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**Abstract** The sawtooth mapping is a family of uniformly hyperbolic, piecewise linear, area-preserving maps on the cylinder. We construct the resonances, cantori, and turnstiles of this family and derive exact formulas for the resonance areas and the escaping fluxes. These are of prime interest for an understanding of the deterministic transport which occurs the stochastic regime. The resonances are shown to fill the full measure of phase space.

Developing techniques to describe transport in Hamiltonian systems remains a challenging problem. Although significant progress has been made during the last few years<sup>1</sup>, a complete understanding is still lacking. The study is plagued by the divided phase space structure of typical Hamiltonian systems: the coexistence of both regular and irregular motions. Instead of looking at mixed systems, in this study we address the problem for a family of uniformly hyperbolic systems: the sawtooth map. We outline here the preliminary theoretical results. We hope to present a detailed study later.

The sawtooth map is derived by linearizing the standard map about the latter's fixed points. It is piecewise linear; the nonlinearity arises from the discontinuity on the dominant symmetry line. In the parameter range we will consider, the map is everywhere hyperbolic and possesses the nice property that the Lyapunov exponent is constant over the whole phase space. The sawtooth map has already been studied extensively by various authors<sup>2</sup>. We first discuss its Poincare-Birkhoff orbits. Although most of the results have been obtained before, we have no knowledge of any prior discussion of the minimax periodic orbits which are necessary to construct the turnstiles of the map. Next we construct the turnstiles and resonances and derive exact formulas for the fluxes and resonance areas. These two quantities are of prime interest according to the theory of transport<sup>1</sup>. Some of the results have been independently obtained by I. Dana, N. Murray and I.C. Percival<sup>3</sup>.

The sawtooth map on the cylinder  $[0,1) \times \mathbb{R}$  is defined by

$$\begin{aligned} p_{t+1} &= p_t + k(x_t - 1/2) \\ x_{t+1} &= x_t + p_{t+1} \quad \text{mod}(1) \end{aligned} \quad (1a)$$

or equivalently

$$x_{t+1} - 2x_t + x_{t-1} = k(x_t - 1/2) \quad \text{mod}(1) \quad (1b)$$

where  $k$  is the parameter of nonlinearity. In our coordinate system, the map is discontinuous on the line  $x=0$ . This is also the dominant symmetry line<sup>4</sup>, as will be shown later. We will only consider positive values of

the parameter  $k$ .

The sawtooth map can be lifted to the plane  $\mathbb{R}^2$ ; we denote the angle coordinate lifted to the real line by  $u$ . Then an orbit is equivalent to the stationary state of the following action

$$W(\{u_t\}) = \sum_t \left\{ (u_{t+1} - u_t)^2/2 + k/2 [\text{saw}(u_t - 1/2)]^2 \right\} \quad (2)$$

where the sawtooth function is

$$\begin{aligned} \text{saw}(u) &= u & -1/2 < u < 1/2 \\ \text{saw}(u+1) &= \text{saw}(u) \end{aligned} \quad (3)$$

The Poincare-Birkhoff orbits of an area preserving map are those periodic configurations which minimize or minimaximize the action (2). These will be referred to as the minimizing and minimax orbits, respectively. A fundamental theorem by Aubry and Le Daeron<sup>5</sup> states that for a minimizing orbit with rotation frequency  $\nu$ , there exists a phase constant  $\alpha$  such that the configuration points  $u_t$  belong to the interval  $[M_t, M_{t+1}]$ , where  $M_t = \text{int}(\nu t + \alpha)$ , or more precisely

$$u_t = x_t + M_t \quad (4)$$

This implies we can rewrite eq (1b) to obtain

$$x_{t+1} - 2x_t + x_{t-1} - k(x_t - 1/2) = -b_t \quad (5)$$

where  $\{b_t\}$  is called the symbol sequence of the minimizing orbit by Percival and Vivaldi<sup>2</sup>; it is determined by the frequency through

$$b_t = M_{t+1} - 2M_t + M_{t-1} \quad (6)$$

A periodic orbit of type  $(m,n)$  satisfies  $u_{t+n} = u_t + m$ . Note that the symbol sequence of a periodic orbit is also periodic. Solving eq(5) for an

