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Exact and Almost Exact Solutions to Vlasov-Maxwell System

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Abstract

Exact and almost exact solutions to the Vlasov-Maxwell systems describing a variety of plasma configurations with density, temperature and current gradients, are presented. Possible consequences of these solutions are also discussed.

I. Introduction

Exact nonlinear solutions to coupled field theories are generally rare. For the Vlasov-Maxwell system (describing a collisionless plasma) the list of reported exact solutions is quite short; there are the electrostatic solutions of the Bernstein-Green-Kruskal (BGK)[1] type, and the magnetic solutions of the kind obtained by 1) Pfirsch,[2] 2) Laval,[3] Pellat and Vuillemin, 3) Marx[4] and 4) Harris[5]. The magnetic solutions were constructed to model the behavior of laboratory plasma containment devices. It was assumed that only the plasma density was a function of space while plasma temperature and current were taken to be spatially uniform. The process leads to the well-known Bennet pinch' density profiles in the cylindrical geometry, and to the strongly localized $\text{sech}^2 x/\delta$ (where δ is some appropriate length) profiles in the slab model.

A more realistic description of the current laboratory plasmas, however, would require the inclusion of the temperature as well as the current gradients (i.e., gradients in current which are in addition to the automatic gradients due to the density dependence of current.) This paper is an attempt to incorporate these essential elements in the theory, i.e., we construct exact and almost exact solutions to the Vlasov-Maxwell system describing plasmas with gradients. The calculations can be carried out both in the slab and in the cylindrical geometry; the slab solution can also be made into time-dependent travelling pulse solutions.

In Sec. II, we deal with the simplest case of a plasma with only density gradients. A part of this section is in the nature of a review, and the rest contains new results including solutions with nonzero electric field.

In Sec. III, we introduce current gradients by imparting spatial dependence to the drift speed u though the plasma temperature is kept constant. It is found that the maintenance of an exact equilibrium demands that the temperatures along and perpendicular to the direction of the current be different. A wide variety of equilibrium profiles (the density n, the current j, the self-consistent magnetic field b) emerge as the anisotropy parameter $\Delta T = T_{||} - T_{\perp}$ is varied over a range.

The problem of finding an exact solution becomes very difficult when the temperature gradients are also taken into consideration. We were, however, able to construct an exact infinite series solution to the Vlasov-Maxwell system. This series can be readily truncated for plasma existing in most of the fusion related machines in which the electron drift speed u_e (drifts which cause the current) is much smaller than the electron thermal speed $v_e(u_e/v_e \ll 1)$. Deriving equilibrium profiles for a variety of configurations is the subject matter of Sec. IV. Depending upon the nature of equilibrium currents, we obtain pinch-like (current only in the axial direction) or tokamak-like (current in both the axial and azimuthal directions) profiles. It is interesting to note that for all of these magnetically confined plasmas, there exists a natural scale length $\delta \sim (c/\omega_{pe})(v_e/u_e)$ which is a scaled version of the collisionless skin depth (c/ω_{pe}), and is the principal control parameter of the equilibrium. For the present day high temperature laboratory plasmas $\delta \gg c/\omega_{pe}$ because $v_e/u_e \gg 1$. It would seem obvious that the profiles of fusion machines should be better understood in terms of this intrinsic parameter than the mirror radius a of the machine provided $\delta < a$ (If $a < \delta$, then the edge effects will be far too important to be neglected).

In Sec. V, we give a brief summary and make some speculations based on our time-dependent solutions of Sec. IIA.

II. General Formalism

Throughout this paper, we deal with a two-component plasma (electrons and ions) embedded in an external axial magnetic field $\mathbf{B}_{\text{ext}} = B_0\hat{e}_z$. Although equilibrium studies in cylindrical geometry form the bulk of this work, we have also dealt with the time-dependent problem in Cartesian geometry. We shall later speculate on the meaning of these time-dependent solutions.

A. Time-Dependent Solutions in Cartesian Coordinates

Assuming that the field quantities vary only in the x-direction $(\partial/\partial y = 0 = \partial/\partial z)$, the relevant equations describing the system can be written as

$$\frac{\partial f_e}{\partial t} + v_x\frac{\partial f_e}{\partial x} - \frac{e}{m_e}\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times (\mathbf{B} + \hat{e}_z B_0)\right] \cdot \frac{\partial f_e}{\partial \mathbf{v}} = 0 \tag{1a}$$

$$\frac{\partial f_i}{\partial t} + v_x\frac{\partial f_i}{\partial x} + \frac{e}{m_i}\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times (\mathbf{B} + \hat{e}_z B_0)\right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \tag{1b}$$

$$\frac{\partial E_x}{\partial x} = 4\pi\rho = -4\pi e \int d\mathrm{v}(f_e - f_i) \tag{2}$$

$$\frac{\partial B_x}{\partial x} = 0 \tag{3}$$

$$\hat{e}_x \times \frac{\partial}{\partial x}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4}$$

$$\hat{e}_x \times \frac{\partial}{\partial x}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} = -\frac{4\pi e}{c}\int d\mathrm{v}\ \mathrm{v}(f_e - f_i) \tag{5}$$

where Eqs. (1a) and (1b) are the Vlasov equations for the evolution of the electron (f_e) and ion (f_i) distribution function in the presence of the external field $\mathbf{B}_{\text{ext}}$, and the self-consistently generated fields **E** and **B** which obey Maxwell's equations (2-5). In Eqs. (1-5), $-e$ is the charge of the electron, m and M are respectively the electron and ion masses, and ρ and **J** are the plasma charge and current densities.

We begin by proposing travelling displaced Maxwellian distributions for the particles

$$f_{e,i} = \frac{n_0}{\pi^{3/2} v_{e,i}^3} \exp\left[-\frac{(\mathrm{v} - \mathbf{u}_{e,i})^2}{v_{e,i}^2}\right] \; g_{e,i}(x - Ut) \tag{6}$$

where n_0 is a measure of the ambient density, $v_{e,i} = 2T_{e,i}/m_{ei}$, $\mathrm{u}_{e,i}$ are respectively the thermal speeds and the drift velocities, and $T_{e,i}$ are the temperatures. In this simple case, the entire space-time dependence is contained in the factor $g_{e,i}$ which depends only on the variables

$$\eta = x - Ut \; . \tag{7}$$

Making use of Eq. (6), one obtains expressions for the charge and current densities (explicit arguments will be dropped),

$$\rho = -en_0(g_e - g_i) \; , \tag{8}$$

$$\mathbf{J} = -en_0(g_e \mathbf{u}_e - g_i \mathbf{u}_i) \; , \tag{9}$$

to be substituted into the right-hand sides of Eqs. (2) and (5) which become ($\partial/\partial t = -U d/d\eta$ and $\partial/\partial x = d/d\eta$)

$$\frac{dE_x}{d\eta} = -4\pi e n_0(g_e - g_i) \tag{10}$$

$$\frac{d}{d\eta}\left[E_y - \frac{U}{c}B_z\right] = 0 \tag{11}$$

$$\frac{d}{d\eta}\left[E_z + \frac{U}{c}B_y\right] = 0 \tag{12}$$

$$\frac{dB_z}{d\eta} = \frac{4\pi e n_0}{c}\left[g_e u_y^e - g_i\, u_y^i\right] + \frac{U}{c}\frac{dE_y}{d\eta} \tag{13}$$

$$\frac{dB_y}{d\eta} = -\frac{4\pi e n_0}{c}\left(g_e u_z^e - g_i u_z^i\right) - \frac{U}{c}\frac{dE_z}{d\eta}\ , \tag{14}$$

and $B_x = 0$. Equations (11) and (12) immediately yield

$$E_y = \frac{U}{c}(B_z + B_0) \tag{15}$$

$$E_z = -\frac{U}{c}B_y \tag{16}$$

where $U/c\,B_0$ is the integration constant accounting for the external magnetic field. Equations (15) and (16) clearly indicate that the electromagnetic fields arrange themselves in such a manner that there is no effective electric field in a frame moving with the plasma (In the main text we deal only with the nonrelativistic approximation; the relativistic generalization is provided in Appendix B).

With the realization that the drift speed of the particles is given by

$$\mathrm{u}_{e,i} = \hat{e}_z u_z^{e,i} + \hat{e}_y\ u_y^{e,i} + \hat{e}_x U\ , \tag{17}$$

substituting Eq. (6) into (1a) and (1b) leads to the differential equations which must be satisfied by g_e and g_i,

$$\frac{1}{g_e}\frac{dg_e}{d\eta} = \frac{e}{cT_e}\left[B_y u_z^e - (B_z + B_0)u_y^e\right] - \frac{e}{T_e}E_x \tag{18}$$

$$\frac{1}{g_i}\frac{dg_i}{d\eta} = \frac{e}{cT_i}\left[-B_y u_z^i + (B_z + B_0)u_y^i\right] + \frac{e}{T_i}E_x\ . \tag{19}$$

Equations (18-19) are the manifestation of the constraints imposed by the Vlasov equation and are to be solved in conjunction with Eqs. (10)-(11) for the electromagnetic fields and the particle distribution functions.

At this stage, it is important to distinguish between two important limits of the system. The first limit is when charge neutrality ($g_e = g_i$) prevails at all times implying $E_x = 0$, and the problem reduces to an essentially magnetic one with only inductive electric fields present. The second limit is when the charge separation electric field E_x dominates, and the system is almost equivalent to a Vlasov-Poisson system. We find that in the latter case, i.e., the Poisson-like (P) situation, it is not easily possible to construct nonsingular continuous solutions. As a result, we shall not discuss the P solutions in the main text. However, we do provide some details about the Vlasov-Poisson system in Appendix A.

Our main concern here is the second type of solution where the magnetic field (or the forces due to it) dominates the charge separation electric field. This is indeed the most relevant case for the confined plasmas.

Substituting Eqs. (15) and (16) into (13) and (14), it is clear that for $u_y^i/u_y^e = u_z^i/u_z^e \equiv \mu$, $B_z = (-u_y^e/u_z^e)B_y$, and only one of the two equations remains independent. With appropriate redefinitions, the set of Eqs. (10), (14), (18) and (19) becomes

$$\frac{dE}{d\eta} = -\frac{1}{\lambda_D^2}(g_e - g_i) \tag{20}$$

$$\frac{db}{d\eta} = -\frac{1}{\delta_e^2}(g_e - \mu g_i) \tag{21}$$

$$\frac{1}{g_e}\frac{dg_e}{d\eta} = -E + b \tag{22}$$

$$\frac{1}{g_i}\frac{dg_i}{d\eta} = \frac{1}{c}(E - \mu b) \tag{23}$$

where $E = eE_x/T_e$, $b = (ecu_e^2/u_z^e T_e)B_y$, $u_e^2 = (u_y^e)^2 + (u_z^e)^2$, $\tau = T_i/T_e$, $\lambda_D = (T_e/4\pi n_0 e^2)^{1/2}$ is the electron Debye length, and $\delta_e = (T_e/4\pi n_0 e^2)^{1/2}(c/u_e)\gamma_U^{-1}$ $\left[\gamma_U^{-2} = 1 - U^2/c^2\right]$ can be viewed as enhanced Debye length or modified collisionless skin depth. However we may choose to interpret δ, it plays the same crucial role in the magnetic problem as λ_D plays in the electrostatic problem. We shall see later that δ is the fundamental intrinsic scale length for magnetically confined plasmas.

To illustrate the full potential of this system, let us consider a partially neutralized plasma with

$$g_i = \alpha g_e \tag{24}$$

where α is independent of η. Incorporation of Eq. (24) immediately leads to two algebraic constraints equations:

$$E = \frac{\tau + \mu}{1+\tau} b \equiv \kappa b \tag{25}$$

relating E and b, and

$$\alpha = \frac{\kappa - \delta_e^2/\lambda_D^2}{\mu\kappa - \delta_e^2/\lambda_D^2} \tag{26}$$

which fixes α in terms of the intrinsic plasma parameters. The rest of the system is described by two differential equations

$$\frac{db}{d\eta} = -\frac{(1-\alpha\mu)}{\delta_e^2} g_e \equiv -\frac{1-\mu}{\delta_e^2(1-\kappa\mu\delta_e^2/\lambda_D^2)} g_e \ , \tag{27}$$

$$\frac{1}{g_e}\frac{dg_e}{d\eta} = \frac{1-\mu}{1+\tau} b \ , \tag{28}$$

that are readily combined to yield

$$\frac{d}{d\eta}\frac{1}{g_e}\frac{dg_e}{d\eta} = -\frac{(1-\mu)^2}{(1+\tau)\delta_e^2}\frac{1}{(1-\kappa\mu\lambda_D^2/\delta_e^2)} g_e \tag{29}$$

which has a nonsingular confined solution ($\eta = x - Ut$)

$$g_e = \mathrm{sech}^2 \frac{\eta(1-\mu)(1-\kappa\mu\lambda_D^2/\delta_e^2)^{-1/2}}{\sqrt{2}|\delta_e|(1+\tau)^{1/2}} \equiv \mathrm{sech}^2 \frac{\eta}{\bar{\delta}} \tag{30}$$

provided $1 > \kappa\mu\lambda_D^2/\delta_e^2 = \kappa\mu(\mu_e^2/c^2)(1-U^2/c^2)^{-1}$, a condition which is quite easy to satisfy; for $\tau > -\mu > 0$, it is automatically true. Notice that in the opposite limit, the solution for $g_e \sim \mathrm{cosech}^2\eta/\bar{\delta}$ which is always singular somewhere or the other. There is a wealth of information in Eqs. (25)-(30) concerning the interplay of plasma parameters leading to acceptable physical solutions. We do not pursue this subject in detail here because the main task of this paper is to elucidate the nature of magnetically confined plasmas. From now on, therefore, we shall concentrate on the special case where the charge separation electric field $E_x = 0$. Clearly, $E_x = 0$ implies $g_e = g_i = g$, and $\kappa = 0 (b \neq 0)$ which implies $\mu = u_z^i/u_z^e = u_y^i/v_y^e = -\tau$, i.e., the ions and the electrons maintaining charge neutrality, flow in opposite directions (add to each other's current) with drift speeds proportional to their temperatures. The entire set of field quantities has the following simple structure,

$$g_e = g_i = g = \mathrm{sech}^2 \frac{x - Ut}{|\delta|} \ , \tag{31}$$

$$
\begin{aligned}
b &\equiv (ecu_e^2/u_z^e T_e)B_y = -\frac{2}{|\delta|}\tanh\frac{x-Ut}{|\delta|}\ , && (32)\\
B_z &= -(u_z^e/u_z^e)B_y\ , && (33)\\
E_y &= \frac{U}{c}(B_z+B_0)\ , && (15)\\
E_z &= -\frac{U}{c}B_y\ , && (16)
\end{aligned}
$$

where$|\delta| = \sqrt{2}|\delta_e|(1+\tau)^{-1/2} = (c/\omega_{pe})(v_e/u_e)(1+\tau)^{-1/2}\gamma_U^{-1}$ is the length scale of profiles. These are travelling pulse-like solutions with a propagation speed U in the x-direction. In the laboratory frame, one sees the inductive electric fields, which, however, vanish in the frame moving with the pulse. In fact, the pulse speed U is picked up precisely to insure that there are no electric fields in the moving frame. For $B_0 = 0$, the electromagnetic fields become purely transverse $\mathbf{E}\cdot\mathbf{B} = 0$, and become more and more light like as $U \to c$. But to deal with $U \to c$ limit properly, we must develop a relativistic theory of the plasma and not depend upon the current nonrelativistic treatment. This is done in Appendix B. Further discussion of the preceding results is deferred to Sec. V.

