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The Electron Fokker-Planck Equation for Collisions with Ions in a Magnetized Plasma

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For those electrons whose Larmor radius is less then the Debye length, by averaging over binary collisions with ions and using a guiding-center approximation, a simple collision operator has been obtained which reproduces accurately the moments obtained from the existing more complex operators.

It has generally been recognized that the Fokker-Planck equation derived by Landau[1] and subsequent workers, based on binary collisions in the absence of a magnetic field, is not correct for those electrons whose Larmor radius is less than the Debye length. A substantial fraction of the electron velocity distribution will fall in this category when $\eta \equiv \Omega_e/\omega_{pe} \tilde{>} 1$, where Ω_e and ω_{pe} are the electron cyclotron and plasma frequencies. (Tokamak plasmas satisfy this condition, for example.) The approaches to a magnetic Fokker-Planck equation[2,3,4,5,6] have involved averaging the particle response to the Fourier spectrum of the fluctuating fields associated with the discrete particle nature of the plasma. These equations contain more physics than the effect of particle collisions. The frictional drag and energy loss due to radiation by the test particle is included, which is important for super-thermal test particles, and the formulae can be applied to cases where the amplitude of the fluctuating fields is higher than thermal if the spectrum is known. However, the Fokker-Planck operators in these equations are very complex, involving many integrals. They present problems even to computers[7] and make analytic use extremely difficult.

Until recently, the one application of these equations to a stable plasma was to determine the rate of energy transfer from a Maxwellian electron velocity distribution to a Maxwellian ion distribution.[4,5,8] The result for the case where $\eta \gg 1$ is the standard Spitzer formula, $3n_e\nu_e(m_e/m_i)(T_e - T_i)$, but with the Coulomb logarithm ($\ell n\,\Lambda$) in the electron collision frequency ν_e being replaced by $\ell n(\Lambda/\eta) + \frac{1}{2}\,\ell n\eta\,\ell n(m_i/m_e)$. This expression is not accurate for η of order unity but the work of Baldwin and Watson[8] confirms that the correction is small for such plasmas. More recently, Matsuda has used the Rostoker formula[2] to compute the friction and diffusion experienced by a test electron due to interactions with a Maxwellian ion distribution. Because of computational difficulties artificially small values of the ion-electron mass ratio were taken ($m_i/m_e = 100$ to 400). The one component of the various friction and velocity diffusion components which was markedly different from the zero magnetic field case was the friction parallel to the magnetic field ($\mathbf{B}$) experienced by an electron with low $v_{||}$ but average $v_\perp$, where the subscripts refer to components $||$ and $\perp$ to $\mathbf{B}$. Extrapolating from her results, Matsuda estimated that for a test electron with $v_{||} = (T/m_i)^{1/2}$, $v_\perp = (T/m_e)^{1/2}$, $\eta = 5$ and the normal proton-electron mass ratio, the friction parallel to $\mathbf{B}$ is 10^3 times larger than predicted by the zero magnetic field collision operator. This result is clearly of importance to tokamaks and other toroidal plasmas since trapped electrons spend part of their trajectories with low velocity relative to the ions and since it is the component of friction parallel to $\mathbf{B}$ which drives neoclassical transport.

The more complex Fokker-Planck operators when applied to the case of small magnetic field and thermal particle energies reduce to expressions agreeing with the binary collision operators. A study was therefore made of binary collisions for the strong magnetic field case. A simple formula has resulted which reproduces accurately the moment results referred to above.

For simplicity only collisions between electrons and hydrogen ions are considered. For values of the impact parameter (p) less than the electron Larmor radius for a given electron velocity, the magnetic field will have a negligible effect. The collision operator for these collisions will be the same as for no magnetic field,[9,10] except that the maximum collision parameter is the electron Larmor radius ($\rho_e \equiv m_e v_\perp/eB$) instead of the Debye length (λ_D). Thus

$$C_{ei}(f_e, f_i)_{p<\rho_e} = -\Gamma \ell n \left(\frac{\rho_e}{p_m}\right) \frac{\partial}{\partial v_\alpha} \left(\frac{m_e}{m_i} f_e \frac{\partial H}{\partial v_\alpha} - \frac{1}{2} \frac{\partial^2 G}{\partial v_\alpha \partial v_\beta} \frac{\partial f_e}{\partial v_\beta}\right) , \tag{1}$$

where $\Gamma = 4\pi e^4/m_e^2$, H, G are the well-known Rosenbluth potentials

$$\left.\begin{aligned} H(\mathrm{v}) &= \int d^3\mathrm{v}_i f_i/|\mathrm{v}_i - \mathrm{v}| \\ G(\mathrm{v}) &= \int d^3\mathrm{v}_i f_i|\mathrm{v}_i - \mathrm{v}| \end{aligned}\right\} \tag{2}$$

and $\mathbf{v}$ is the particular electron velocity. If quantum mechanical effects are not important, an average value for p_m (the minimum value of p) is $e^2/3T_e$ and $\ell n(\rho_e/p_m) \simeq \ell n(\Lambda/\eta)$.

