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Nonlinear Evolution of a Force-Free Arcade Field Driven by Shear Flow

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Abstract

The nonlinear evolution of a force-free magnetic field driven by slow shear flow on a boundary is investigated analytically and numerically. The Woltjer-Taylor theory is generalized to obtain the time-dependent magnetohydrodynamic (MHD) solution for the nonlinear force-free fields driven by shear flow. This solution agrees well with our resistive MHD simulations, which show that the force-free arcade field in a solar coronal plasma evolves quasi-statically through a series of force-free states before reconnection occurs.

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Since the Sun provides a natural laboratory, the dynamics of solar magnetic fields and plasmas can play an important role in providing an understanding of the behavior of basic plasmas. A crucial difference between a laboratory plasma on Earth and a solar plasma involves the problem of boundary conditions.

Taylor[1] proposed a theory for the relaxation of plasma discharges[2] bounded by electrically conducting walls. In Taylor's theory, a slightly resistive plasma is assumed to relax to an equilibrium of minimum magnetic energy subject to the invariance of the magnetic helicity. The state of minimum energy is given by the linear force-free relation[3]

$$\nabla \times \mathbf{B} = \alpha_T \mathbf{B} \ , \tag{1}$$

where $\mathbf{B}$ is the magnetic field and α_T *is a single constant having the same value on all field lines* (Taylor's conjecture). However, observations of the force-free arcade fields in the solar corona indicate that α_T varies in space.[4] Taylor's conjecture needs to be modified if α_T has spatial and temporal dependence as a result of energy supplied through the boundary. Such is the case for coronal plasmas.[5–7] (A similar situation may be found in the reversed field pinch sustained by an external electric field.[8–10]) Instead of Eq. (1), therefore, we consider a nonlinear force-free field,[11]

$$\nabla \times \mathbf{B} = \alpha(\mathbf{x}, t)\mathbf{B} \ , \tag{2}$$

where α is a scalar function of position $\mathbf{x}$ and time t.

A useful method for treating basic plasma problems is to separate relatively slow processes from much faster processes that typically occur on the Alfvén transit time τ_A. For the coronal arcade problem, we assume that the fluid shear motion on the boundary is slow, but not as slow as the resistive time τ_R, i.e., $\tau_A \ll \tau_v \ll \tau_R$, where $\tau_v = L/v$, with L the characteristic length and v the fluid velocity. For coronal conditions, the plasma thermal energy is small compared with the magnetic energy, and the magnetic field may be assumed to be in a force-free state of quasi-static evolution through a series of equilibria. Heyvaerts and Priest[7] applied Taylor's conjecture to the coronal plasma and discussed the coronal heating to be expected from complex reconnection processes. They assumed that α is uniform even before reconnection takes place. Vekstein[12] subsequently gave an explicit form of the linear solution of Eq. (1) under the assumption of uniform α. These theories[7,12] can explain neither the solar observations of nonuniform α nor our resistive magnetohydrodynamic (MHD) simulations. The aim of the present Letter is to extend the Woltjer-Taylor theory and to obtain the time-dependent MHD solution of Eq. (2). In particular, our interest concerns on the effects of slow shear motion on the force-free state specified by $\alpha(\mathbf{x}, t)$.

From Faraday's and Ohm's laws, we have

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}) \ , \tag{3}$$

where $\mathbf{J} = \nabla \times \mathbf{B}$ is the current density and η is the resistivity. Integration of Eq. (3) yields $\partial_t \mathbf{A} = \mathbf{v} \times \mathbf{B} - \eta \mathbf{J} + \nabla G$, where $\mathbf{A}$ is the vector potential for the magnetic field ($\mathbf{B} \equiv \nabla \times \mathbf{A}$), and $G(\mathbf{x}, t)$ is an arbitrary gauge function. The velocity $\mathbf{v}$ is specified at the boundary.