B. Equilibrium Solutions in Cylindrical Geometry

In Cartesian geometry, it was possible to obtain travelling (time dependent) solutions because the equations were invariant to the galilean transformation $x' \to x - Ut$. This property does not pertain in cylindrical geometry, which is of primary interest to the plasma confinement experiments. Thus time dependence will be dropped now ($U = 0$), and only equilibrium solutions will be presented. For the time being, we shall also put $u_\theta = 0$, i.e., there no currents in the azimuthal direction. Thus, the plasma has density gradients in the radial (r)-direction and has a current in the axial (z)-direction. (In Sec. IV, we shall reintroduce u_θ to simulate the tokamak currents.) The calculation in the cylindrical geometry follows the same logic as the Cartesian calculation, and we can derive the appropriate coupled nonlinear equations ($g_e = g_i = g$) for the density profile g, and the normalized magnetic field b, (all other self-consistent fields are zero)

$$\frac{1}{g}\frac{dg}{dr} = b\,, \tag{34}$$

$$\frac{1}{r}\frac{d}{dr}rb = -\frac{2}{\delta^2}g\,, \tag{35}$$

where the only change is the $d/dx \rightarrow r^{-1}d/dr$. All the symbols have exactly the same meaning, and we have chosen to deal with the case $E_x = 0$ ($\tau = \mu$), although inclusion of E_x is straightforward. Equations (34) and (35) led to the famous 'Bennet' solutions for a self-confining pinch,

$$g = \frac{1}{\left(1+\frac{r^2}{4\delta^2}\right)^2} \tag{36a}$$

$$b = -\frac{1}{\delta^2}\frac{r}{\left(1+\frac{r^2}{4\delta^2}\right)} \tag{36b}$$

characterized by the same fundamental length δ. Notice that the externally applied magnetic field B_0 does not appear in the pinch equilibrium studies. However, its presence is absolutely essential for the pinch stability, and hence will be considered to be always there.

It is interesting to compare and contrast the results of the Cartesian (for $U = 0$) and the cylindrical calculation. For small values of x, Eqs. (31) and (32) reduce exactly to (36) and(37) as they should, because for small values of $x(r)$, the cylindrical curvature is not important, and the problem is locally Cartesian. However, for distances $r \geq \delta$, the solutions can differ considerably; 1) Eq. (36) leads to an algebraic fall to zero while Eq. (31) predicts a stronger exponential fall for g, and 2) Eq. (37) predicts a slowly falling b while Eq. (32) predicts the field to become a constant for large x. However, the general structure of the problem is the same and if the problem at hand does not depend crucially on the details of the solution, our time-dependent Cartesian solutions could be effectively used in providing a dynamical (as different from static, or equilibrium) understanding.

In the simple problem considered so far, we had proposed the solutions to the Vlasov equation in the form of a displaced Maxwellian. The choice of a near Maxwellian is, of course, warranted by theoretical as well as experimental considerations. We would continue to be led by these considerations as we deal with more and more complicated cases. The alternative route of writing the solution of Vlasov equation in terms of the constants of motion, though extremely general, is not the obvious choice here, because we understand (to some degree) the nature of the solutions we are seeking in real space and velocity variables and not so much in the space spanned by the invariants. In Appendix C, it is shown that the simple problems could be treated either way, but for the problems where temperature and other gradients are present, it becomes very difficult to think of what combination of the invariants should be taken to model a discharge. The most important object, however, is to seek and find a solution of the problem by any correct and consistent approach.

III. Plasma with Density and Drift Speed Gradients

In addition to g, we now let the drift speed of the particles become a function of the radius r, i.e., $u_\alpha \equiv u_{\alpha 0}\varphi_\alpha(r)$, where $\varphi_\alpha(r)$ is the profile of the α-th specie, and $u_{\alpha 0}$ is the drift speed at the center. We still keep the temperatures T_α (equivalently, the thermal speed v_α) independent on r. It is straightforward to see that a displaced Maxwellian of the type Eq. (6) with $u_{e,i}$ dependent on r will not allow a solution in the velocity space because now the $v_r \partial f/\partial r$ has additional terms, quadratic in the velocity variables, which cannot be compensated by the force term. The required compensating force terms, however, are readily generated by introducing a temperature anisotropy: the particles have different temperatures in the direction along the current (T_z), and perpendicular to the current ($T \neq T_z$) suggesting the distribution

$$f_\alpha = \frac{n_0 g(r)}{\pi^{3/2} v_\alpha^2 v_{\alpha z}} \exp\left\{-\frac{v_r^2 + v_\theta^2}{v_\alpha^2} - \frac{[v_z - u_{\alpha 0}\varphi_\alpha(r)]}{v_{\alpha z}^2}\right\} \tag{37}$$

which represents a neutral plasma ($g_e = g_i = g$) carrying a current in the z-direction. Substituting Eq. (37) into the equilibrium Vlasov equation ($\partial/\partial t = 0$), and carrying out the usual algebra, one obtains the following set of coupled nonlinear ordinary differential equations (in the variables g, $\varphi_e = \varphi_i = \varphi$, and b with $\varphi(0) = 1$),

$$\frac{1}{g}\frac{dg}{dr} = \frac{\varphi}{\lambda}\frac{d\varphi}{dr} \tag{38}$$

$$\frac{d\varphi}{dr} = \lambda b \tag{39}$$

$$\frac{1}{r}\frac{d}{dr} rb = -\frac{2}{\delta^2} g\varphi \tag{40}$$

where $\lambda = \Delta T_\alpha / m_\alpha \mu_{\alpha 0}^2$ with $\Delta T_\alpha = T_\alpha - T_{\alpha z}$ is a measure of the anisotropy. Equations (38)-(40) are derived along with the constraints $\tau = -u_{i0}/u_{e0}$ (to keep $E_x = 0$), and $\Delta T_i/T_i = (\tau m_i/m_e)\Delta T_e/T_e$, which relates the electron and the ion anisotropy. Notice that the ion anisotropy is much stronger than the electron anisotropy. We shall see later that the Eqs. (38-40) allow extremely interesting behavior for $\lambda \simeq 1$. In a typical hot plasma, $\lambda \simeq 1$ implies $\Delta T_e/T_e = m_e u_{e0}^2 / m_e v_e^2 \simeq m_e/m_i$ which will lead to $\Delta T_i/T_i \sim \tau$, which is not unreasonably large. We must stress, however, that the situation with gradients in drift speed and no gradients in the thermal speed is not of much interest to thermonuclear plasmas, its applications have to be sought elsewhere.

Equations (38)-(40) can be combined into a single second-order equation

$$\frac{1}{r}\frac{d}{dr} r \frac{d\varphi}{dr} = -\frac{2}{\delta^2}\lambda\varphi \, e^{(\varphi^2-1)/2\lambda} \tag{41}$$

which does not allow close form solutions except in the limits $\varphi \simeq 1$. The equation is, however, trivially, solved on the computer and yields a rich variety of profiles as λ is varied. In Figs. 1-4, we present respectively the profiles of the density g, the magnetic field B, the drift speed φ and the current $g\varphi$ for a variety of positive ($T > T_z$) values of λ. The same set of variables is plotted in Figs. 5-8 for negative ($T < T_z$) values of λ. As λ takes larger and larger positive values, one observes more and more departure from the $\lambda = 0$ [Eqs. (36) and (37)] profiles: g decreases monotonically with $x = r/\delta$, and B, starting from zero, rises to a maximum at $x = 2$, and then slowly goes to zero. For larger values of λ, we see that B becomes oscillatory (going below zero also) with the wavelength of oscillations decreasing with λ. The other fields also show oscillations. Of particular interest is the fact that the density g becomes less and less peaked showing loss of confinement. This is a direct consequence of the fact that the confining magnetic field (now oscillatory) becomes considerably smaller than its $\lambda = 0$ value.

For negative values of λ, the richness exhibited by the $\lambda > 0$ case is missing. As λ decreases, the maximum of the confining magnetic field B_θ becomes larger, and comes nearer and nearer to the center. Consequently, the plasma density g becomes more and more peaked. Since B_θ has opposite sign to b (to keep the conventional current in the positive z-direction), negative value of λ would imply $d\varphi/dr > 0$ for $B_\theta > 0$ [see Eq. (39)]. Thus φ increases away from the center as seen in Fig. 7.

In this section, we have seen that there seems to be a fundamental connection between the existence of gradients in drift speeds of the particles (without temperature gradients), and the existence of temperature anisotropy; the connection is provided by the constraints imposed by the Vlasov equation. A whole variety of equilibrium profiles result for various values of the anisotropy parameter. The profile range from highly peaked to the oscillatory ones.

IV. Plasmas with General Gradients

We now deal with the general problem where the plasma has gradients in all the relevant macroscopic variables: the density, the current, and the temperature. Without any loss of generality, we introduce the gradients explicitly in the density $n = n_0 g$, and the temperature (through the thermal speed) $v_\alpha = (2T_\alpha/m_\alpha)^{1/2} \equiv v_{\alpha 0}\psi_\alpha$ where g and ψ are the profile factors [at $r = 0$, $g(0) = 1 = \psi_\alpha(0)$]; the gradients in the drift speed and the current will automatically follow. Because of its relevance to the laboratory plasmas, we must make a distinction between two of the principal kinds of experiments: the pinches (z-pinch), and the tokamak. In this paper, we take the view that the nature of the equilibrium depends upon the plasma currents (in a nonlinear system, all these things are, of course, self-consistent). Thus we shall begin by postulating that z-pinches are characterized by a plasma current only in the z-direction (i.e., direction of the strong externally applied magnetic field) while the tokamaks have currents in the z as well in the θ-direction; the details of the current and other profiles will naturally be determined by a self-consistent solution of the problem. We must emphasize here that although we are using the words z-pinch and tokamak, our relatively simple analytical models based on Vlasov-Maxwell equilibria could not possibly be a complete (or anywhere near complete) description of these complicated systems. The mandate of this paper is to expose some generic features of these systems when equilibrium currents are specified. In this entire paper the use of z-pinch and tokamak should be taken with strong qualifications.

A. A Simple z-pinch

This is an extension of the problem discussed in the last section. The plasma is embedded in a strong external field $B_0\hat{e}_z$, and has a current along the z-direction giving rise to the self-magnetic field $\mathbf{B} = B_\theta \hat{e}_\theta$. The equilibrium is obtained by solving the Vlasov equation in conjunction with the z-component of the Ampere's law. Unlike the simpler systems discussed in the earlier sections, a plasma with temperature gradients does not allow a one-term drifting Maxwellian as a solution. Fortunately, the infinite series expansion (species index is suppressed, and u_0 is a constant measure of the drift speed)

$$f = \frac{n_0 g}{\pi^{3/2} v_0^3 \psi^3}\left[1 + \frac{2u_0}{v_0}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\left(\frac{v_z}{v_0}\right)^n \left(\frac{v}{v_0\psi}\right)^{2m}\right]\exp\left[-\frac{v^2}{v_0^2\psi^2}\right] \tag{42}$$

converts the Vlasov equation into ordinary differential equations in g and ψ in addition to providing relations which determine all C_{nm} in terms of the plasma parameters, and $C_{10} = 1$, $C_{||} = \beta$ where β is some constant to be fixed by constraints outside our analysis. Equation (42) thus represents an exact solution to the Vlasov equation. The details of this algebraically complicated calculation are given in Appendix D, where explicit forms for some C_{nm}'s are also presented.

It turns out that all C_{nm} for which $m \geq n+1$ are identically zero, and the remaining $C_{nm} \sim (u_0/v_0)^n$, where u_0/v_0 is the ratio of the central drift speed to the central thermal speed. Clearly this series will be totally useless (if not downright meaningless) unless $u_0/v_0 \ll 1$. This is indeed the case for high temperature laboratory fusion plasmas. In fact, the existence of this small parameter $[u_0/v_0 \sim (m_e/m_i)^{1/2}]$ was the primary motive for seeking a series solution in (u_0/v_0).

After having demonstrated that a genuine exact series solution is possible, we invoke the smallness of u_0/v_0, and truncate the series keeping terms only to order u_0/v_0. Thus we have the simplified distribution function (species index restored, $g_e = g_i = g$)

$$f_\alpha = \frac{n_0}{\pi^{3/2} v_{0\alpha}^3}\frac{g}{\psi_\alpha^3}\left\{1 + \frac{2u_{0\alpha}}{v_{0\alpha}^2} v_z \left[1 + \beta_\alpha\left(\frac{v}{v_{0\alpha}}\psi_\alpha\right)^2\right]\right\}\exp\left(-\frac{v^2}{v_{0\alpha}^2\psi_\alpha^2}\right) + 0(u_0^2/v_0^2) \tag{43}$$

which, when substituted into the equilibrium Vlasov equation yields

$$\frac{1}{g}\frac{dg}{dr} - \frac{3}{\psi_\alpha}\frac{d\psi_\alpha}{dr} = \frac{q_\alpha u_{\alpha 0}}{cT_{\alpha 0}} B_\theta \tag{44}$$

and

$$\frac{2}{\psi_\alpha}\frac{d\psi_\alpha}{dr} = \frac{q_\alpha u_{\alpha 0}}{cT_{\alpha 0}} B_\alpha B_\theta \ . \tag{45}$$

Equations (44) and (45) lead to the algebraic relations connecting g with ψ_α,

$$g = \psi_\alpha^{3+2/\beta_\alpha}$$

in addition to the differential equations [one for the electrons and one for the ions]

$$\frac{1}{g}\frac{dg}{dr} = -\frac{q_\alpha u_{\alpha 0}}{cT_{\alpha 0}}[1 - (3/2)\beta_\alpha]B_\theta \ ,$$

requiring $u_{i0}(1 - 1.5\beta_i) = (1 - 1.5\beta_e)\tau_0\mu_{e0}$ for consistency. One could, in principle, carry out the calculation by choosing different constants β_α for the electrons and ions. However, it is expected that the electron and ion temperature profile factors should be quite close to each other in long-lived equilibria. It is physically reasonable therefore to assume $\beta_e = \beta_i = -\beta(\beta > 0)$ implying $\psi_e = \psi_i = \psi$ related to g by

$$g = \psi^{3-\frac{2}{\beta}} \ . \tag{46}$$

The consistency constraint takes on its usual form $u_{i0}/u_{e0} = -\tau_0$ where τ_0 is the ion-electron temperature ratio at the center. We have chosen β_α to be negative because the density profiles are, in general, less steep than the temperature profiles $(T \sim \psi^2)$ for most thermonuclear plasmas. The choice of β_α simplifies the differential equation relating ψ with B_θ,

$$\frac{1}{\psi}\frac{d\psi}{dr} = -\frac{\beta}{2}\frac{2eu_{e0}}{m_e v_{e0}^2}\frac{B_\theta}{c} \equiv -\frac{\beta}{2}\, b \tag{47}$$

which along with Eq. (46) represents the manifestation of the Vlasov equations with the ansatz Eq. (43). Following the standard procedure, we use Eq. (43) to calculate the total plasma current

$$J_z = +en_0 g\psi^2\,(5\beta/2 - 1)\,(1+\tau)u_{e0} \tag{48}$$

which is to be substituted into the z-component of Ampere's law to yield [after using Eq. (46)]

$$\frac{1}{r}\frac{d}{dr}rb = \frac{2}{\delta^2}\,(5\beta/2 - 1)\psi^{5-2/\beta} \ . \tag{49}$$