To treat the collisions with $p > \rho_e$, only the electron's lowest order guiding-center velocity $v_{||}$ will be considered which is parallel to the magnetic field $\mathbf{B}$. The cyclotron motion of the electron is neglected and it is assumed that the electron will suffer significant acceleration and displacement only in the direction of $\mathbf{B}$, acceleration and movement perpendicular to $\mathbf{B}$ being higher order in ρ_e. For the ions it is assumed that $\rho_i \gg \lambda_D$ so that an ion's trajectory in the absence of collisions can be taken as a straight line. A moving frame of reference is chosen in which the electron is at the origin with zero guiding center velocity. The z-axis is chosen parallel to the relative velocity of an approaching ion. In this moving frame the ion velocity is $u\mathbf{i}_z = (\mathbf{v}_i - \mathbf{v}_{||})$ and the ion's initial position is taken as $x = p\cos\varphi$, $y = p\sin\varphi$, $z = -\infty$. The magnetic field is taken in the xz-plane; $\mathbf{B} = \mathbf{i}_x B\sin\alpha + \mathbf{i}_z B\cos\alpha$ with $\cos\alpha = u_{||}/u$, where $u_{||}$ is the component of $u\mathbf{i}_z$ parallel to $\mathbf{B}$. Neglecting in lowest order the recoil and change in velocity of both the electron and ion, the force on the electron parallel to $\mathbf{B}$ when the ion is at position z is

$$F_{||} = \frac{e^2 p\cos\varphi\sin\alpha + e^2 z\cos\alpha}{(z^2+p^2)^{3/2}} \; . \tag{3}$$

Since $dt = dz/u$ the change in the electron guiding-center velocity due to this single collision is

$$\Delta v_{||} = \int_{-\infty}^{+\infty} F_{||}dz/m_e u = \frac{2e^2\cos\varphi\sin\alpha}{m_e u p} \; . \tag{4}$$

When averaging $\Delta v_{||}$ over φ for collisions with fixed u and p, the average is zero but not the average of $(\Delta v_{||})^2$. The number of collisions per unit time with impact parameter p is $2\pi u p dp dn_i$ with $dn_i = f_i d^3 v_i$. Hence, integrating p over the range ρ_e to λ_D

$$\left\langle (\Delta v_{||})^2 \right\rangle = \Gamma dn_i \ell n\left(\frac{\lambda_D}{\rho_e}\right)\left(\frac{1}{u} - \frac{u_{||}^2}{u^3}\right) \; ,$$

since $\sin^2\alpha = 1 - (u_{||}^2/u^2)$. Finally, integrating over the ion velocity distribution, since

$$\left[1 - (u_{||}^2/u^2)\right]/u = \partial^2 u/\partial v_{||}^2 \; , \; \left\langle (\Delta v_{||})^2 \right\rangle = \frac{4\pi e^4 \ell n\left(\frac{\lambda_D}{\rho_e}\right)}{m_e^2}\frac{\partial^2 G(v_{||})}{\partial v_{||}^2} \tag{5}$$

where $G(v_{||})$ is identical with the Rosenbluth potential defined in Eq. (2), except that $\mathbf{v}$ is replaced by $\mathbf{v}_{||}$.

Proceeding to higher order in the small parameter e^2/mu^2p and considering first the acceleration and displacement of the electron, when the ion is at position z the electron's velocity is

$$\Delta v_{||} = \int_{-\infty}^{z} F_{||} dz/m_e u = \frac{e^2}{m_e u}\left\{\frac{\cos\varphi \sin\alpha}{p}\left[1 + z(z^2+p^2)^{-1/2}\right] - \cos\alpha (z^2+p^2)^{-1/2}\right\} .$$

The electron will be displaced from the origin a distance $\Delta\ell_{||}$ parallel to $\mathbf{B}$ given by

$$\Delta\ell_{||} = \int_{-\infty}^{z} \Delta v_{||} dz/u ,$$

and since the relative velocity is now $(u\mathbf{i}_z - \Delta\mathbf{v}_{||})$, the element of time is $dz/(u - \Delta v_{||}\cos\alpha)$ to first order in $\Delta v_{||}$.