The magnetic helicity is defined by $K = \int \mathbf{A} \cdot \mathbf{B} d^3\mathbf{x}$ and its time evolution is given by

$$\begin{aligned} \partial_t K \ &= \ \oint G\mathbf{B} \cdot d\mathbf{S} + \oint (\mathbf{A} \cdot \mathbf{v}) \mathbf{B} \cdot d\mathbf{S} - \oint (\mathbf{A} \cdot \mathbf{B}) \mathbf{v} \cdot d\mathbf{S} \\ &\quad - \oint \eta \mathbf{J} \times \mathbf{A} \cdot d\mathbf{S} - 2 \int \eta \mathbf{B} \cdot \mathbf{J} d^3\mathbf{x} \ , \end{aligned} \tag{4}$$

where $\oint d\mathbf{S}$ is the surface integral at any surfaces on which the fields become discontinuous. The first term on the right-hand side of Eq. (4) does not vanish because the coronal fields are not bounded by magnetic surfaces; the second term represents the helicity increment injected into the corona by footpoint motion; the third term vanishes because $\mathbf{v} \cdot d\mathbf{S} = 0$ on the surface; and the fourth and fifth terms correspond to helicity damping due to the resistivity, which is usually very small but nonzero. Equation (4) contains Woltjer's theorem,[3] $\partial_t K = 0$, in the case of a bounded ideal MHD plasma without shear flow.

We adopt Cartesian coordinates (x, y, z), where $x(\geq 0)$ corresponds to the vertical height from the photospheric surface. The system is taken to be periodic in the y- and z-directions, with the periodic lengths being L_y and L_z. Hereafter all quantities are normalized in terms of the characteristic length L_y, the Alfvén time τ_A, and the magnitude of the initial magnetic field B_0.

The possible application to the practical boundary-value problem will be confined to the two-dimensional case in which all physical quantities are independent of the z-coordinate. The quasi-state force-free field is given by

$$\mathbf{B} = \nabla\psi \times \hat{\mathbf{z}} + B_z\hat{\mathbf{z}} \ , \tag{5}$$

where $\psi(x, y, t) = A_z$ is the flux function, which satisfies $\mathbf{B} \cdot \nabla\psi = 0$ and the boundary condition $\lim_{x\to\infty} |\psi| = 0$. Note that α depends on position only through ψ; the functional form of $\alpha = \alpha(\psi)$ remains to be specified.

Let us consider a topology-preserving linear transformation (or Möbius transformation):

$$\alpha^2(\psi) = \beta^2 \frac{a + \psi}{a' + \psi} \ , \tag{6}$$

where β, a, and a' are adiabatic constants. When $a = a' = 0$, Eq. (6) reduces to the constant-α theory, for which $\alpha = \beta$. Since $B_z = \int d\psi\alpha$, we obtain

$$B_z = \left\{ \sqrt{(\beta\psi + 1)(\beta\psi + \epsilon)} + \frac{1}{2}\ell n \left| \frac{\sqrt{\beta\psi + 1} - \sqrt{\beta\psi + \epsilon}}{\sqrt{\beta\psi + 1} + \sqrt{\beta\psi + \epsilon}} \right| \right\} \mathrm{sgn}(\psi) \ , \tag{7}$$

where $\epsilon = a\beta^2/a'$. Noting that the magnetic energy $W = \frac{1}{2}\int (B_z^2 + |\nabla\psi|^2) d^3x$ must be finite, which is ensured by having $B_z \to 0$ with ψ at infinity, we conclude that $a = 0$ for this semi-infinite problem. Then, taking $a' = 1/\beta$, we obtain a single-parameter functional form $\alpha(\psi, t) = \beta(t) \left[|\psi|/(|\psi| + \beta^{-1}) \right]^{1/2}$. From the force-free relationship, we have a nonlinear Grad-Shafranov type of equation,

$$\nabla^2\psi + \beta^2\psi - \beta \left(\frac{\beta\psi}{\beta\psi + 1} \right)^{1/2} \ln \left| \sqrt{\beta\psi + 1} + \sqrt{\beta\psi} \right| = 0 \ , \tag{8}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2$. The geometrical parameter $\beta(t)$ represents the quasi-static stretch of the magnetic field line in the vertical direction due to the shear flow and may therefore be assumed to be a monotonically increasing function of time, before sudden relaxation. For the linear stage ($\mid \psi\beta \mid \ll 1$), Eq. (8) becomes approximately $\nabla^2\psi \simeq 0$, whose solution for any arcade configuration is given by

