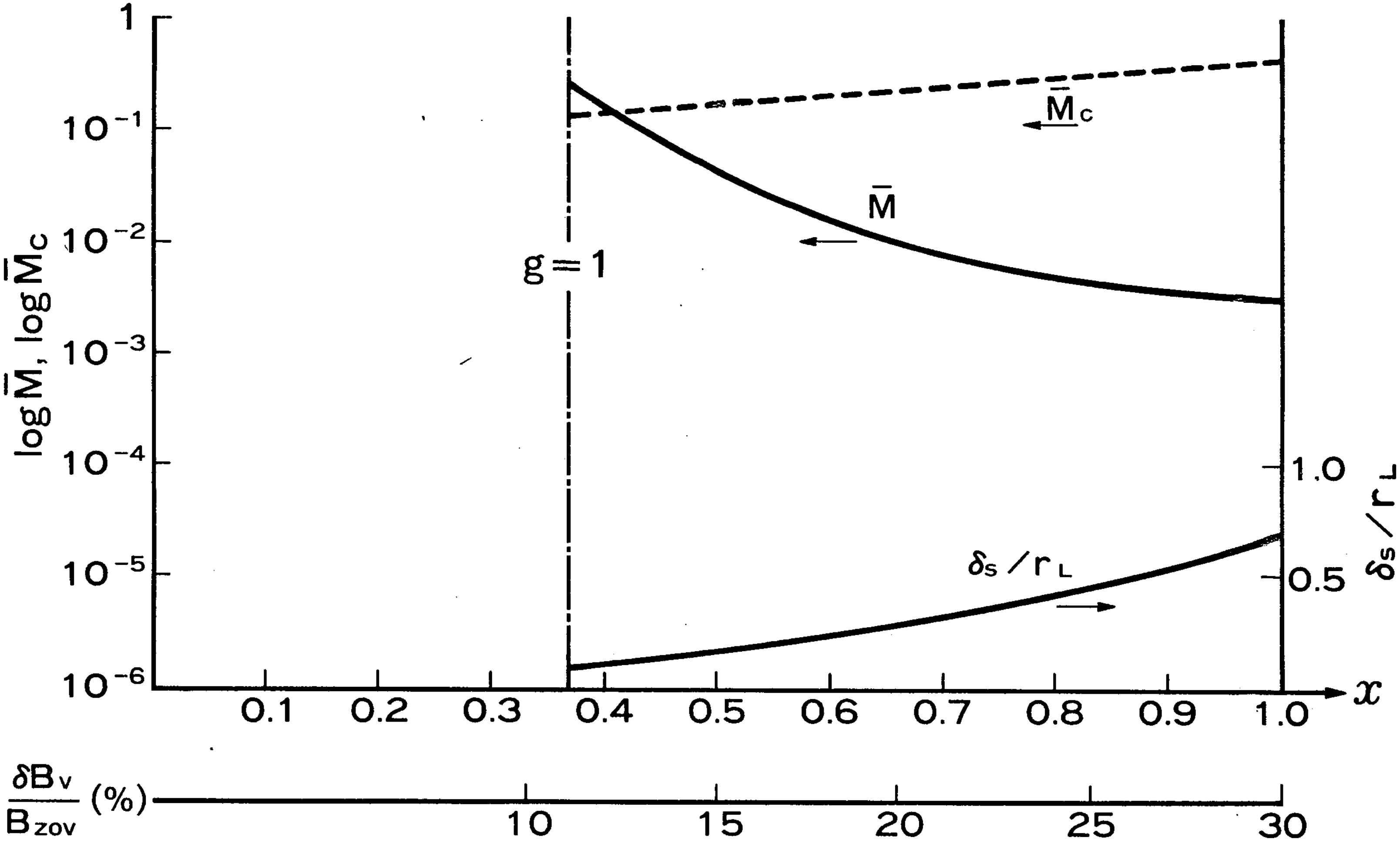


Fig. 3