(m,n) periodic configuration, we find<sup>2</sup>

$$x_t = \sum_{s=0}^{n-1} \frac{(\lambda^{-s} + \lambda^{-n+s})}{(1 - \lambda^{-n})} b_{s+t} + 1/2 \quad (7)$$

where

$$\lambda = 1 + (k + \sqrt{D})/2, \quad D = k^2 + 4k \quad (8)$$

In fact, this configuration is minimizing. It is easy to see that any minimax periodic orbit must have one and only one point at the maximum of the potential  $x=0$ , because if this is not true then the second variation of the action is

$$\delta^2 W(\{u_t\}) = 1/2 \sum_{s=0}^{n-1} \sum_{t=0}^{n-1} H_{s,t} \delta u_s \delta u_t \quad (9)$$

$$H_{s,t} = -\delta_{s,t-1} - \delta_{s,t+1} + (2+k)\delta_{s,t} \quad (10)$$

where the indices  $s, t$  are assumed to be cyclic modulo  $n$ , the period of the orbit. Thus  $\delta^2 W > 0$ , for  $k > 0$ , i.e. this configuration is always a local minimum of the action. However, according to Aubry and Le Daeron<sup>5</sup>, an (m,n) minimax configuration exists, each point of which lies between corresponding (m,n) minimizing configuration. The equation of motion for minimax configurations is still given by (1) except at  $x=0$ , where we take the force term to be zero, since minimax configurations must be symmetric for the sawtooth map.

In the following discussion, it is convenient to let  $x_0$  be the leftmost point of the orbit in the unit interval  $[0,1)$ . Then we claim that the minimax orbit  $\{x'_t\}$  of rotation number  $m/n$  has the same symbol sequence as the minimizing (m,n) orbit  $\{x_t\}$  and is given by

$$x'_0 = (x_0 + x_\varphi - 1)/2 = 0$$

$$x'_t = (x_t + x_{t+\ell})/2 \quad t \not\equiv 0 \pmod{n} \quad (11)$$

where  $x_{t+\ell}$  is the left neighboring point of  $x_t$ , and  $\ell$  is the unique integer determined by

$$m\ell \equiv -1 \pmod{n}, \ell < n$$

or

$$m\ell - nj = -1, \ell < n \quad (12)$$

The geometrical meaning of (11) is that the points on the minimax periodic orbit are exactly halfway between the neighboring points of the corresponding minimizing periodic orbit.

Now we show that the orbit given by eq(10) indeed has the same symbol sequence as the minimizing orbit. Let  $M_t$  be

$$M_t = \text{Int}\{1/2n + (m/n)t\} \quad (13)$$

it follows from eqs (12) and (13)

$$\begin{aligned} M_{t+\ell} &= M_t + j & t \not\equiv 0 \pmod{n} \\ M_\ell &= M_0 + j - 1 = j - 1 \end{aligned}$$

or equivalently

$$\begin{aligned} b_{t+\ell} &= b_t & 1 < t < n-1 \\ b_{1+\ell} &= b_1 - 1 \\ b_{n-1+\ell} &= b_{n-1} - 1 \\ b_\ell &= b_0 + 2 \end{aligned}$$

This implies that eq(5) is satisfied for  $1 < t < n-1$

$$x'_{t+1} - 2x'_t + x'_{t-1} - k(x'_t - 1/2) = - (b_t + b_{t+\ell})/2 = -b_t$$

We need also verify eq(5) for  $t = 1$  and  $t = n-1$ . It suffices to consider the case  $t = 1$

$$x'_2 - 2x'_1 + x'_0 - k(x'_1 - 1/2) = - (b_1 + b_{1+\ell} + 1)/2 = -b_1$$

The equation for the minimax orbit at  $t = 0$  is automatically satisfied

$$x'_{-1} + x'_1 = 1 = -b_0$$

The gap function at time  $t$  for the minimizing periodic configuration  $\{x_t\}$  is defined to be

$$\xi_t = x_t - x_{t+\ell} + \delta_{0,t} \quad (14)$$

The equation satisfied by  $\xi_t$  is

$$\begin{aligned} & \xi_{t+1} - 2\xi_t + \xi_{t-1} - k\xi_t \\ &= (-b_t + b_{t+\ell} + \delta_{0,t+1} + \delta_{0,t-1} - 2\delta_{0,t}) - k\delta_{0,t} \\ &= -k\delta_{0,t} \end{aligned} \quad (15)$$

hence for a period  $n$  orbit

$$\xi^{(n)}_t = k(\lambda^{-t} + \lambda^{-n+t})/(\sqrt{D}(1 - \lambda^{-n})) \quad (16)$$

where we use the superscript  $n$  to emphasize that the gap function depends only on the period of the orbit, and not its frequency. Formula (16) was first obtained by Percival and Vivaldi by a continuity argument<sup>2</sup>.