We remark here that $u_{e0} > 0$ does not exactly equal the drift speed of the electrons, but is of that order. In that sense, it is different from u_{e0} used in Sec. III. Equations (47) and (49) can be combined to yield for $Q = (5 - 2/\beta)\ell n\, \psi$,

$$\frac{1}{r}\frac{d}{dr}r\frac{dQ}{dr} = -\frac{2}{\delta_{\text{eff}}^2}\, e^Q \tag{50}$$

where $\delta_{\text{eff}} = \delta|5\beta/2 - 1|^{-1}$. Equation (50) is readily solved to obtain (the profiles of) the thermal speed,

$$\psi_e = \psi_i = \psi = (e^Q)^{\frac{1}{5-2/\beta}} = \left[1 + \frac{r^2}{4\delta_{\text{eff}}^2}\right]^{-\frac{2}{5-2/\beta}} , \tag{51}$$

the temperature

$$T = \psi^2 = \left[1 + \frac{r^2}{4\delta_{\text{eff}}^2}\right]^{-\frac{4}{5-2/\beta}} , \tag{52}$$

the density

$$g = \left[1 + \frac{r^2}{4\delta_{\text{eff}}^2}\right]^{-\frac{2(3-2/\beta)}{(5-2/\beta)}} , \tag{53}$$

the current

$$j \sim g\psi^2 \sim \left[1 + \frac{r^2}{4\delta_{\text{eff}}^2}\right]^{-2} \tag{54}$$

and the self-consistent magnetic field

$$B_\theta \sim b = \frac{(5\beta/2 - 1)}{\delta_{\text{eff}}^2} \frac{r}{1 + \frac{r^2}{4\delta_{\text{eff}}^2}} . \tag{55}$$

Notice that for a confined plasma, i.e., where the plasma density and temperature, separately, decrease away from the center, we must choose $\beta > 2/3$. Further fixing of β must come from additional information; boundary conditions or possibly the experiment. Equations (50)-(51), the main result of this section, represent a class (labelled by β) of exact, nonlinear, self-consistent solutions for the plasma and field parameters relevant to a z-pinch in which the drift speed u_0 is much smaller than the thermal speed v_0 of the particles. As expected, there is only one length scale $\delta_{\text{eff}} \sim \delta$. Further discussion of the solutions will be done at the end of Sec. IVB.

B. A Simple Tokamak

In order to simulate a tokamak-like discharge (which has already been opened up into a cylinder of length $2\pi R_0$), we must include currents in θ-direction as well. In fact, considerable more freedom ensues because now the strong external magnetic field $B_0\hat{e}_z$ can also start playing some role in the equilibrium force balance. A necessary consequence of this addition is that the constraint $u^z_{i0}/u^z_{e0} = -\tau$ disappears, which is very important, because the toroidal (z)-current in the tokamaks is largely due to the electrons. For simplicity, we shall assume that the electron current is in the z-direction while the ion current is essentially in the θ-direction (In Appendix E, we solve the problem with electron current in both θ- and z-directions. The algebra becomes more complicated, but precious little is gained in perspective). Since we have a strong external magnetic field in the z-direction, we shall neglect the self-consistent B_z, and keep only B_θ. Following the methods of Sec. IVA, we can construct infinite series solution for the distribution functions. Since in all high temperature tokamaks, the drift speed u_0 is indeed much smaller than the thermal speed v_0, we can simply use the truncated distributions which are correct to order u_0/v_0.

In the context of the preceding discussion, we make the following ansatzs for the electron and the ion distribution functions:

$$f_e = \frac{n_0 g}{\pi^{3/2} v_{0e}^2 \psi_e^3} \left\{ 1 + \frac{2u_{0e}}{v_{0e}^2} v_z \left(1 - \frac{\beta v^2}{v_{0e}^2 \psi_e^2} \right) \right\} e^{-\frac{v^2}{v_{0e}^2 \psi_e^2}} , \tag{56}$$

and

$$f_i = \frac{n_0 g}{\pi^{3/2} v_{0i}^3 \psi_e^3} \left\{ 1 + \frac{2V_i}{v_{0i}^2} v_\theta \left(1 - \frac{\mu v^2}{v_{0e}^2 \psi_i^2} \right) \right\} e^{-\frac{v^2}{v_{0i}^2 \psi_i^2}} , \tag{57}$$

where, of course, we have chosen $g_e = g_i = g$ to ensure that the charge separation electric field $E_r = 0$. Substituting Eqs. (56) and (57) into the Vlasov equations, we obtain [$\sim 0(u_0/v_0)$],

$$\frac{1}{g}\frac{dg}{dr} - \frac{3}{\psi_e}\frac{d\psi_e}{dr} = b \tag{58}$$

$$\frac{2}{\psi_e}\frac{d\psi_e}{dr} = -\beta b \tag{59}$$

$$\frac{1}{g}\frac{dg}{dr} - \frac{3}{\psi_i}\frac{d\psi_i}{dr} = \frac{eB_0}{cT_{i0}} V_i \tag{60}$$

$$\frac{2}{\psi_i}\frac{d\psi_i}{dr} = -\frac{eB_0}{cT_{i0}} \mu V_i \tag{61}$$

where the two electron equations (58)-(59) are the same as the corresponding pinch equations of Sec. IVA, while the ion equations (60) and (61) are very different. The system (58)-(61) reduces to the following relationships between the gradients

$$g = \psi_e^{3-2/\beta} = \psi_i^{3-2/\mu} , \tag{62}$$

an expression for the ion poloidal (azimuthal) drift speed

$$V_i = \frac{cT_{i0}}{eB_0}\frac{2}{\mu}\frac{1}{\psi_i}\frac{d}{dr}\psi_i \equiv \frac{cT_{i0}}{eB_0}\frac{2}{\mu}\frac{3-2/\mu}{3-2/\beta}\frac{1}{\psi_e}\frac{d\psi_e}{dr}$$

which is nothing but the manifestation of diamagnetic drifts, and the independent Eq. (59). Since the current in the z-direction is purely from electrons, it is given by Eq. (48) without the factor $(1+\tau)$. Thus, mathematically, the tokamak system is reduced to the pinch system with the exception that the constraint $u^z_{0i}/u^z_{0e} = -\tau$ is no longer there (in fact, there is no u^z_i in the problem), and that the relevant tokamak δ, $\delta_t = (c/\omega_{pe})(v_{e0}/u_{e0})|5\beta/2-1|^{-1}$. If we further believe that the electron and ion temperature gradients are the same, then $\psi_e = \psi_i(\mu = \beta)$. Thus, the tokamak profiles are precisely those given in the set (51)-(55) with δ_{eff} replaced by δ_t. If $\mu \neq \beta$, then we need additional information to fix μ, though the functional form of ψ_i is easily determined [Eq. (62)], when ψ_e is known. Even at the cost of being repetitive, we spell out the tokamak profiles $(\beta > 2/3)$

$$g = \left(1+\frac{r^2}{4\delta_t^2}\right)^{-2\frac{(3\beta-2)}{(5\beta-2)}}, \tag{63}$$

$$T = \psi^2 = \left(1+\frac{r^2}{4\delta_t^2}\right)^{-\frac{4\beta}{5\beta-2}}, \tag{64}$$

$$J_z = en_0(5\beta/2-1)u_{e0}\left(1+\frac{r^2}{4\delta_t^2}\right)^{-2}, \tag{65}$$

$$B_\theta = \frac{cT_{e0}}{eu_{0e}}\frac{[5\beta/2-1]^{-1}}{\delta_t^2}\frac{r}{1+\frac{r^2}{4\delta_t^2}}, \tag{66}$$

$$q = \frac{rB_0}{R_0B_\theta} = q_0\left[1+\frac{r^2}{4\delta_t^2}\right], \tag{67}$$

and

$$V_i = (cT_{i0}/eB_0)(2/\beta)\frac{1}{\psi_e}\frac{d\psi_e}{dr}, \tag{68}$$

where $q_0 = (eB_0/cT_{e0})(u_{e0}/R_0)\delta_t^2$ is the safety factor at the plasma center. In Appendix E, we have solved the slightly more complicated problem with additional electron current (diamagnetic) in the θ-direction. The form of the solution remains unchanged. Hence a discussion of Eqs. (63)-(68) is adequate to elucidate the nature of the equilibrium.

We first notice that the equilibrium profiles for the z-pinch-like and the tokamak-like equilibria are exactly alike; these are all some power of a Lorentzian with a scale length δ which depends only upon the plasma parameters, and a constant (β) which has to be given by the boundary conditions or the experiment. The primary difference between the two cases is that the pinch solution has a stringent condition on the electron and ion currents, i.e., they must flow in opposite directions with magnitudes proportional to their temperatures while there is no such constraint on the tokamak-like profiles. Furthermore, in the pinch equilibrium, the strong external magnetic field, though essential for stability, plays no significant role in equilibrium. The stability role for B_0 is unaltered for the tokamak, but it also does acquire some (although a minor) role in equilibrium; the ion azimuthal drift U_i depends on B_0 [Eq. (68)]. For most other considerations, the two can be treated alike.

Although we have introduced azimuthal ion (also electron in the Appendix E) current, we have neglected the self-consistent field B_z. This is justified by comparing Eq. (69) with (59) which gives in the tokamak ordering ($|B_\theta|/B_0 \ll 1$)

$$\left|\frac{V_i}{u_{e0}}\right| = \tau_0 \frac{|B_\theta|}{B_0} \ll 1 \ . \tag{69}$$

Thus the self-consistent field $B_z \ll B_\theta \ll B_0$, and can be easily neglected. This surely does not imply that the current, itself, is unimportant. In fact, it is the coupling of this current with the large B_0 which constitutes the force term in the ion Vlasov equation.

Let L denote the length scale of macroscopic variation of plasma parameters: density, temperature, etc. [$L \sim 2\delta_t$ from Eqs. (63)-(68)], then it is easy to see from Eq. (59) that

$$\left|\frac{u_{e0}}{v_0}\right| = \frac{v_e}{(eB_\theta/m_e c)}\frac{1}{L} = \frac{\rho_{e\theta}}{L} \ll 1 \ , \tag{70}$$

i.e., the expansion parameter used in the series solution for the Vlasov equation equals the ratio of the electron poloidal gyroradius to the size of the system; an extremely small parameter in the tokamak ordering.

From a knowledge of the central density, temperature, and the drift speed [$u_{d0} = u_{e0}(5\beta/2 - 1)$] one can calculate the basic confinement scale length $L_c = 2\delta_t = 2(c/\omega_{pe})(v_{e0}/u_{d0})$. This paper is essentially a theoretical paper, and we do not do any curve-fitting with particular tokamaks (which, in detail, are much more complicated than the theoretical tokamak presented in this paper). However, for reference, we calculated L_c based on standard data[6]) for a few typical discharges: For TEXT $L_c \sim 6$cm, and for Alcator-C, $L_c \sim 4$cm; in either case $L_c < a$, the minor radius of the machine.

Guided by our desire to simulate a totally confined system, we found that $\beta > 2/3$ insures that all our profiles for the plasma parameters (g, ψ, J_z) are monotonically decreasing. However, the self-consistent magnetic field B_θ shows a maximum at $r = 2\delta_t$ (the safety factor q is monotonically increasing as expected). This might suggest a possible experimental determination of $2\delta_t$, i.e., measure the radial position where the poloidal field peaks.

All these profiles are derived for a very simplified system. Actual machines not only are much more complicated, but also have a large number of other features [like gas puffing, collisions, time dependent phenomena] not in our theory. It is quite straightforward to show that in our ordering the collisions are u_0^2/v_0^2 smaller than the terms kept (say, $v_r \frac{\partial f}{\partial r} \sim v_e f L_c^{-1}$), and should appear in the 2nd order theory (if the basic theory is zero order). But clearly, at large time scales, all these phenomena (including anomalous collisions) etc. could have strong effects on the profiles that we have calculated. However, the zero order confinement profiles must serve as the basic substrata on which other processes (not included here) act, and if one were to make attempts to theoretically derive the profiles (like attempts to derive them from transport theories based on anomalous heat conductivity etc.), one should be forced to reckon with the existence of fundamental confinement profiles with a scale length $2\delta_t$ which existed before the instabilities and other such phenomena came into play. It must be stressed that, in the realistic machines, our solutions will provide a basic framework only if $2\delta_t < a$, where a is the mirror radius of the machine. If $a < 2\delta_t$, then the edge effects etc. could be far too important to allow simple forms Eqs. (63)-(68).

V. Summary and Discussion

Sections IIB and III were essentially a prelude to Sec. IV. Therefore, the discussion will be limited to the findings of Secs. IIA (time-dependent solutions), and IV. We have already discussed some of the salient features of the class of general solutions (calculated in Sec. IV) for a plasma with inhomogeneities [Eqs. (63)-(68)]. We found that after making reasonable assumptions like $\psi_e = \psi_i$ (i.e., the ion and the electron profiles are parallel), the set of profiles was unknown to a constant β. Several properties of the system could be discussed essentially independent of what β may be; the shape of B_θ and J_z depend on β only through the magnitude of δ_t. Many other properties, however, are determined crucially by β. First and foremost is the fact that we require $\beta > 2/5$ for the temperature to fall away from the center, and then $\beta > 2/3$ for the density also to fall away from the center. Thus, $\beta > 2/3$ insures the confinement of particles as well as heat. Although this is the most interesting case for laboratory plasmas, other interesting situations can arise either in the laboratory or in the astrophysical plasmas where one deals with plasmas where one of the parameters could increase away from the center. An appropriate adjustment of β could simulate such situations indicating the scope of applicability of this theory. As an example $2/3 > \beta > 2/5$ gives confined temperature and increasing density.

Although the confinement constraints restricts the value of β, they are not sufficient to determine it. Some additional information is necessary. One example is the ohmically heated discharges, where the current goes as $T^{3/2} \sim \psi^3$ implying $g = \psi$ and $\beta = 1$. The fixing of β determines the relative sharpness of the density profiles and temperature. If the plasma is heated or created in a different manner, β, and hence the ratio of the profiles would be different. The most important conclusion of the Sec. IV calculations is to point out that Eqs. (63)-(68) provide a set of fundamental equilibrium profiles characterized by the scale length $2\delta_t = (2c/\omega_{pe})(v_{0e}/u_d)$ which must serve as the standard state on which any perturbations are imposed; δ_t must figure in calculations which predict experimental profiles.