For the ion the only Δ term which contributes to first order in m_e/m_i is the recoil in the direction of the impact parameter given by

$$\Delta p_i = -\int_{-\infty}^{z} \frac{dz}{u} \int_{-\infty}^{z} \frac{e^2 p dz}{m_i u (z^2+p^2)^{3/2}} .$$

The more accurate expression for $\left\langle \Delta v_{||}\right\rangle$ is therefore

$$\left\langle \Delta v_{||}\right\rangle =$$

$$dn_i \int_{\rho_e}^{\lambda_D} dp\, pu \frac{e^2}{m_e} \int d\varphi \int_{-\infty}^{+\infty} dz \left\{\frac{\left[(p+\Delta p_i)\cos\varphi - \Delta\ell_{||}\sin\alpha\right]\sin\alpha + \left[z - \Delta\ell_{||}\cos\alpha\right]\cos\alpha}{(u - \Delta v_{||}\cos\alpha)|\mathbf{r}_i - \Delta\boldsymbol{\ell}_{||}|^3}\right\} \tag{6}$$

with $\mathbf{r}_i$ the corrected position of the ion. Linearizing the integrand in Eq. (6) with respect to the Δ terms and performing the integrals

$$\left\langle \Delta v_{||}\right\rangle = -\Gamma dn_i \ell n\left(\frac{\lambda_D}{\rho_e}\right)\left[\frac{3}{2}\left(\frac{u_{||}}{u^3} - \frac{u_{||}^3}{u^5}\right) + \frac{m_e}{m_i}\frac{u_{||}}{u^3}\right] \tag{7}$$

and integrating over the ion velocity distribution

$$\left\langle \Delta v_{||}\right\rangle = \Gamma \ell n\left(\frac{\lambda_D}{\rho_e}\right)\left[\frac{1}{2}\frac{\partial^3 G(v_{||})}{\partial v_{||}^3} + \frac{m_e}{m_i}\frac{\partial H(v_{||})}{\partial v_{||}}\right] . \tag{8}$$

Following the usual procedure[10] to obtain the standard Fokker Planck type operator, namely

$$-\frac{\partial}{\partial v_\alpha}\left(\frac{F_\alpha}{m_e} f_e - D_{\alpha\beta}\frac{\partial f_e}{\partial v_\beta}\right) ,$$

the only non-zero components of $D_{\alpha\beta}$ and F_α are $D_{\|\|}$ and $F_\|$ given by

$$D_{\|\|} = \frac{1}{2}\left\langle(\Delta v_\|)^2\right\rangle ,$$

$$\frac{F_\|}{m_e} = \left\langle \Delta v_\| \right\rangle - \frac{1}{2}\frac{\partial}{\partial v_\|}\left\langle(\Delta v_\|)^2\right\rangle = \Gamma \ell n\left(\frac{\lambda_D}{\rho_e}\right)\frac{m_e}{m_i}\frac{\partial H(v_\|)}{\partial v_\|} .$$

Whence

$$C_{ei}(f_e, f_i)_{p>\rho_e} = -\Gamma \ell n\left(\frac{\lambda_D}{\rho_e}\right)\frac{\partial}{\partial v_\|}\left[\frac{m_e}{m_i} f_e \frac{\partial H(v_\|)}{\partial v_\|} - \frac{1}{2}\frac{\partial^2 G(v_\|)}{\partial v_\|^2}\frac{\partial f_e}{\partial v_\|}\right] \tag{9}$$

and the total collision operator is the sum of Eqs. (1) and (9).

For a Maxwellian ion velocity distribution

$$\frac{\partial H(v_\|)}{\partial v_\|} = -\frac{\hat{n}_i}{v_\|^2} , \quad \frac{1}{2}\frac{\partial^2 G(v_\|)}{\partial v_\|^2} = \frac{\hat{n}_i T_i}{|v_\||^3 m_i} , \quad \frac{1}{2}\frac{\partial^3 G(v_\|)}{\partial v_\|^3} = -\frac{\hat{n}_i \left\langle v_i^2 \right\rangle}{v_\|^4}$$

where $\hat{n}_i$ is the number of ions within the sphere in velocity space with radius $|v_\||$ and $\langle v_i^2 \rangle$ is the average of v^2 for these ions. Using these expressions, the rate of transfer of energy from Maxwellian ions to Maxwellian electrons for large η is found to be the standard Spitzer formula with $\ell n\Lambda$ replaced by $\ell n(\Lambda/\eta) + \frac{1}{2}\ell n(m_i/m_e)\ell n\eta$, agreeing with result from the more complex operators.