An immediate consequence of eq(16) is that the leftmost point of the minimizing orbit is given by

$$x^{(n)}_0 = \xi^{(n)}_0/2 = k(1 + \lambda^{-n})/(2\sqrt{D}(1-\lambda^{-n})) \quad (17)$$

By taking  $n$  to infinity, we obtain the gap function for both cantori and homoclinic orbits

$$\xi^{(\infty)}_t = k\lambda - |t|/\sqrt{D} \quad (18)$$

Formula (18) has interesting implications, e.g. the fractal dimension of any cantorus is zero. This fact was first discovered for the golden mean cantorus in the standard map, and later was proved for any hyperbolic cantorus<sup>6</sup>.

The flux across the  $(m,n)$  minimizing orbits is obtained by constructing the turnstile of the orbit<sup>1</sup>. We connect the neighboring points of the  $(m,n)$  minimizing orbit by straight line segments, then iterate these line segments one step backwards. Note that these line segments pass through the  $(m,n)$  minimax orbit (fig. 1). Due to the discontinuity of the sawtooth map on its dominant symmetry line, one line segment is broken to two pieces (fig. 1). The region bounded by the dominant symmetry line, the broken pieces and the original line segments is defined as the flux region, or turnstile. It is shown as the two shaded triangles in fig. 1. Hence the area of either one of the two triangles gives us the flux  $\Delta W$  across the  $(m,n)$  minimizing orbit in one iteration of the map

$$\Delta W(m,n) = 1/4 \Delta p x_0$$

where  $\Delta p$  is the discontinuity in  $p$  at  $x = 0$ . By eq(1b)

$$\Delta p' = \Delta p - k$$

Since the discontinuity maps to a continuous segment after one iteration,  $\Delta p' = 0$ , so

$$\Delta p = k$$

thus using eq(17) we obtain



$$\Delta W(m,n) = k^2 (1 + \lambda^{-n}) / (8\sqrt{D} (1 - \lambda^{-n})) \quad (19)$$

The flux across a cantorus or a homoclinic orbit is given by letting  $n$  tend to infinity

$$\Delta W_C = k^2 / 8\sqrt{D} \quad (20)$$

One can also find the flux by taking the action difference of the corresponding minimax and minimizing orbits<sup>1</sup>

$$\begin{aligned} \Delta W(m,n) &= W(\{x'_t\}) - W(\{x_t\}) \\ &\quad n-1 \\ &= 1/8 \left\{ 2k\xi_0 - \sum_{t=0}^{n-1} [ (\xi_{t+1} - \xi_t)^2 + k \xi_t^2 ] \right\} \end{aligned}$$

which, by eq(16), yields eq(19) again.

An  $(m,n)$  resonance is, roughly speaking, the region bounded by the stable and unstable manifolds of the minimizing  $(m,n)$  periodic orbit<sup>1</sup>. For the sawtooth map, the stable and the unstable manifolds of any periodic orbit are straight line segments: the slope does not depend on the period of the orbit. Thus the resonances of the sawtooth map are constructed simply by drawing the unstable(stable) manifolds of the points on the periodic orbit till they intersect the stable(unstable) manifolds of the neighboring points; these defines the upper and lower partial separatrices of the resonances. Some of the resonances of the sawtooth map are shown in fig. 2. Note they are all parallelograms.