We now go back to the Cartesian time-dependent solutions of Sec. IIA to make some speculations about plasma dynamics in a confined system. We begin by noticing that the entire set of calculations in Sec. III and Sec. IV could be made time-dependent if we had chosen to work in the Cartesian geometry. In fact, the solutions for the tokamak-like case will be $[u_{e0} > 0, \delta_t > 0, \beta > 2/3]$

$$g = \left[\operatorname{sech}^2 \frac{x - Ut}{\delta_t}\right]^{\frac{3\beta-2}{5\beta-2}} \tag{71}$$

$$T = \psi^2 = \left[\operatorname{sech}^2 \frac{x - Ut}{\delta_t}\right]^{\frac{2\beta}{5\beta-2}} \tag{72}$$

$$B_y = \frac{cT_{e0}}{eu_{0e}} \frac{(5\beta/2 - 1)^{-1}}{\delta_t} \tanh \frac{x - Ut}{\delta_t} \tag{73}$$

along with the inductive electric fields,

$$E_z = -\frac{U}{c} B_y \tag{74}$$

$$E_y = \frac{U}{c} B_0 = \text{const.}\ , \tag{75}$$

which are a necessary consequence of the time dependence. In Eqs. (71)-(75) U is the plasma flow velocity in the x-direction.

The question arises: can we exploit the above solutions to say anything about the plasma dynamics in cylindrical systems for which we have no time-dependent solutions? We first notice that both in the Cartesian and the cylindrical geometry, we have two ignorable directions, y and z in the Cartesian system, and θ and z in the cylindrical system. Thus one would expect that the basic nature of the solutions in the two systems should be the same. Of course, they could vary widely in details. We had already remarked in Sec. IIB that for small values of the argument, the two solutions are exactly alike, vindicating our conjecture. Our inability to find a time-dependent solution in cylindrical coordinates is a mathematical inability, but based on the shared symmetries, physical considerations suggest that we use the Cartesian solution to draw some qualitative conclusions about the dynamics of a cylindrical plasma.

With the preceding preamble, we now analyze Eqs. (71)-(75), in particular, Eqs. (73)-(74). Notice that B_y is an odd function of $\eta = x - Ut$ implying that if $E_z < 0$, then $U > 0$ for $\eta > 0$, and $U < 0$ for $\eta < 0$, i.e., the velocity U is always pointed away from the center $\eta = 0$ as long as $E_z < 0$. For initial times, $\eta = x$ is to be identified with cylindrical r allowing us the simple extrapolation that in an equivalent cylindrical system U is always radially outwards as long $E_z < 0$ and $B_\theta > 0$ (which is indeed the case). Clearly, if $E_z > 0$, the plasma moves radially inwards giving rise to pinching.

We have thus come to an important conclusion that there exist dynamical solutions to the Vlasov-Maxwell system which in the laboratory frame move radially outwards (inwards) [with a speed U determined by Eq. (74)] provided they have an associated $E_z < 0$ (> 0). There is nothing in our analysis which can fix the speed U. In fact, U must be a function purely of the mechanism of plasma creation. It is possible that the plasmas born in similar conditions will choose nearby values for U. Clearly, $U = 0$, implies that there are no local inductive electric fields created during plasma formation.

We now make the bold conjecture that an upper limit on the lifetime of a discharge is determined at the very moment of its creation. It is simply $t_c = a/U$, where a is the minor radius of the machine; just the time taken for the discharge to move out. We cannot predict U (or E_z) but we can make estimates of E_z for some standard plasma lifetimes. Using (73)-(74) and $U = (a/t_c)$, we note that

$$E_z = -\frac{U}{c} B_\theta = -\frac{a}{ct_c} B_\theta \sim \frac{\sqrt{2\pi}}{2} \frac{a}{ct_c} (n_0 T_{e0})^{1/2} \tag{76}$$

which for typical parameters, $n_0 = 10^{-4}$, $T_{e0} = 1\,\mathrm{keV}$, $a = 25\,\mathrm{cm}$ becomes [E_z in Volts/cm]

$$E_z \simeq -\frac{10^{-4}}{t_c} \,. \tag{77}$$

Let us see if the above formula sheds some light on two apparently disjoint phenomena: the major-disruption and the 'anomalous transport'.

In a major disruption, the plasma goes to the wall in a very short time $\sim 10\mu\,\mathrm{secs}$, and a very large negative voltage spike is seen. For $t_c = 10\mu\,\mathrm{secs}$, $E_z = -10$, leading to a voltage spike $V = -2\pi R_0 \times 10 \simeq -6000$ volts for $R_0 = 100\,\mathrm{cms}$. The velocity for this case $U \simeq 25/10^{-5} \sim 25 \times 10^5\,\mathrm{cm/sec}$. Neither of these numbers is absurd.

'Anomalous transport' on the other hand is a slow process which results in plasma energy loss in 50-100 msecs. For this case, Eq. (79) yields $E_z \simeq -10^{-3}$ volts/cm indicating that a reasonably small amplitude electric field could indeed self-consistently provide enough U to result in a plasma loss in 50-100 msec.

Thus it seems possible to view both the disruptive losses and the slower 'anomalous loss' as limits of the same process. The plasma likes to be in a configuration with no electric fields; it moves radially outwards precisely to achieve this end because in the moving frame the electric fields are transformed away. The characteristic speed U depends upon the magnitude of the electric fields it is born with; for large enough E_z, the discharge disrupts (goes to the wall in a short time), and for low enough E_z, it gently (comparatively) moves towards the wall giving rise to 'anomalous transport'. It is conceivable that this simple scenario may have something to do with reality.

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Figure Captions

1. Density (profile factor) g versus $x = r/\delta$ for several positive values of the anisotropy parameter $\lambda = \Delta T/mu_0^2$, $\Delta T = T - T_z$. Large λ results in a loss of confinement. This set of parameters will be used in Figs. 1-4.

2. Drift speed u versus $x = r/\delta$. Observe the oscillations as λ becomes large; the current changes its direction.

3. The self-consistent magnetic field $B_\theta = B$ versus $x = r/\delta$. The field can reverse for higher values of λ; it also becomes smaller in magnitude resulting in loss of confinement, c.f., the broader density profiles in Fig. 1 as λ increases.

4. The current profile factor $u{*}g$ versus $x = r/\delta$.

5. Density profile factor g versus $x = r/\delta$ for several negative values of the anisotropy parameter λ. As $|\lambda|$ becomes large, the density profile becomes more peaked. Same set of parameters is used in Figs. 5-8.

6. Drift speed u versus $x = r/\delta$. The drift speed is increasing away from the center.

7. The magnetic field $B_\theta = B$ versus $x = r/\delta$. The field becomes more peaked, and the peak shifts towards the plasma center as $|\lambda|$ increases.

8. The current profile factor $u{*}g$ versus x. The drift speed u increases away from the center while g becomes more and more peaked resulting in a current profile (and hence the accompanying magnetic field) which is more and more peaked near the center as $|\lambda|$ increases.

Appendix A

The pure Vlasov-Poisson problem (with no plasma current and the self-magnetic field) is completely defined by

$$\frac{dE}{d\eta} = -\frac{1}{\lambda_D^2}(g_e - g_i) \tag{A.1}$$

$$\frac{1}{g_e}\frac{dg_e}{d\eta} = -E \tag{A.2}$$

$$\frac{1}{g_i}\frac{dg_i}{d\eta} = \frac{E}{\tau} \tag{A.3}$$

which are the $[b = 0, u = 0]$ limit of Eqs. (21)-(23). Equations (A.2) and (A.3) lead to the constraint

$$g_i g_e^{1/\tau} = \text{const.} \equiv \mu^2 \tag{A.4}$$

which can be used to derive a single equation $(g_e \equiv g, \zeta = \eta/\lambda_D)$

$$\frac{d}{d\zeta}\frac{1}{g}\frac{dg}{d\zeta} = \left(g - \mu^2 g^{-1/\tau}\right) . \tag{A.5}$$

On one integration (A.5) yields

$$\frac{1}{2}\left(\frac{1}{g}\frac{dg}{d\zeta}\right)^2 = \left[g + \frac{\mu^2}{\tau}g^{-1/\tau}\right] + C \tag{A.6}$$

where C is an integration constant. Equation (A.6) is readily solved

$$\int \frac{dg}{g\left[C + g + \frac{\mu^2}{\tau}g^{-1/\tau}\right]^{1/2}} = \pm 2(\zeta + \zeta_0) \tag{A.7}$$

where ζ_0 is another constant of integration. For arbitrary τ, one must do the integral numerically, but for $\tau = 1$, nice revealing analytical forms are possible. Putting $\tau = 1$, $\mu = 1$ (μ can be easily absorbed in the redefinition of g; thus there is no loss of generality), we rewrite the system in terms of $Q = \ell n\, g$

$$\frac{1}{2}\left(\frac{dQ}{d\zeta}\right)^2 = \left[e^Q + e^{-Q} + C\right] , \tag{A.8}$$

and

$$\int \frac{dQ}{(e^{Q}+e^{-Q}+C)^{1/2}} \;=\; \pm 2^{1/2}(\zeta+\zeta_0)\,. \tag{A.9}$$

The constant C must be determined by the boundary condition on $e^{Q}=g$, or the electric field $E \sim -dQ/d\zeta$. The detailed form of the solution depends on this choice. Following independent cases exhaust the range of C:

(a) $C=-2$, which could correspond to the boundary condition that as $\zeta \to \infty$, $dQ/d\zeta = 0$, $Q \to 0 (g=1)$. The solution is $[\bar{\zeta} = (\zeta+\zeta_0)/\sqrt{2}]$

$$g_e = e^{Q} \;=\; \coth^2\bar{\zeta} \tag{A.10}$$

$$g_i = g_e^{-1} \;=\; \tanh^2\bar{\zeta} \tag{A.11}$$

$$E \sim -\frac{dQ}{dr} \;=\; \frac{1}{\sqrt{2}}\frac{\operatorname{cosech}^2\bar{\zeta}}{\coth\bar{\zeta}}\,. \tag{A.12}$$

Since g and g^{-1} occur symmetrically, the roles of g_i and g_e could be interchanged. This feature will hold for all the cases considered.

(b) $C=2$, which corresponds to a particular relationship between g and E at some point, say the origin. The solution is

$$g_e \;=\; e^{Q} = \frac{1+\sin 2\bar{\zeta}}{1-\sin 2\bar{\zeta}} \tag{A.13}$$

$$g_i \;=\; g_e^{-1} \tag{A.14}$$

$$E \;=\; -2\sqrt{2}\,\sec 2\bar{\zeta} \tag{A.15}$$

(c) $-2 < C < 2$; after a fair amount of algebra, the solutions come out to be in terms of Jacobian elliptic functions. There are two cases: For $C<0$, we have $[C/2 = \lambda]$

$$g+\frac{1}{g} \;=\; 2\frac{1-|\lambda| Sn^2 2\bar{\zeta}}{1-Sn^2 2\bar{\zeta}}\,, \tag{A.16}$$

$$E \;=\; -2\frac{[1-|\lambda|]^{1/2}}{Cn\, 2\bar{\zeta}} \tag{A.17}$$

with the argument of elliptic functions $\kappa = [(1+|\lambda|)/2]^{1/2}$, and for $C > 0$ $[\kappa = [(1-\lambda)/2]^{1/2}$

$$g^{1/2} + \frac{1}{g^{1/2}} = \frac{2}{Sn\, 2\bar{\zeta}}\,, \tag{A.18}$$

$$E = -2\sqrt{2}\left[\frac{1}{Sn^2\, 2\bar{\zeta}} - \frac{1-\lambda}{2}\right]^{1/2}\,. \tag{A.19}$$

Equations (A.15) and (A.18) could be readily solved for g.

(d) $C > 2$ case gives $\left[\kappa = (C-2)^{+1/2}(C+2)^{-1/2}\right]$

$$e^Q = g = \frac{1 + Sn(1+C/2)^{1/2}\bar{\zeta}}{1 - Sn(1+C/2)^{1/2}\bar{\zeta}} \tag{A.20}$$

from which $E = -dQ/d\zeta$ can be computed.

(e) The last case is $C < -2$ for which the solutions are $\left[\kappa = 0.5(|C|+2)^{-1/2}\right]$

$$g + \frac{1}{g} = 2\frac{\lambda - Sn(1+|\lambda|)^{1/2}\bar{\zeta}}{1 - Sn(1+|\lambda|)^{1/2}\bar{\zeta}}\,, \tag{A.21}$$

$$E = = -2\left[\frac{\lambda - Sn[1+|\lambda|]^{1/2}\bar{\zeta}}{1 - Sn(1+|\lambda|)^{1/2}\bar{\zeta}} - \lambda\right]^{1/2}\,. \tag{A.22}$$

It is to be noted that the solutions for $C > 2$, and $C < 2$ continuously go to each other through the solution at $C = 2$. The most important conclusion is that for all values of C, one or the other quantity of interest (g_e, g_i, E) becomes singular at some point. It seems a small amount of self-magnetic field may be needed to remove the singularity.

Appendix B

If the system is characterized by large drift and flow speeds (which may be the case for many astrophysical applications), then one must use the relativistically correct Vlasov equation; the distribution function $f \equiv f(\mathbf{x}, \mathbf{p}, r)$ ($\mathbf{p}$ is the momentum) obeys

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + q \left[\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right] \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \tag{B.1}$$

where

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}}{m[1 + p^2/m^2c^2]^{1/2}} . \tag{B.2}$$

This formalism will introduce algebraic complications when attempts are made to calculate plasma current etc. The difficulty can be readily overcome if one assumes that the only relativistic velocities are the drifts $\mathbf{u}$ and the flow $\mathbf{U}$, while the thermal speed $v_e \ll c$. Notice that this situation is of no interest to fusion plasmas. With this assumption γ can be replaced by $\gamma_0 = [1 + (m^2c^2)^{-1}(\mathbf{p}_{\hat{U}} + \mathbf{p}_{\mathbf{u}})^2] \equiv [1 - (\mathbf{U} + \mathbf{u})^2/c^2]^{-1/2}$, and can be taken out of the integration whenever $\mathbf{p}$ integrations need to be done. The entire analysis in the text remains valid; if we use the above value for γ_0 and replace ω_{pe} by its relativistic form $\omega_e = (4\pi n_0 e^2/\gamma_0 m)^{1/2}$ which will modify δ_e^2 to $\delta_{er}^2 = \gamma_0^{-2}(c^2/\omega_e^2)(v_e^2/u_e^2)$.