$$\psi = \cos(ky)\exp(-kx) \ , \tag{9}$$

where $k = 2\pi$ (with $L_y = 1$). Likewise, for the nonlinear stage ($\mid \psi\beta(t) \mid \gg 1$), we have $\nabla^2\psi + \beta^2\psi \simeq 0$, and

$$\psi = \cos(ky)\exp\left[-x\sqrt{k^2 - \beta^2}\,\right] \ , \tag{10}$$

where the time dependence of ψ is contained in $\beta(t)$, with $0 \leq \beta(t) \leq k$. Since Eq. (10) coincides with Eq. (9) when $\mid \beta \mid \ll k$, it may be employed as an approximate flux function to represent the general force-free arcade field solution of Eq. (8); deviations of order $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^{-1})$ for small and large β, respectively, have been estimated by perturbation theory.

Using Eq. (10), we then evaluate the relevant terms in the evolution equation, Eq. (4), to determine the time dependence of $\beta(t)$. The expression for the magnetic helicity can be reduced to a single integral over the y-coordinate, numerical analysis of which shows that $K(\beta)$ can be well approximated by

$$K(\beta) \approx \frac{\beta^{1.25}/4}{\sqrt{k^2 - \beta^2}} \tag{11}$$

over the range $1 < \beta \leqq 2\pi$. Also, for $\beta < 1$, the expression $K(\beta) \cong 0.04\beta^{3/2}$ can be derived exactly. If we take the shear flow on the boundary ($x = 0$) to have a velocity given by

$$\mathbf{v}(0, y, t) = v(t)\sin(2ky)\hat{z} \ , \tag{12}$$

then the helicity evolution equation becomes

$$\partial_t K(\beta) \cong \frac{k}{2}v(t) + \mathcal{O}(\eta_0) \tag{13}$$

where η_0 is the resistivity at $x = 0$. The first term on the right-hand side of Eq. (13) represents the driving source due to the shear flow, whereas the second represents the small damping term due to the finite resistivity on the photosphere, $\eta_0 = S^{-1}$, usually negligibly small, where $S(= \tau_R/\tau_A)$ is the Lundquist number. If we approximate $\beta^{1.25} \cong k^{0.25}\beta$ in the expression of Eq. (11) for $K(\beta)$, since $\beta \simeq k$ in the nonlinear stage, and take $v(t) \equiv V_0 t^n$, a tractable MHD solution of Eq. (13) can be derived:

$$\beta(t) = \frac{k^2 \xi_n(t)}{\sqrt{1 + (k\xi_n(t))^2}} , \tag{14}$$

where

$$\xi_n(t) = \hat{V}_0 t^{n+1}/(n+1) \quad \text{for} \quad \eta_0 = 0$$

with $\hat{V}_0 = 0.69V_0$. Equation (14) is a nonlinear solution to the problem of describing the dynamics of the nonlinear force-free state driven by shear flow, approximately correct in the limits both of $\beta \ll 1$ and $\beta > 1$. Using this MHD solution, we obtain the evolution of the magnetic energy :

$$W(t)/W(0) \cong \sqrt{1 + (k\xi_n(t))^2} , \tag{15}$$

Thus, the incremental magnetic energy is an explicit function of the footpoint displacement $\xi_n(t)$.

In our resistive MHD simulations, we numerically solved Eq. (3) and the equation of motion with constant density, where the convective and viscosity terms were neglected as being small. This set of coupled MHD equations was integrated in a cubic box of size $L_x \times L_y \times L_z$ with $64 \times 64 \times 4$ grid points (the number of grids in the z-direction being immaterial) by use of a semi-implicit code.[14] The upper boundary at $x = L_x$ is an artificial numerical boundary, placed sufficiently far away, typically at $L_x/L_y = 10$, with $L_y = 1 = L_z$, so as not to affect the evolution of the configuration. The grid in the x-direction is chosen to be exponential. The initial field, which can be obtained from Eq. (9), is referred to as an arcade. In our simulation runs we use the shear velocity profile given in Eq. (12), with