Combining Eqs. (1) and (9) and determining the friction on a test electron one reproduces Eq. (8) plus a contribution from Eq. (1). For a Maxwellian ion distribution, with $v \gg v_{Ti}$ but $v_\|$ arbitrary,

$$\left\langle \Delta v_\| \right\rangle_m = -\Gamma\left[\frac{n_i v_\|}{v^3}\ell n\left(\frac{\rho_e}{p_m}\right) + \frac{\hat{n}_i \left\langle v_i^2\right\rangle}{v_\|^4}\ell n\left(\frac{\lambda_D}{\rho_e}\right)\right] , \tag{10}$$

where the small m_e/m_i term in Eq. (8) has been neglected and the subscript m denotes the magnetic case. The corresponding expression for zero magnetic field (subscript 0) is

$$\left\langle v_\| \right\rangle_0 = -\Gamma(n_i v_\|/v^3)\ell n\Lambda ;$$

and the ratio of these two expressions is

$$\frac{\left\langle \Delta v_\| \right\rangle_m}{\left\langle \Delta v_\| \right\rangle_0} = 1 + \left[\left(\frac{\hat{n}_i\left\langle v_i^2\right\rangle}{n_i v_{Ti}^2}\right)\left(\frac{v^3}{v_\|^3}\right)\left(\frac{v_{Ti}^2}{v_\|^2}\right) - 1\right]\frac{\ell n\frac{\lambda_D}{\rho_e}}{\ell n\Lambda} . \tag{11}$$

In Table I the numerical values of this ratio are compared with those of Matsuda[7] using the parameters chosen for illustration in her paper. (Note that in Matsuda's notation $v_j = (T/m_j)^{1/2}$. For $v_\perp = (T/m_e)^{1/2}$, λ_D/ρ_e is $\eta/\sqrt{2}$ and for $v_\parallel/(T/m_i)^{1/2} = 1,3$ the quantity $\hat{n}_i \langle v_i^2 \rangle / n_i v_{Ti}^2 = 0.220, 1.5$, where $v_{Ti}^2 = 2T/m_i$.) With the exception of the $\eta = 1$ cases, where the simple theory presented here does not apply, there is close agreement; the difference exceeds 15% in only one case. For the correct hydrogen mass with $v_\perp = (T/m_e)^{1/2}$, $v_\parallel = (T/m_i)^{1/2}$, $\eta = 5$ the ratio in Eq. (11) is 1059, close to Matsuda's extrapolated value. The corresponding ratios for the other components of the friction and for components of diffusion are found to be close to unity as found by Matsuda.

Thus, by treating the magnetized electrons ($\rho_e < \lambda_D$) as one-dimensional guiding-center particles and averaging over binary collisions a simple collision operator has been obtained [Eq. (1) plus Eq. (9)], which reproduces the moment results for particles with thermal energies which were obtained from the more complex operators.

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[1] L.D. Landau, Zh. Eksp. Teor. Fiz., **7**, 203 (1937).

[2] N. Rostoker and M.N. Rosenbluth, Phys. Fluids **3**, 1 (1960); N. Rostoker, Phys. Fluids **3**, 922 (1960).

[3] D. Montgomery, L. Turner, and G. Joyce, Phys. Fluids **17**, 954 (1974).

[4] V.P. Silin, Zh. Eksp. Teor. Fiz., **43**, 1813, (1962) (Soviet Physics JETP, **16**, 1281 1963).

[5] S. Ichimaru and M.N. Rosenbluth **13**, 2778 (1973).

[6] M.H.A. Hassan and C.J.H. Watson, Plasma Phys. **19**, 237 (1976).

[7] K. Matsuda, Phys. Rev. Lett. **49**, 1486 (1982).

[8] D.E. Baldwin and C.J.H. Watson, Plasma Phys. **19**, 517 (1977).

[9] M.N. Rosenbluth, W.M. MacDonald, and D.L. Judd, Phys. Rev. **107**, 1 (1957).

[10] B.A. Trubnikov in "Reviews of Plasma Physics," edited by M.A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 105.

TABLE I

The Ratio $\langle \Delta v_{\|} \rangle_m / \langle \Delta v_{\|} \rangle_0$
for $T_e = T_i = T$ and $v_{\perp} = (T/m_e)^{1/2}$

$\left(\frac{m_i}{m_e}\right)$	$\eta \equiv \frac{\Omega_e}{\omega_{pe}}$	$\langle \Delta v_{\|} \rangle_m / \langle \Delta v_{\|} \rangle_0$			
		$v_{\|} = 3(T/m_i)^{1/2}$		$v_{\|} = (T/m_i)^{1/2}$	
		Matsuda	Eq. (11)	Matsuda	Eq. (11)
100	1	-		1.9	1
	5	1.81	1.9	9.56	9.62
	10	2.24	2.39	14.4	14.4
	100	-		35.5	30.1
200	1	-		4.87	1
	5	3.56	3.68	26.6	24.5
	10	-		39.9	37.0
300	5	5.89	5.98	49.3	46.1
400	1	2.18	1	14.7	1
	5	8.81	8.71	76.0	70.4
	10	12.5	13.0	113	108
	100	21.6	27.0	280	234