Appendix C

By discussing two examples, we show that it is possible to express the proposed distribution functions in terms of the constants of the motion. We deal with the electron [charge $(-e)$] distribution with current only in the z-direction. The constants of the motion are the Hamiltonian

$$H = \frac{mv^2}{2} = \frac{1}{2m}\left[\mathrm{p} + \frac{e}{c}\mathrm{A}(r)\right]^2 , \tag{C.1}$$

and the canonical momentum in the z-direction

$$p_z = mv_z - \frac{e}{c}A_z(r) . \tag{C.2}$$

It is the through p_z, that one has to introduce r-dependence. The general solution will be

$$f = f(H, p_z) . \tag{C.3}$$

Let us first consider the simple case with only density gradients $g = g(r)$. For this case we had chosen the function $[2T_{e0} = mv_{e0}^2]$ given by Eq. (6) $(U = 0)$,

$$f_e = \frac{n_0}{\pi^{3/2}v_{e0}^3} g(r) \exp\left[-\frac{H}{T_{e0}} + \frac{mu_{e0}v_z}{T_{e0}}\right] \tag{C.4}$$

where the subscript 0 is used to explicitly display that these quantities are constants. From (22), we recall that ($E = 0$, $B_\theta = (\nabla \times A)_\theta = -\partial A_z/\partial r$)

$$\frac{1}{g}\frac{dg}{dr} = \frac{eu_{e0}}{cT_{e0}} B_\theta = -\frac{eu_{e0}}{cT_{e0}}\frac{\partial A_z}{\partial r} \tag{C.5}$$

which yields

$$g = \exp\left[-\frac{eu_{e0}}{cT_{e0}} A_z\right] , \tag{C.6}$$

$$f_e = \frac{n_0}{\pi^{3/2}v_{e0}^3} \exp\left[-\frac{H}{T_{e0}} + \frac{u_{e0}}{T_{e0}}(mv_z) - \frac{e}{c}A_z\right] \equiv f_e(H, p_z) . \tag{A.7}$$

Let us now deal with the more difficult problem when all the gradients are present. The general problem is cumbersome, so we demonstrate the theorem only to the leading order approximation. For this we had chosen

$$f_e = \bar{n}_0 \frac{g}{\psi^3} \left[1 + 2\frac{u_0 v_z}{v_{0\alpha}^2} \left(1 - \beta \frac{v^2}{v_{0e}^2 \psi^2} \right) \right] \exp\left[-\frac{v^2}{v_{0e}^2 \psi^2} \right] \tag{C.7}$$

for which we have from Eqs. (44) and (45) $[h = eu_{e0}/cT_{e0}]$

$$\frac{g}{\psi^3} = \exp[-hA_z] \tag{C.8}$$

and

$$\psi^2 = \exp[\beta h A_z] \; . \tag{C.9}$$

Using (45), we see that

$$\exp\left[-\frac{v^2}{v_{0e}^2}\frac{1}{\psi^2} \right] = \exp\left[-\frac{v^2}{v_{0e}^2} \exp(\beta h_z) \right] = \exp\left[-\frac{v^2}{v_{0e}^2} \left(1 + \beta h_z \frac{v^2}{v_{0e}^2} + \cdots \right) \right] \; . \tag{C.10}$$

Keeping terms only to $\mathcal{O}(A_z)$, $\mathcal{O}\left(\frac{u_0}{v_{0e}}\right)$, etc.,

$$\begin{aligned} f_e &\simeq \bar{n}_0[1 - hA_z] \left[1 + \beta h A_z \frac{v^2}{v_{0e}^2} \right] \left[1 + 2\frac{u_0 v_z}{v_{0e}} \left(1 - \beta \frac{v^2}{v_{0e}^2} \right) \right] \exp\left[-\frac{H}{T_{e0}} \right] \\ &\simeq \bar{n}_0 \left\{ 1 + \left(1 - \frac{\beta H}{T_{e0}} \right) \left(m v_z - \frac{e}{z} A_z \right) \frac{u_{e0}}{T_{e0}} \right\} \exp\left[-\frac{H}{T_{e0}} \right] \equiv f_e(H, p_z) \end{aligned} \tag{C.11}$$

showing that f_e could be seen as a function of the constants of the motion alone. We have carried out the calculation up to second order in A_z, and u_0/v_{0e}, and found that the theorem holds. We have not succeeded in a general proof yet, but we believe that the theorem is true. Clearly, the form used in the text is much more perspicuous, and is suggested by physical intuition.

Appendix D

The plasma has current only in the z-direction, and hence, the self-magnetic field has only the B_θ component. The equilibrium Vlasov equation is

$$v_r\frac{\partial f}{\partial r} + \frac{q}{m}\left(\mathrm{v}\times\hat{\theta}\frac{B_\theta}{c}\right)\cdot\frac{\partial f}{\partial \mathrm{v}} = 0\,. \tag{D.1}$$

For f given in Eq. (42),

$$\frac{\partial f}{\partial \mathrm{v}} = \mathrm{v}F_1 + \hat{z}F \tag{D.2}$$

converting (D.1) to

$$\frac{\partial f}{\partial r} + \frac{q}{mc}B_\theta F = 0\,. \tag{D.3}$$

Using Eq. (42), we calculate $\partial f/\partial r$ and F,

$$\begin{aligned}\frac{\partial f}{\partial r} \propto\ & \left[\frac{d}{dr}\left(\frac{g}{\psi^3}\right) + \frac{g}{\psi^3}\frac{v^2}{v_0^2}\frac{2}{\psi^3}\frac{d\psi}{dr}\right]\left[1 + \frac{2u_0}{v_0}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}C_{nm}\right.\\ & \left.* \left(\frac{v_z}{v_0}\right)^n\left(\frac{v}{v_0\psi}\right)^{2m}\right] - \frac{2u_0}{v_0}\frac{g}{\psi^3}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}C_{nm}\left(\frac{v_z}{v_0}\right)^n\left(\frac{v}{v_0\psi}\right)^{2m}\frac{2m}{\psi}\frac{d\psi}{dr}\,,\end{aligned} \tag{D.4}$$

$$F \propto \frac{g}{\psi^3}\frac{2u_0}{v_0^2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}C_{nm}n\left(\frac{v_z}{v_0}\right)^{n-1}\left(\frac{v}{v_0\psi}\right)^{2m}\,, \tag{D.5}$$

substitute them in (D.3), and obtain after equating equal powers of v_z and v,

$$\frac{\psi^3}{g}\frac{d}{dr}\frac{g}{\psi^3} + \frac{2u_0}{v_0}\frac{q}{mc}B_\theta\,C_{10} = 0\,, \tag{D.6}$$

$$\frac{2}{\psi}\frac{d\psi}{dr} + \frac{2u_0}{v_0}\frac{q}{mc}B_\theta\,C_{11} = 0\,, \tag{D.7}$$

which comes from the terms independent of v_z. It is easily seen that $C_{1m} = 0$, $m \geq 2$, in fact, all $C_{nm} = 0\, m \geq n+1$. Relationships between some other coefficients are

$$C_{20} = -\frac{2u_0}{v_0}C_{10}\,, \tag{D.8}$$

$$C_{21} = \frac{2u_0}{v_0}\left[\frac{C_{11}}{2}C_{10} + (1 - C_{11})C_{\|}\right]\,, \tag{D.9}$$

$$C_{22} = \frac{2u_0}{v_0}\frac{C_{\parallel}^2}{2}\,. \tag{D.10}$$

Each higher coefficient is determined in terms of the lower order coefficients. All of them are determined in terms of $C_{10} = 1$ and $C_{11} = -\beta$. Similar procedure applies for current in other directions.

Appendix E

Let the electrons have current in both z- and θ-direction, while the ions carry current only in θ. Correct to $\mathcal{O}(u_0/v_0)$, the distribution functions are

$$\begin{aligned} f_e &= \frac{n_0 g}{\pi^{3/2} v_{0e}^3 \psi^3} \left\{ 1 + \frac{2u_{0e}}{v_{0e}^2} v_z \left(1 - \beta \frac{v^2}{v_{0e}^2 \psi_e^2} \right) + \frac{2V_e}{v_{0e}^2} v_y \left(1 - \frac{v^2}{v_{0e}^2 \psi^2} \right) \right\} \\ &* \ \exp\left[-v^2 / v_{0e}^2 \psi^2 \right] \end{aligned} \tag{E.1}$$

and f_i is represented by Eq. (57). Using (E.1) and (57), the plasma currents can be calculated. We had shown in the text that while the $\hat{\theta}$ drifts contribute essentially to the force term in the Vlasov equation, only the z-current is important for calculating the self-fields. Substituting (E.1) and (57) into the Vlasov equation, we obtain to leading order [$\psi_e = \psi_i = \psi$, and $eu_{e0}/cT_{e0} = h$] from the electrons,

$$\frac{1}{g}\frac{dg}{dr} - \frac{3}{\psi}\frac{d\psi}{dr} = h\left[B_\theta - \frac{V_e}{u_{e0}} B_0 \right] \tag{E.2}$$

$$\frac{2}{\psi}\frac{d\psi}{dr} = h\left[-\beta B_\theta + a\frac{U_e}{v_{e0}} B_0 \right] \tag{E.3}$$

and from the ions

$$\frac{1}{g}\frac{dg}{dr} - \frac{3}{\psi}\frac{d\psi}{dr} = \frac{h}{\tau}\frac{V_i}{u_{e0}} B_0 \tag{E.4}$$

$$\frac{2}{\psi}\frac{d\psi}{dr} = -\frac{h\mu}{\tau}\frac{V_i}{u_{e0}} B_0 \tag{E.5}$$

consistency constraints imply

$$\frac{V_e}{V_i} = \frac{1}{\tau}\frac{\beta - \mu}{a - \beta} \tag{E.6}$$

$$V_e + V_i/\tau = u_{e0}\, B_\theta / B_0 \tag{E.7}$$

which determine V_e and V_i in terms of the field B_θ (which will be eventually solved), and

$$g = \psi^{3 - \frac{2}{\mu}} \, . \tag{E.8}$$

The remaining independent equation

$$\frac{2}{\psi}\frac{d\psi}{dr} = h\frac{\mu(\beta - a)}{a - \mu}B_\theta = -\frac{\mu(a - \beta)}{a - \mu}b \tag{E.9}$$

needs to be solved in conjunction with Ampere's law.

$$\frac{1}{r}\frac{d}{dr}rb = \frac{2}{\delta^2}(5\beta/2 - 1)\psi^{5-\frac{2}{\mu}} \tag{E.10}$$

giving the same structure of profiles as Eqs. (63)-(68) with β replaced by μ, and $\bar{\delta}_t = (c/\omega_{pe})(v_{0e}/u_{0e})|(5\beta/2 - 1)(5\mu/2 - 1)(a - \beta)/(a - \mu)|^{-1/2}$ which for all a reduces to δ_t if $\beta = \mu$. Thus a simply fixes the ratio of V_e to V_i.

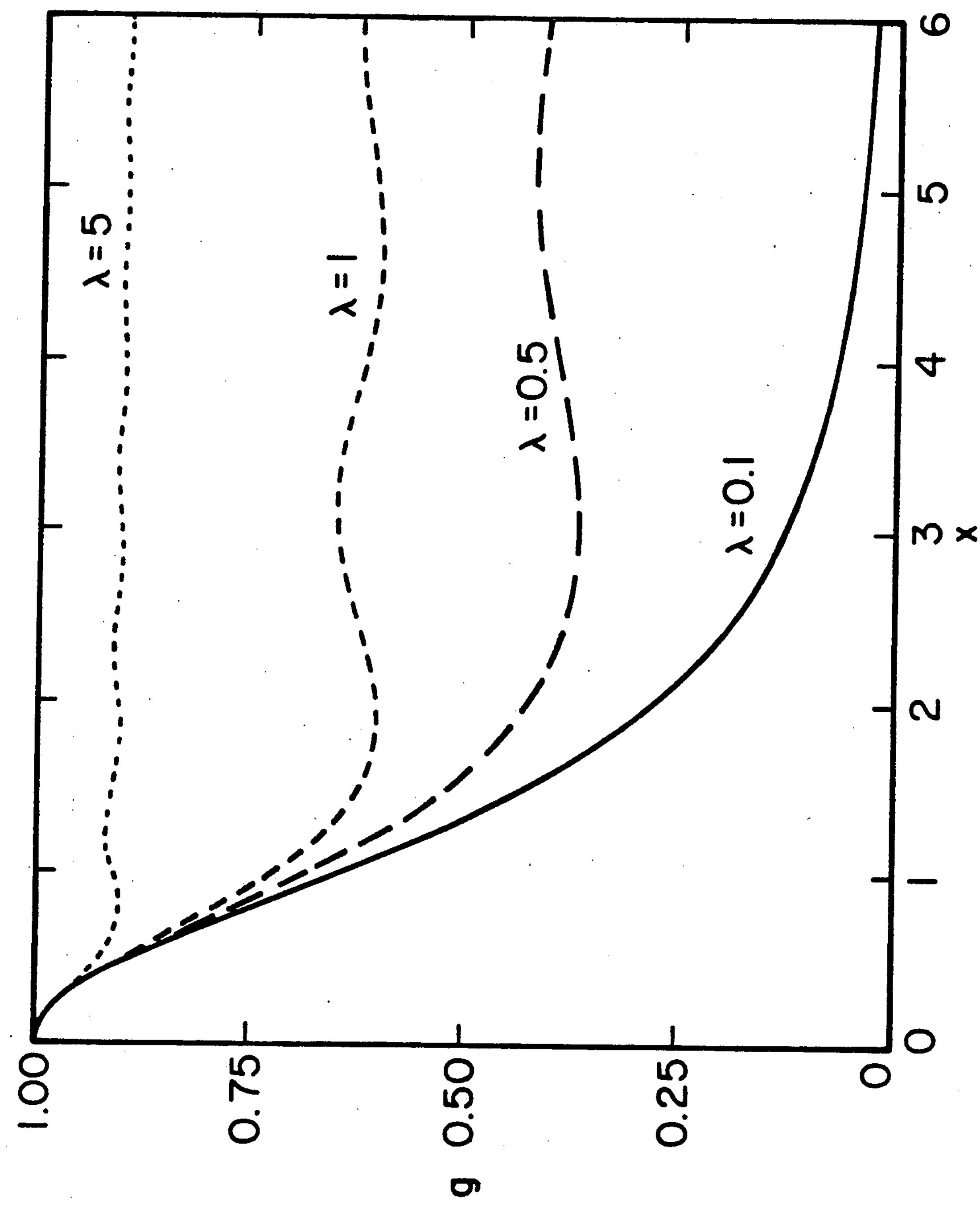
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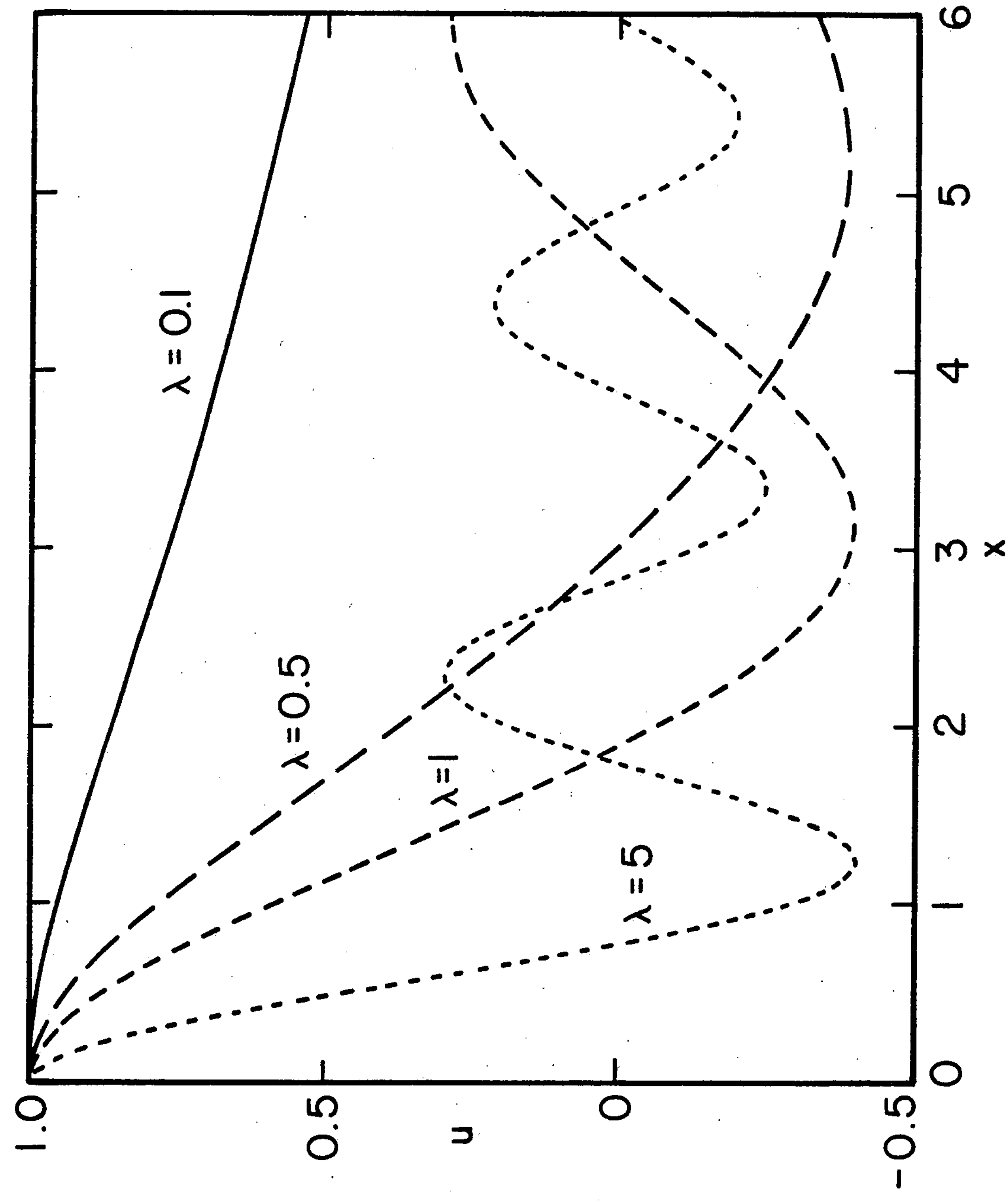
Fig. 1