It is straight forward to calculate the area of an  $(m,n)$  resonance. This is illustrated in fig. 3 for the  $(1,2)$  resonance. For an  $(m,n)$  minimizing orbit, the leftmost point is given by eq(17), that of the homoclinic orbit is obtained by taking  $n$  to infinity:  $h_1 = k/\sqrt{D}$ ,  $h_2 = x^{(n)}_0 - x^{(\infty)}_0 = k/(\sqrt{D} (\lambda^n - 1))$ . Simple geometry gives

$$h = h_2(\tan\alpha + \tan\beta)$$

where  $\tan\alpha$  and  $-\tan\beta$  are the slopes of the unstable and the stable manifolds

$$\begin{aligned} \tan\alpha &= k/(\lambda - 1) \quad \tan\beta = k/(1 - \lambda^{-1}) \\ h(m,n) &= h_2(\tan\alpha + \tan\beta) = k/(\lambda^n - 1) \end{aligned} \quad (21)$$

$$A(m,n) = nh(h_1 + h_2) = nk^2/\sqrt{D}(\lambda^n + \lambda^{-n} - 2) \quad (22)$$

One can verify that formulas (21), (22) also apply to (0,1) and (1,1) resonances if we take them as the same resonance.

Now we can define area as a monotone function of frequency<sup>1,8</sup>. It is continuous at each irrational and is given by the area under KAM curve or cantor; it is discontinuous at each rational and is given by the areas under the upper and lower partial separatrices of the resonance. Fig. 4 shows an area devil's staircase function in the sawtooth map. If the sum of all the jumps of the function is the total variation, it is called a complete devil's staircase. In fact there are two complete staircases in the sawtooth map, one is given by the height function (21), the other by the area function (22). Since these two quantities depend only on  $n$ , we may denote them as  $h(n)$  and  $A(n)$ . It is not difficult to prove these two devil's staircases are complete. For height function, we sum over all the steps

$$\sum_{n=1}^{\infty} \varphi(n) h(n) = 1$$

Where  $\varphi(n)$  is the Euler  $\varphi$ -function<sup>7</sup>, i.e. the number of positive integers not greater than and prime to  $n$

$$k \sum_{n=1}^{\infty} \varphi(n) / (\lambda^n - 1) = k/(\lambda + \lambda^{-1} - 2) = 1 \quad (23)$$

Introducing a new variable  $\theta$ :  $\lambda = e^\theta$ , we can rewrite eq(23) as

$$\sum_{n=1}^{\infty} \varphi(n) / (e^{n\theta} - 1) = 1 / [4\text{sh}^2(\theta/2)]$$

taking the derivative with respect to  $\theta$ , we get

$$\sum_{n=1}^{\infty} \varphi(n)n / \text{sh}^2(n\theta/2) = \text{ch}(\theta/2) / \text{sh}^3(\theta/2)$$

which is exactly the completeness condition for the area staircase

$$\sum_{n=1}^{\infty} \varphi(n)A(n) = 1 \quad (24)$$

The completeness of the area devil's staircase can be proved in a more general way<sup>8</sup>. For area-preserving maps in Frenkel-Kontorova class, Aubry showed that if the lower bound of the Lyapunov exponents of all the minimizing configurations is positive, the staircase is complete. He also showed that if the potential function is at least  $C^4$  (differentiable four times), the staircase is incomplete for sufficiently small parameter value. S. Bullett gives an example of complete area staircase for which the potential function is  $C^1$ , however the boundaries of the resonances are KAM curves with rational frequencies<sup>9</sup>.

The complement set of the resonances is a cantor set. From eq(22), we see that this set again has zero fractal dimension. We expect this to hold generically for supercritical area-preserving maps.

We conclude with a discussion of the application to transport. According to the transport theory<sup>1</sup>, the escape rate out of an  $(m,n)$  resonance is given by

$$\Gamma(m,n) = 2\Delta W_c/A(m,n) \quad (25)$$

This expression appears to capture the major dependence of the escape rate on parameter. However, numerical determination of the escape rate from a single resonance shows abnormal behavior even for primary resonances. The deviations from  $\Gamma(m,n)$  become more prominent when the parameter values  $k$  are small<sup>3</sup>.

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## Figure captions

figure 1: Flux across the (3,8) periodic orbit at  $k = 0.1$ . The crosses are the points on the minimizing orbit. The dots are on the minimax orbit. The shaded region is the turnstile.

figure 2: Some resonances of the sawtooth map at  $k = 0.3$ .

figure 3: Illustration of the calculation of resonance area for the (1,2) resonance.

figure 4: The area devil's staircase function in the sawtooth map at  $k = 0.3$ . Only half the staircase is plotted. Another half can be obtained by the reflection symmetry:  $A(1-\nu) = 1 - A(\nu)$ .