$$v(t) = \begin{cases} v_0 t/\tau_{\mathrm{ramp}} \ , & \text{for } 0 \le t \le \tau_{\mathrm{ramp}} \ , \\ v_0 \ , & \text{for } t > \tau_{\mathrm{ramp}} \ , \end{cases} \tag{16}$$

where τ_{ramp} is the interval driving which the driving velocity is ramped up linearly in time to a specified value of $v_0 = V_0 \tau_{\mathrm{ramp}}$. The resistivity and the initial force-free state were chosen to be uniform in space, with $\alpha(x, y, t = 0) = 10^{-2}$.

We can now directly compare the simulation results with the ideal MHD solution derived earlier. In the simulation, it is convenient to use an average global quantity $\tilde{\alpha}$ in place of $\alpha(x, y, t)$; we will take the B^2-weighted spatial average,

$$\tilde{\alpha}(t) = \frac{\int_0^\infty dx \int_0^1 dy \alpha(x, y, t) \mathbf{B}^2}{\int_0^\infty dx \int_0^1 dy \mathbf{B}^2} \ , \tag{17}$$

which tends to filter away the regions where α is small. Substituting Eqs. (6) and (10) into Eq. (17), and using our numerical fits valid for $\beta \ge 1$, in the nonlinear stage we have $\tilde{\alpha}(t) \approx (19.8\beta) \left[\frac{\pi}{4}\left(k^2 - \frac{1}{2}\beta^2\right) + 0.08\beta^{2.5}\right]^{-1}$.

Figure 1 compares the numerical and various analytical results. Curve C in Fig. 1 shows the ideal MHD ($\eta_0 = 0$) exact nonlinear analytic solution of Eq. (13), with $K(\beta)$ represented by its $\beta < 1$ and $\beta \ge 1$ forms, given earlier. This solution tends to agree with the result of the resistive MHD simulations (curve B). The linear solution[12] (curve A) is clearly valid only at small t. The overall functional dependence of theory is in good agreement with simulation. The main quantitative deviation of our theory, viz., the slightly smaller asymptotic value of $\tilde{\alpha}$ at large times, arises from the fact that Eq. (6) does not include the overshoot feature that can be seen near $\psi \sim 0.3$ in Fig. 2(c). The four values for $\tilde{\alpha}$ indicated by triangles are computed from a "hybrid" B^2-weighted average that uses the simulation results for $\alpha(\psi)$ at four discrete times, but with B^2 obtained from the theoretical flux function, given in Eq. (10); these four points show good agreement with the simulation results of curve B.

Figure 2(a) shows the simulation result for the spatial dependence of $\alpha(x, y, t) \equiv \mathbf{J} \cdot \mathbf{B}/\mathbf{B}^2$ at the last of these four times, namely, $t = 760(\tau_A)$. Note that Fig. 2 suggests that Eq. (6) is approximately valid (apart from the overshoot feature). As can be seen in Fig. 2(b), the behavior of the magnetic flux $\psi(x, y, t)$ is quasi-static before the force-free state breaks down at time t^*. The present theory does not attempt to explain any catastrophic phenomenon after reconnection, corresponding to rapid energy release (on the τ_A time scale). Another nonlinear theory would need to be constructed to explain such phenomena.

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Figure Captions

1. Time-evolution of the nonlinear force-free state specified by α. Curves A, B, and C correspond to the linear theory,[12] the simulation result for $\tilde{\alpha}$, and the ideal MHD solution of Eq. (13), respectively. The triangles indicate the hybrid average values discussed in the text. These simulation results were for $S = 2 \times 10^4$, $v_0 = 0.01$, and $\tau_{\text{ramp}} = 2 \times 10^3$.

2. (a) Spatial dependence of $\mid \alpha(x, y, t) \mid$. (b) Contour profile of the quasi-static flux function $\mid \psi(x, y, t) \mid$. (c) $|\alpha(\psi)|$ as a function of flux ψ. All shown at $t = 760$ for the parameters indicated in Fig. 1.