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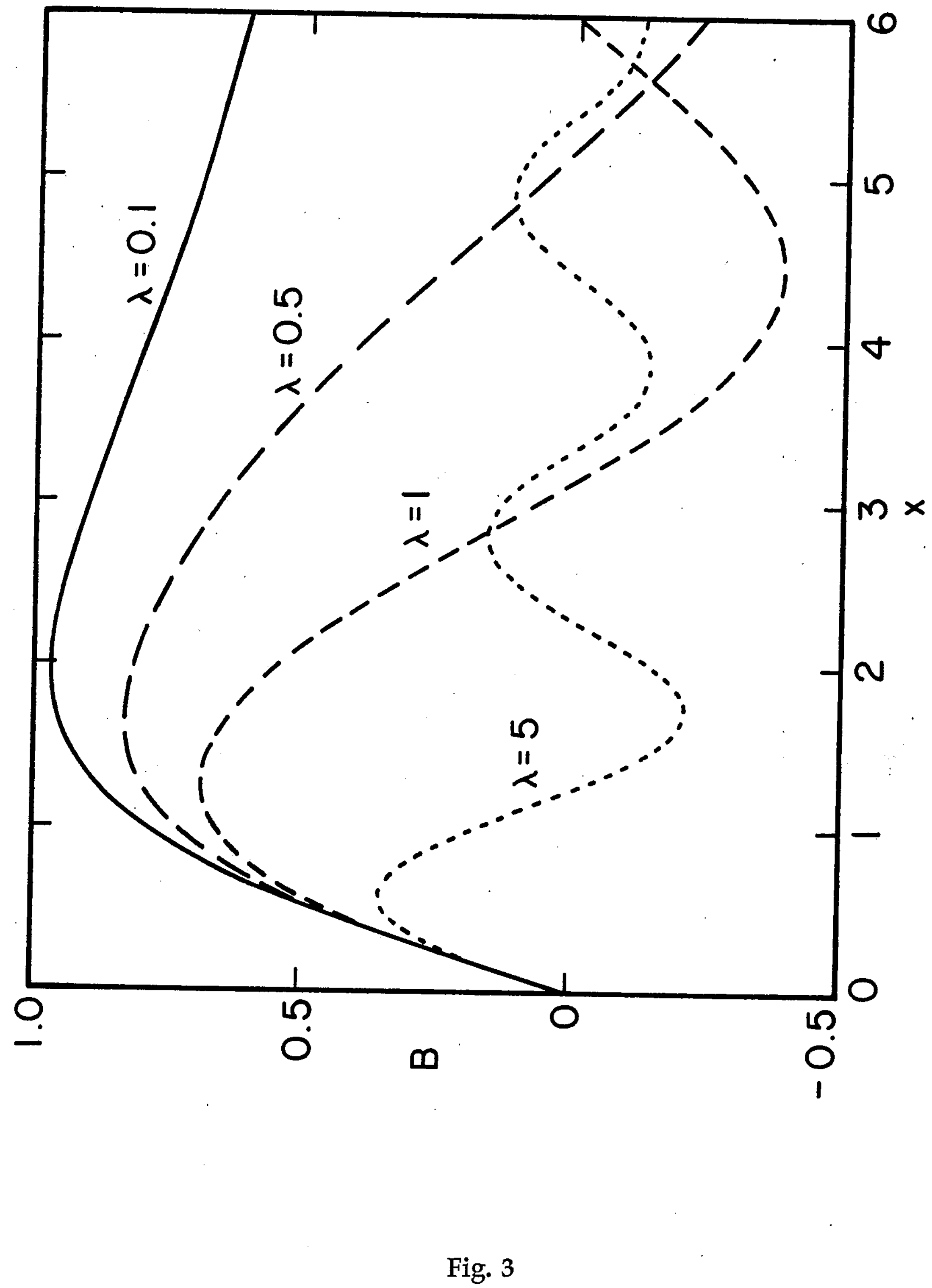

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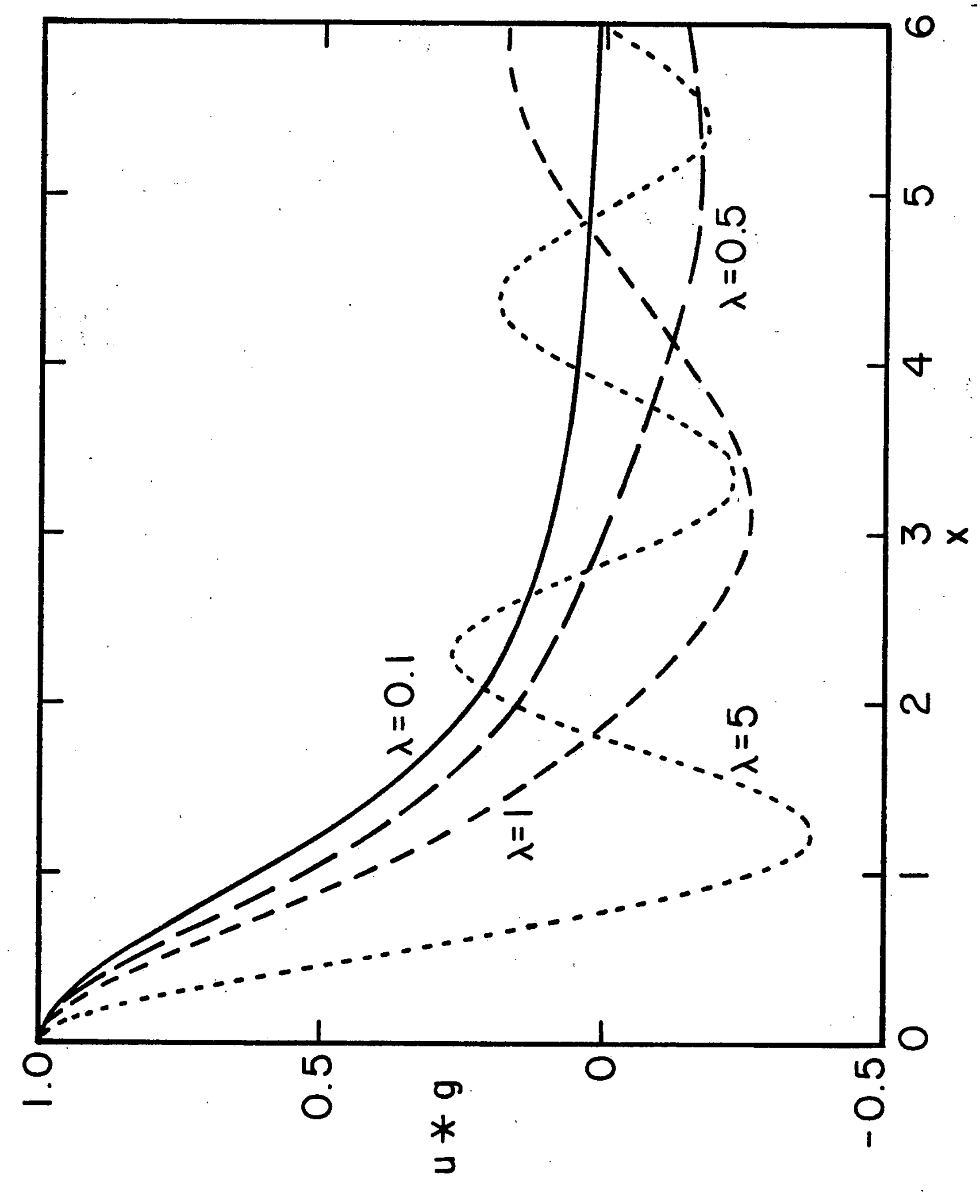

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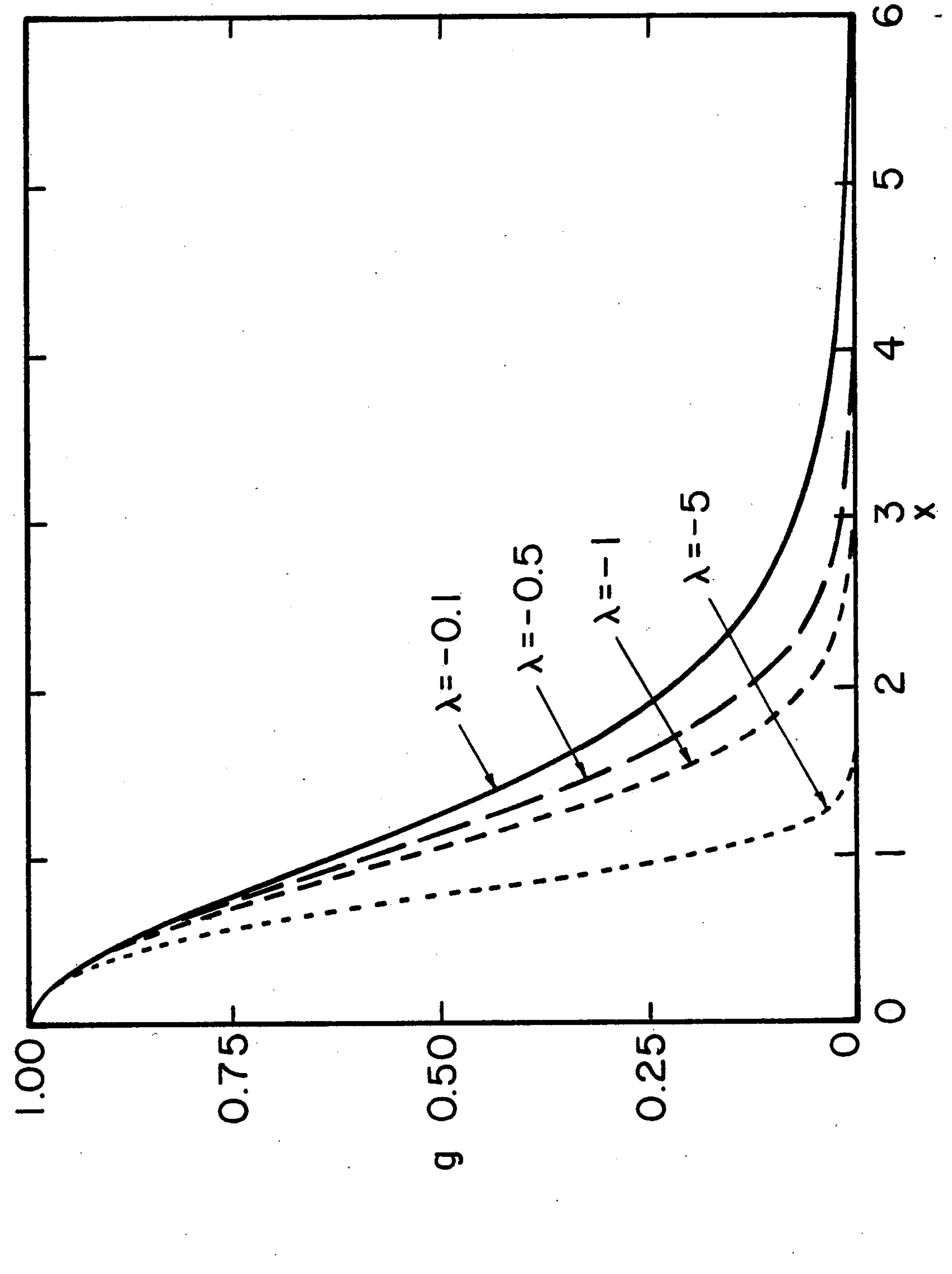
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Fig. 5

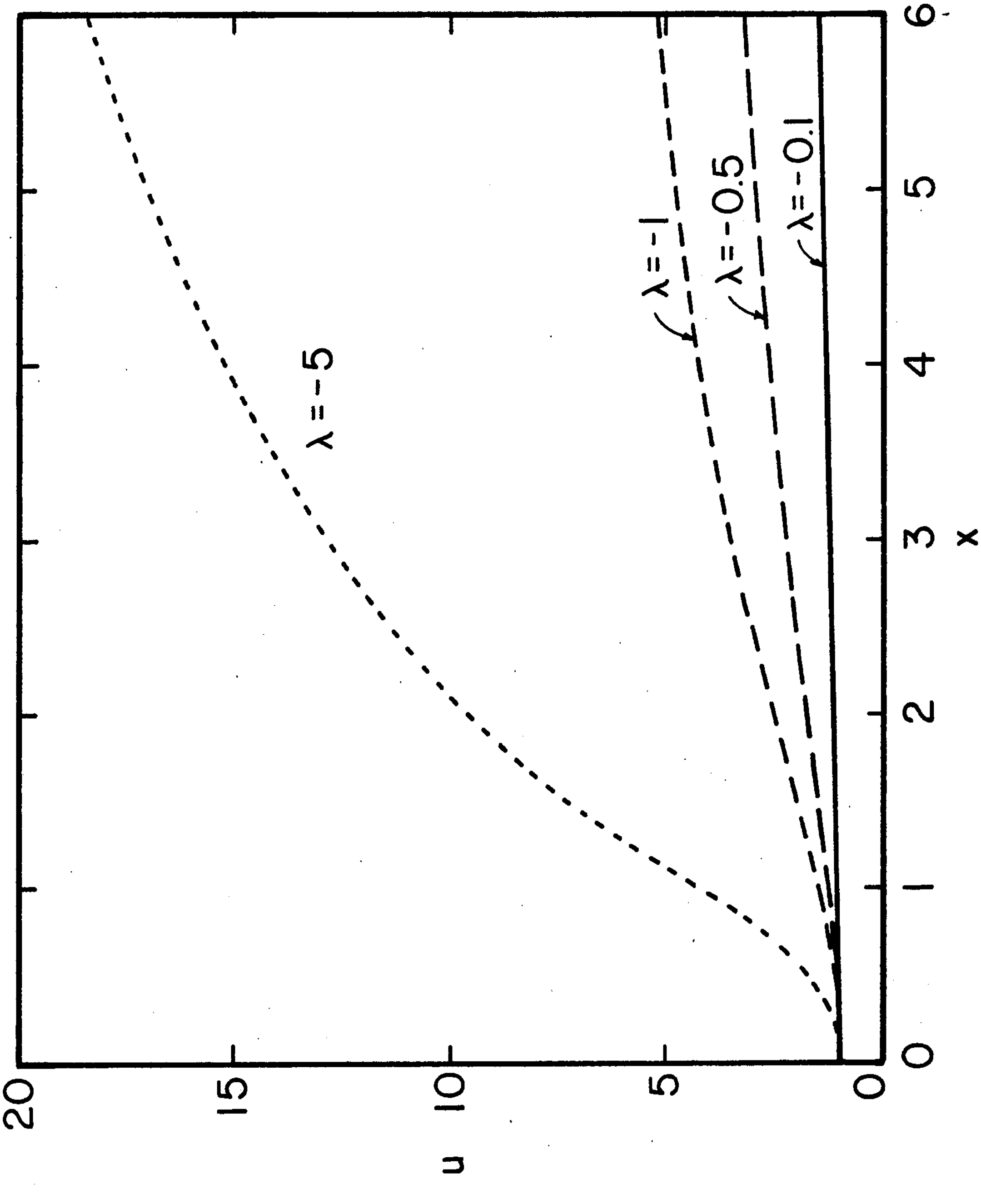

Fig. 6

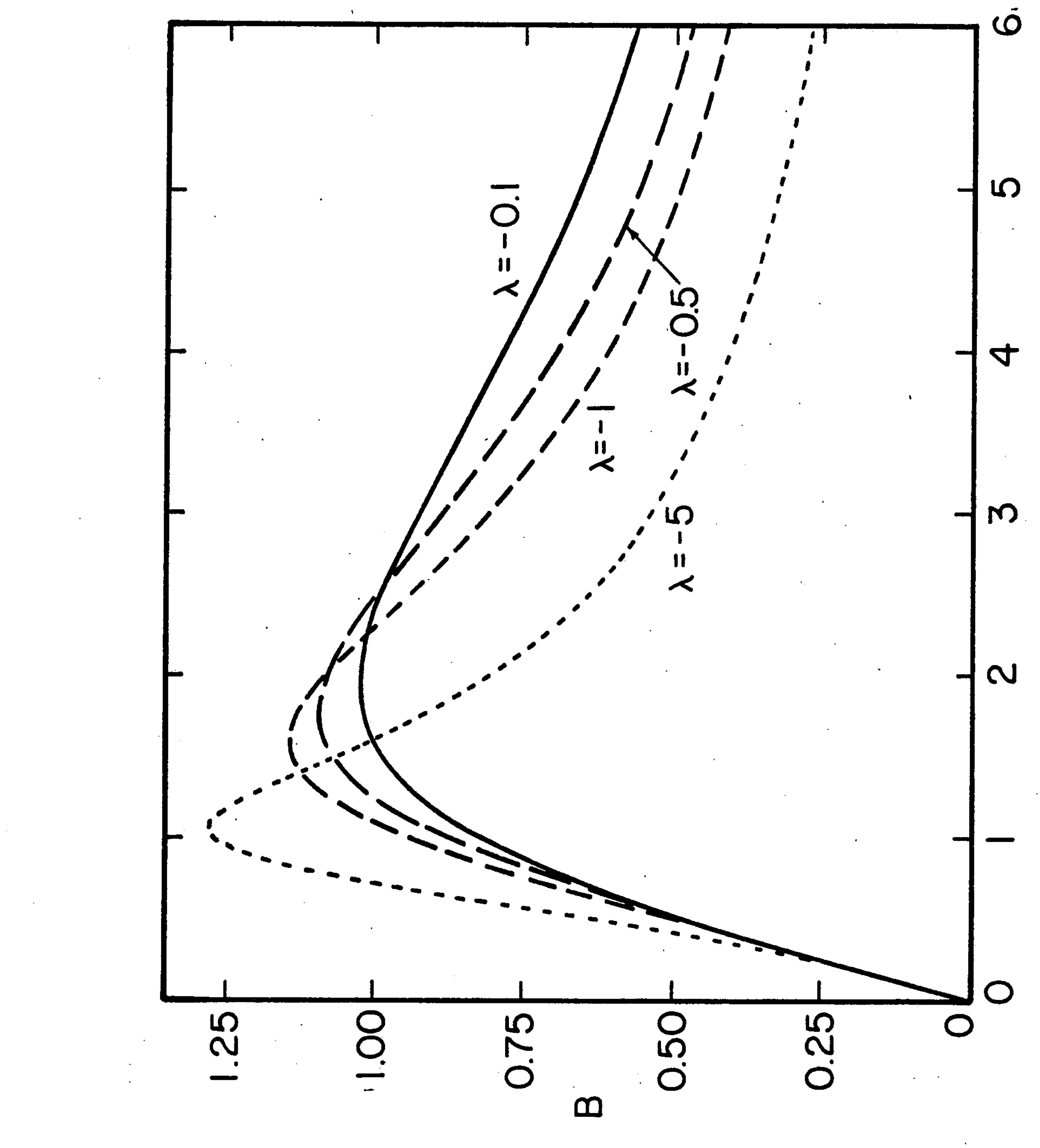
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Fig. 7

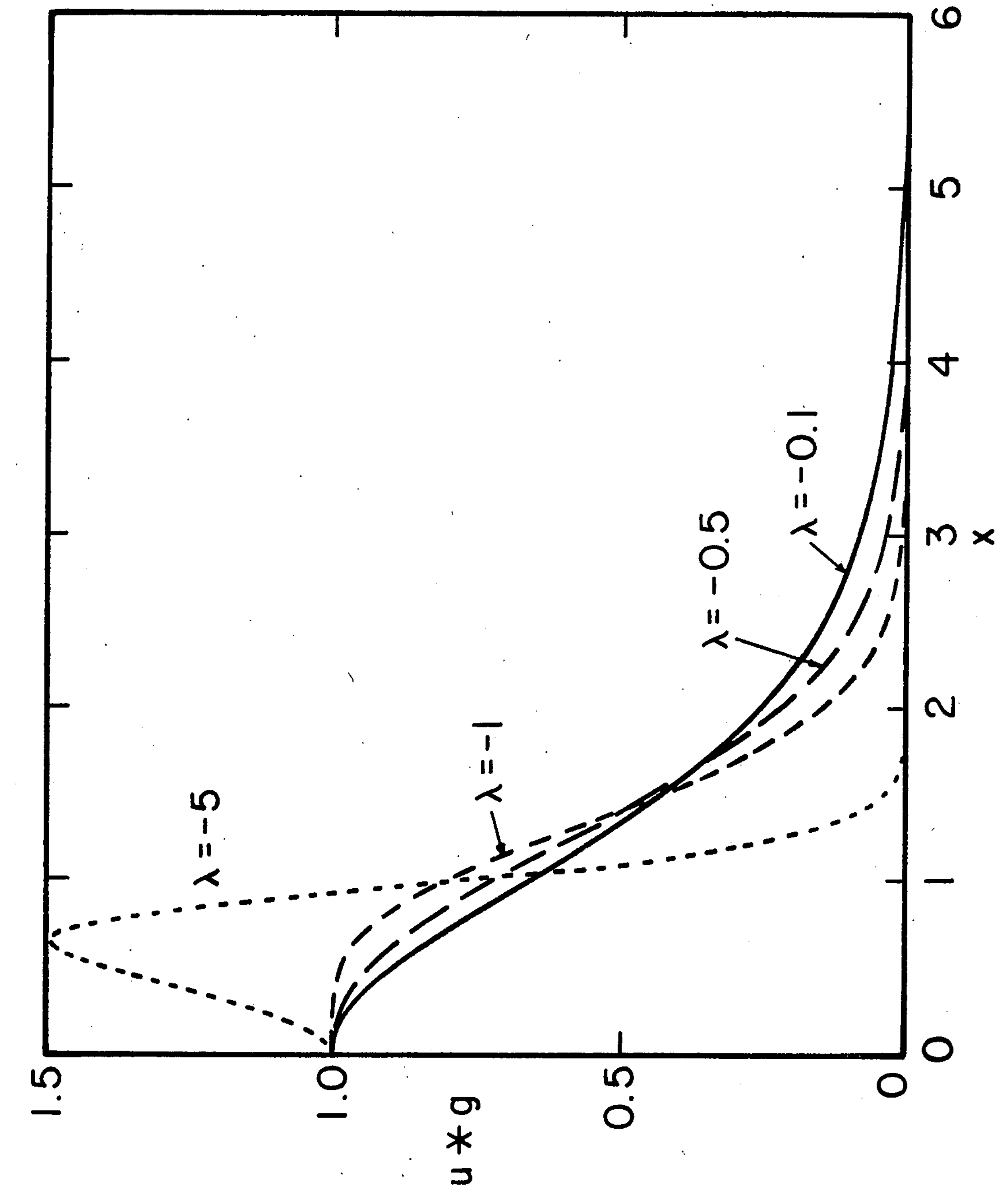
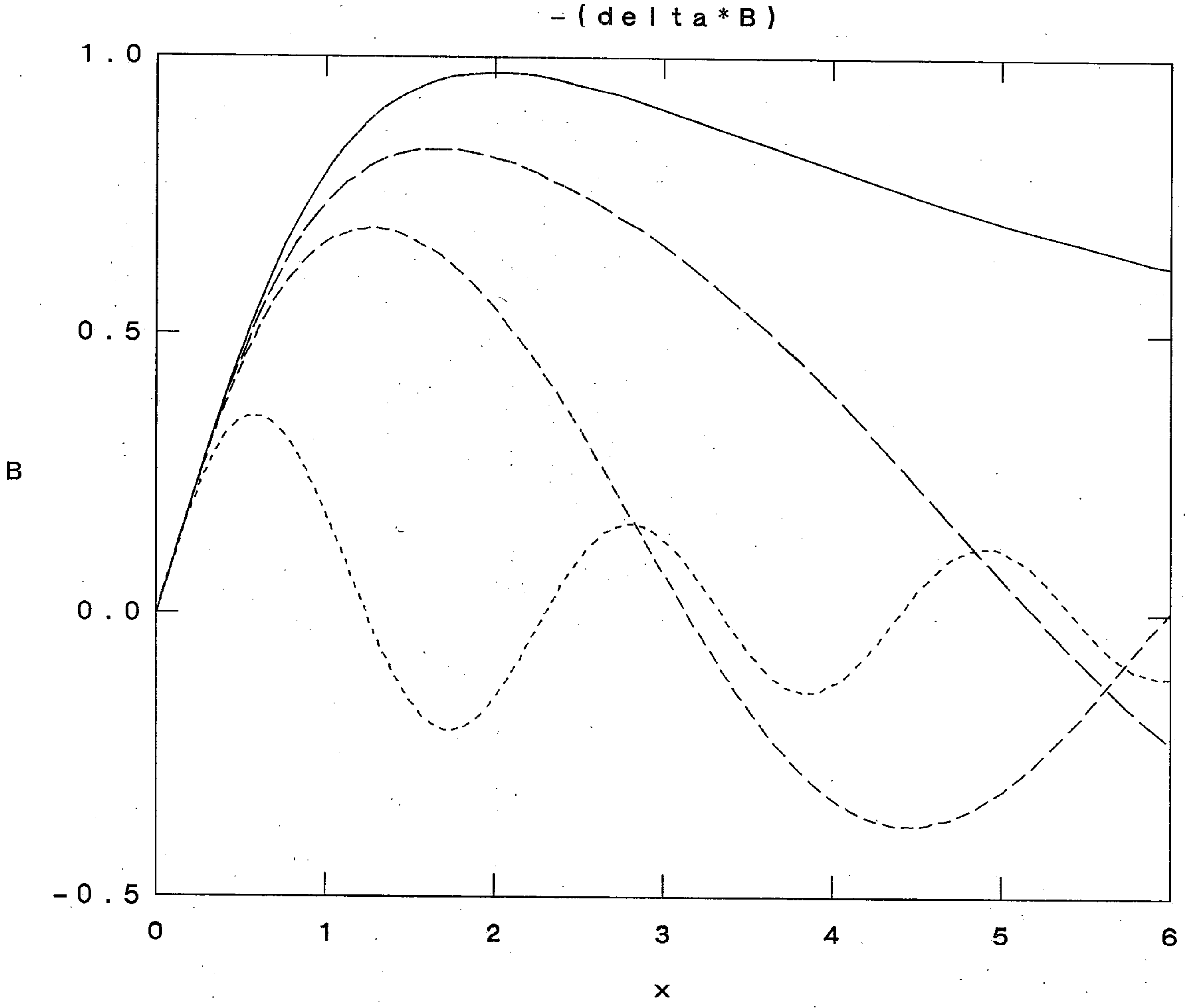