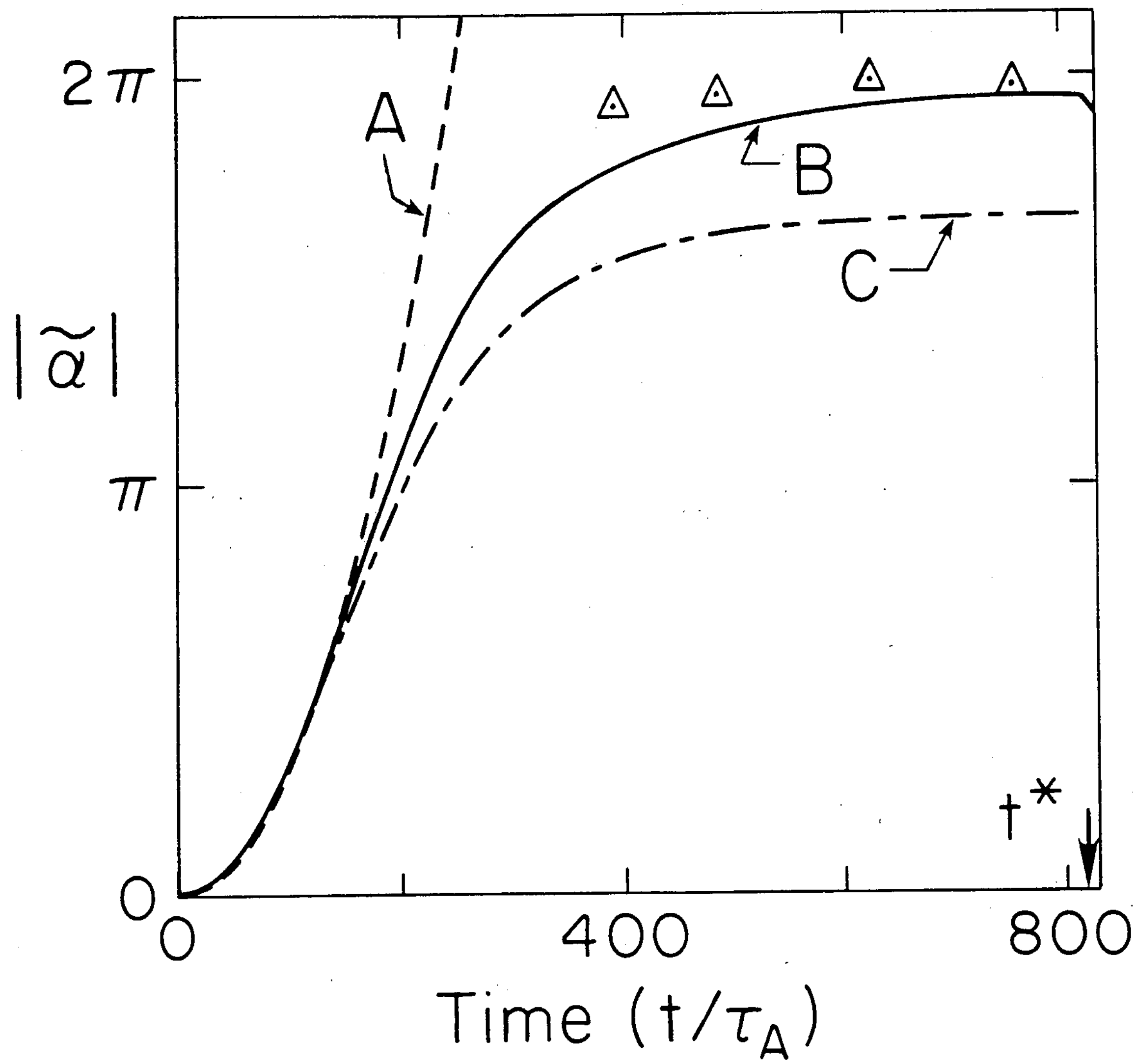
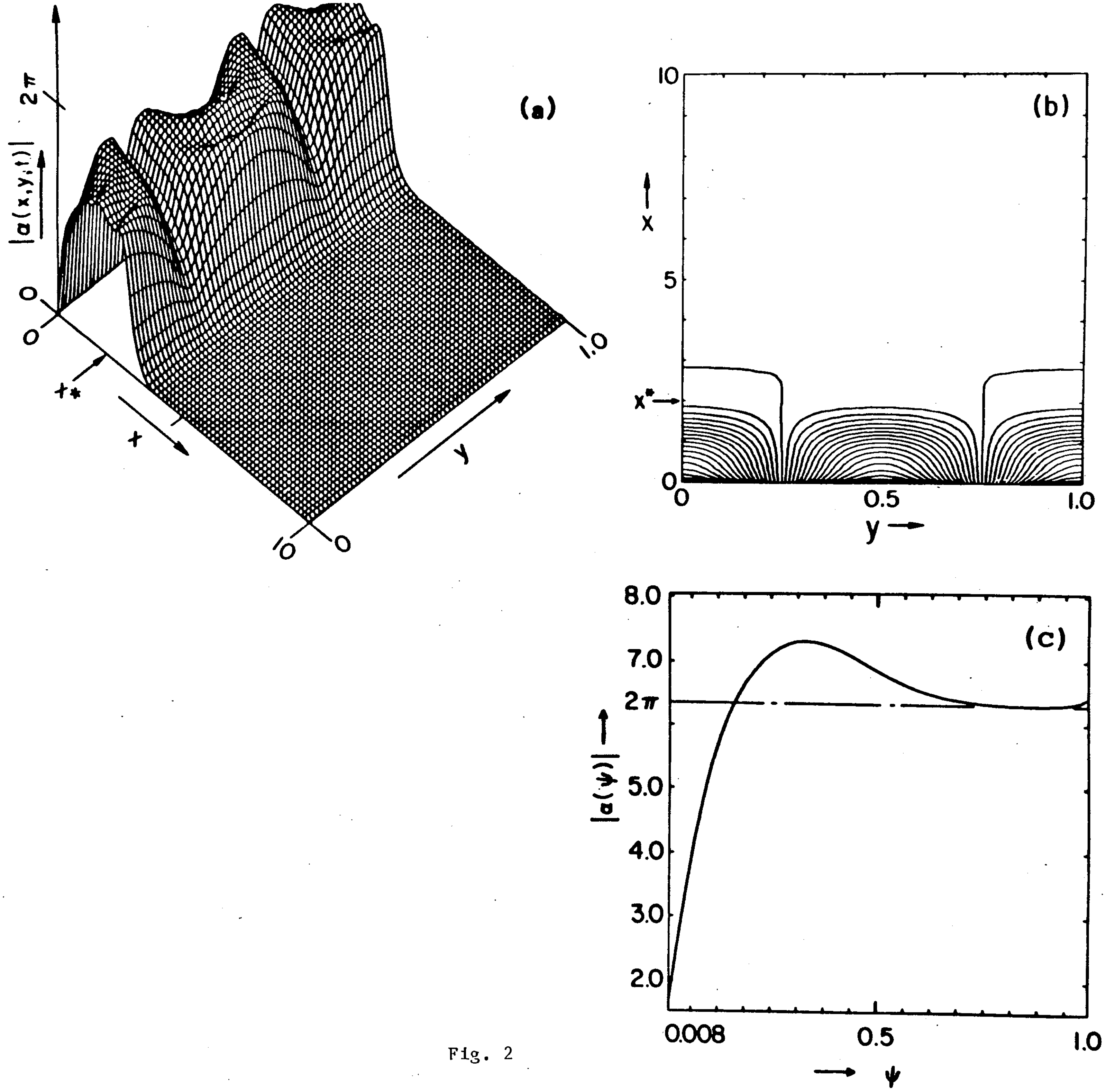


Fig. 1



(a)
2π
|a(x,y,t)|
0
0
x*
x
y
10
0
1.0
(b)
10
x
5
x*
0
0
0.5
1.0
y
(c)
8.0
7.0
2π
5.0
4.0
3.0
2.0
|a(ψ)|
0.008
0.5
1.0
ψ

Fig. 2