Fig. 8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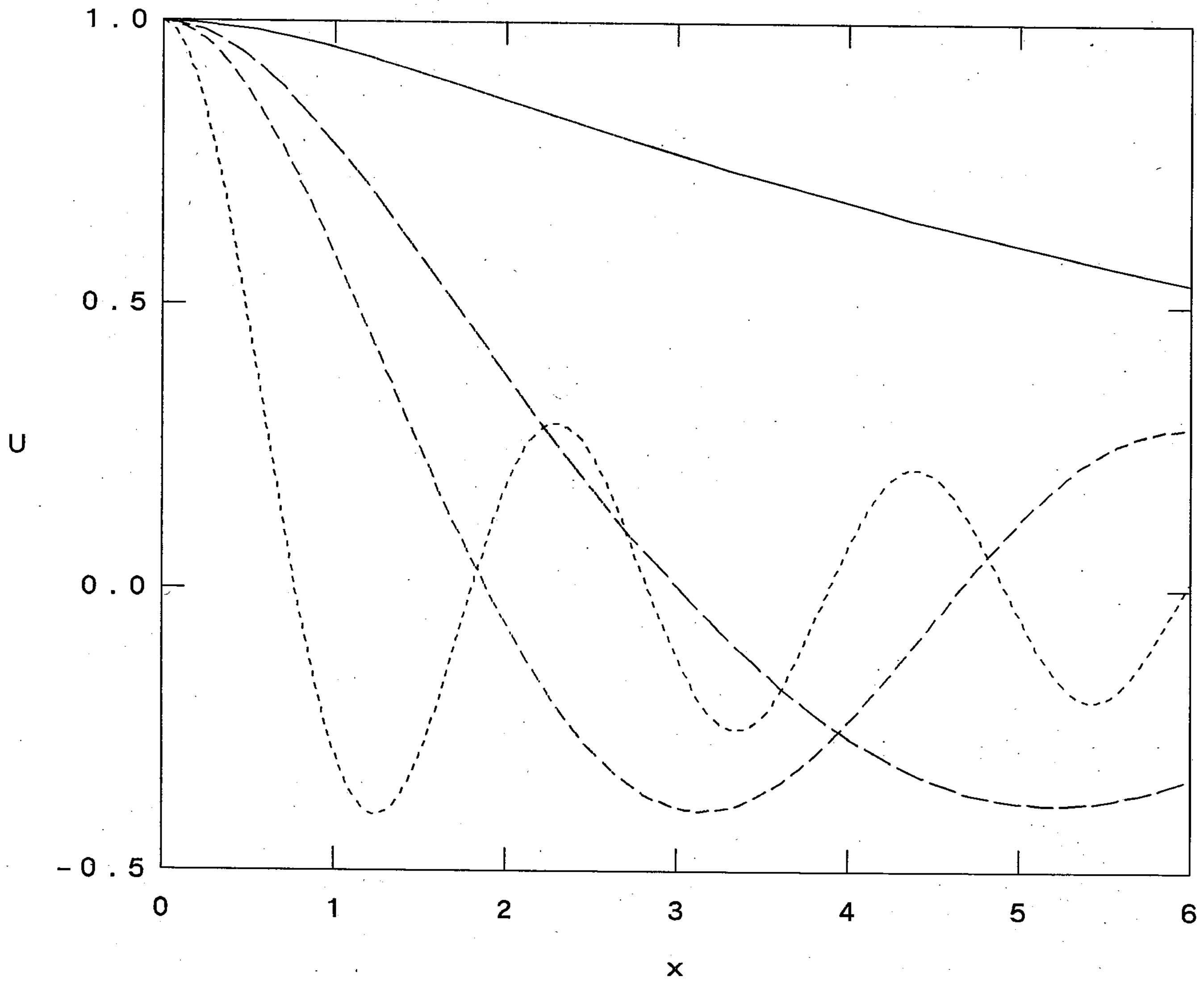
x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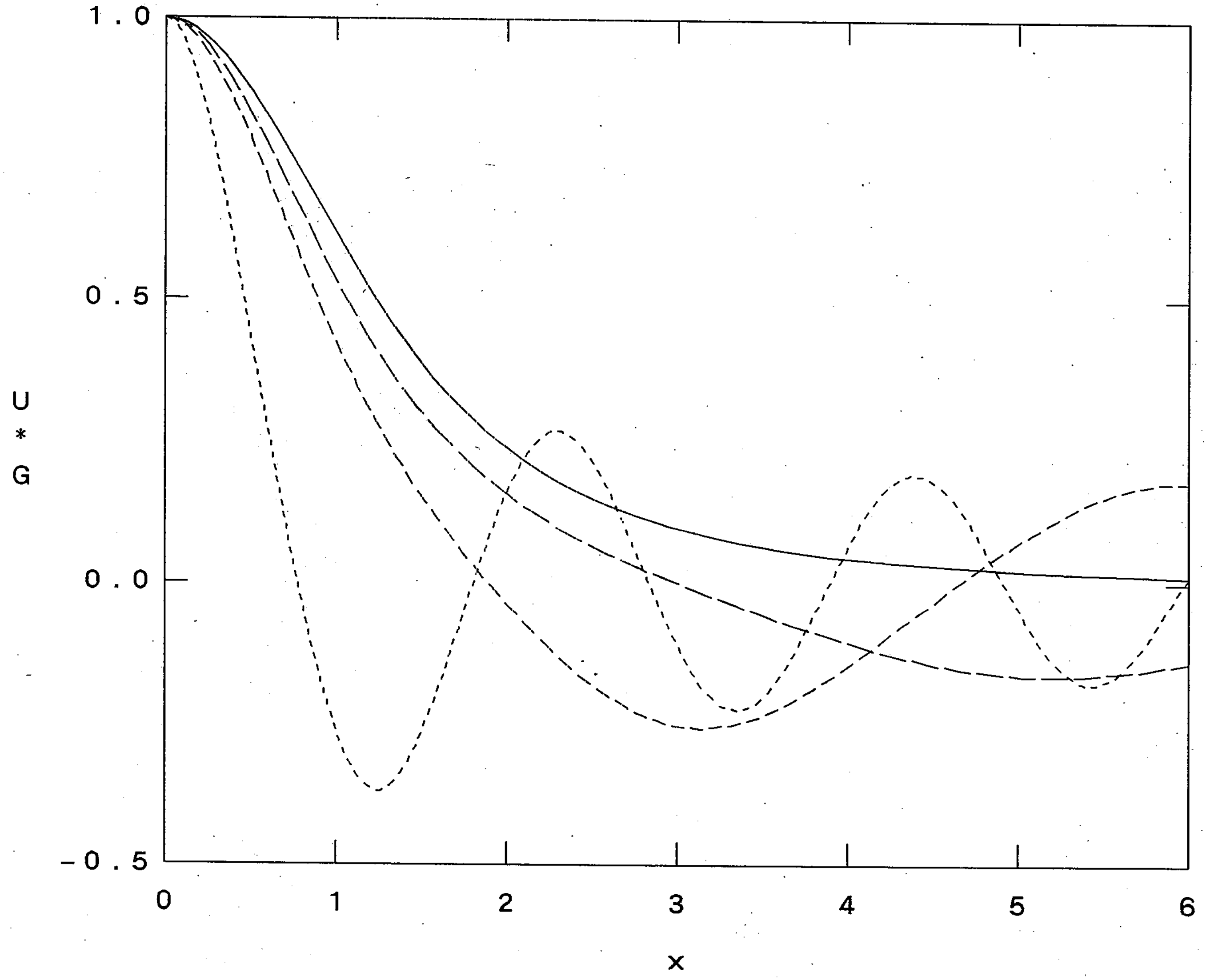
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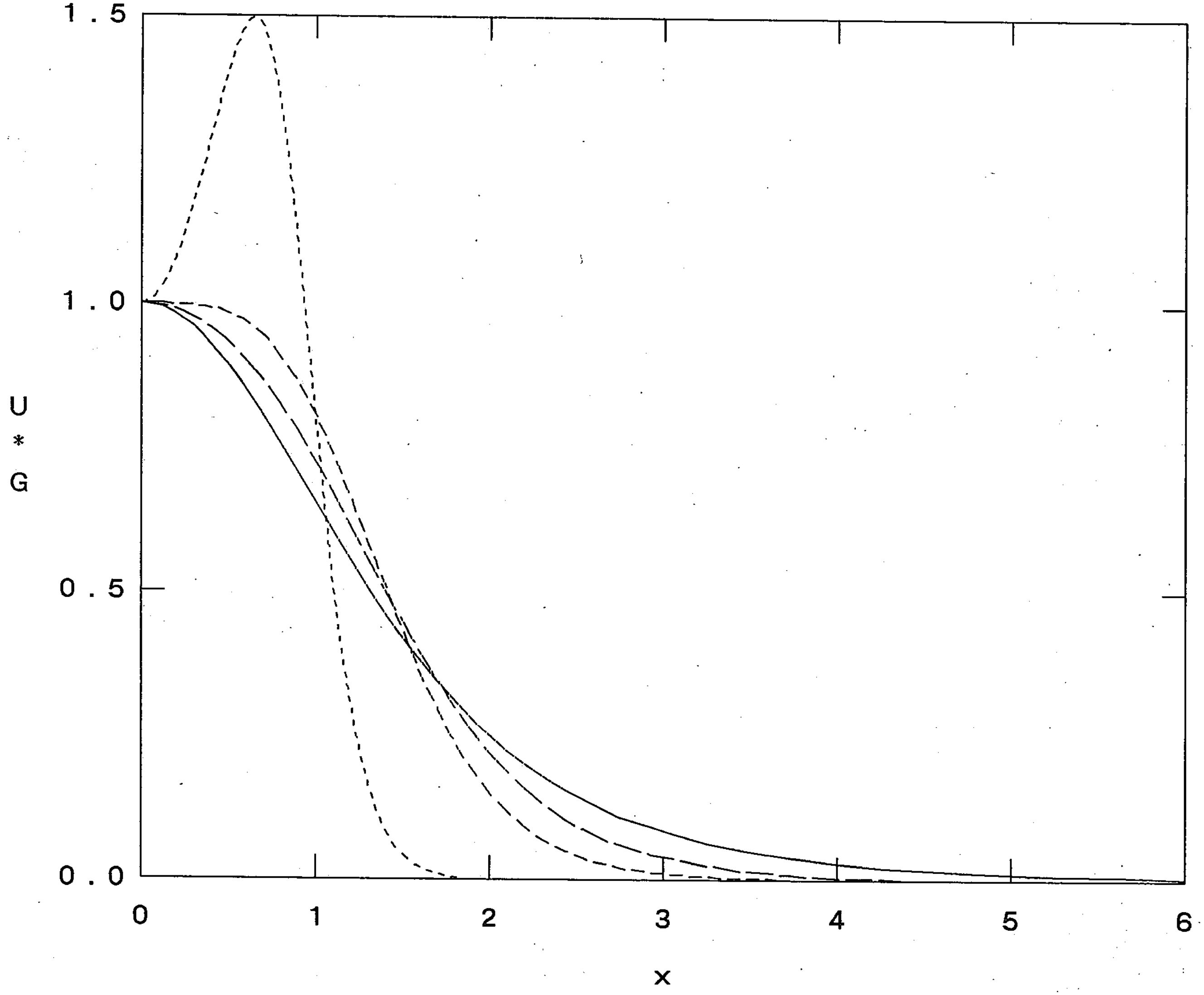
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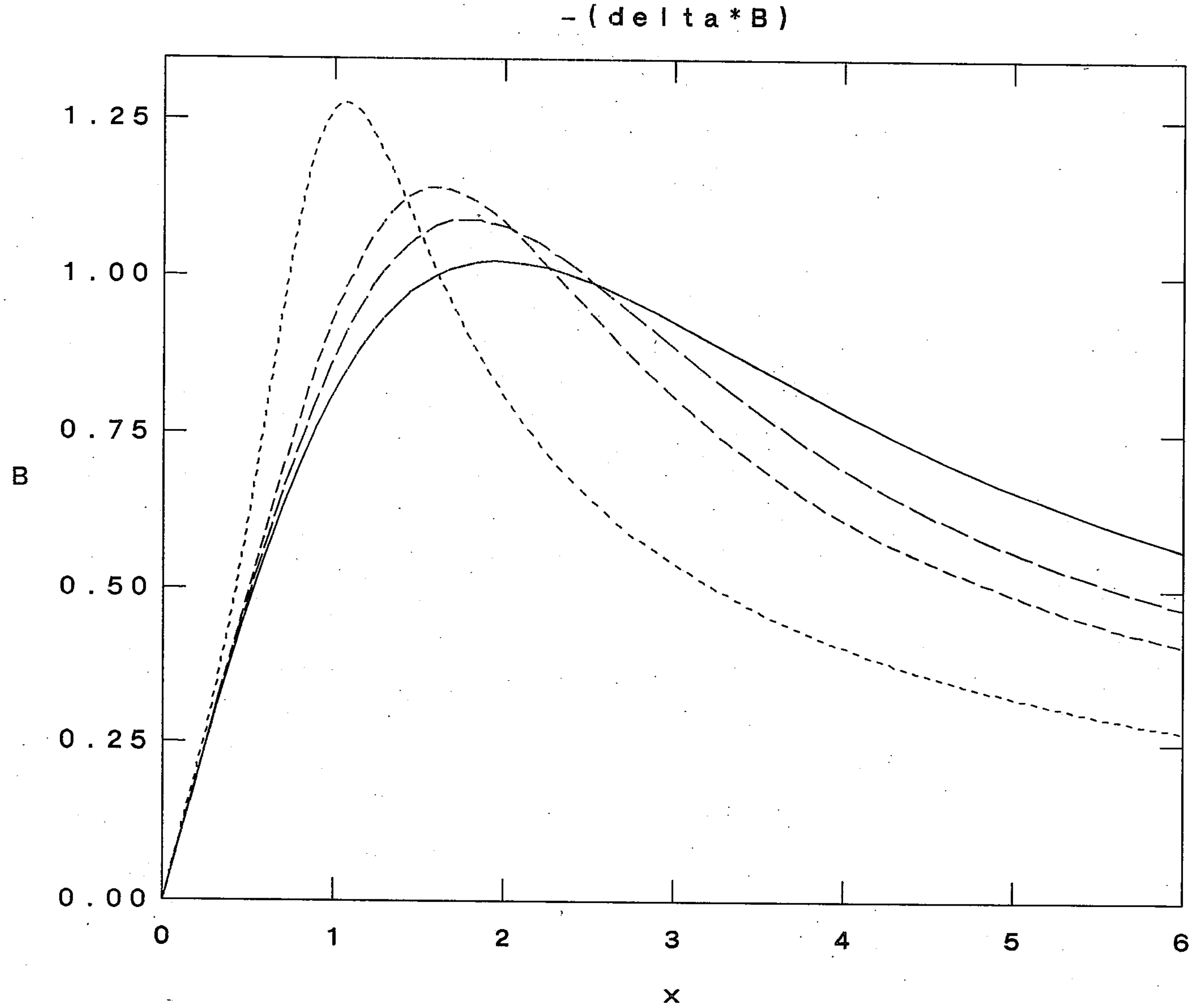
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2
3
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