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Free Energy Expressions for Vlasov Equilibria

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Free Energy Expressions for Vlasov Equilibria

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Abstract

Hamiltonian or variational formulations of the Maxwell-Vlasov equation naturally yield expressions for the free energy available upon perturbation of an equilibrium. The noncanonical Hamiltonian, Hamilton-Jacobi, and Lagrangian formulations are used to obtain such expressions. It is concluded that all interesting equilibria are either linearly unstable or possess negative energy modes.

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I. Introduction

The main result of this paper is that *all* interesting equilibria of the Maxwell-Vlasov system are either linearly unstable or possess negative energy modes. By the latter we mean that the free energy surfaces in the vicinity of an equilibrium are not closed and bounded (in any reference frame), in spite of the existence of linear stability. It has previously been conjectured[1,2,3] that equilibria with negative energy modes are generically susceptible to nonlinear instability; that is, instability due to nonlinearity which occurs for arbitrarily small perturbations about an equilibrium. Moreover, it is believed generally that systems with negative energy modes are structurally unstable to dissipative perturbations of the governing dynamical equations. If a negative energy mode is dissipated, then it loses its spectral stability.

The conclusion we come to is that equilibria for which the "monotonicity-isotropy" condition of inequality (18) is true, have negative energy modes. Here $f_\nu^{(0)}$ is the equilibrium distribution function of species ν and $\mathbf{k}$ is an arbitrary vector. This result depends only on the velocity dependence of the equilibrium distribution function; it is independent of the structure of the equilibrium fields.

It is obvious that in order for us to arrive at the criterion of inequality (18) we must have an expression for the free energy. An important result of this paper is such a free energy expression. This is given by Eq. (68). The word free is used here because the perturbations away from the equilibrium state are required to obey the Hamiltonian constraints.

Our results are obtained within the context of three Hamiltonian/Lagrangian formulations of the Maxwell-Vlasov system. In Sec. II we present results using the noncanonical Hamiltonian formalism.[4–7] This can be viewed as a purely Eulerian variable description—one where particle orbit information does not explicitly appear. Here, after reviewing the formalism, we physically describe the meaning of negative energy perturbations for nonmonotonic equilibria. This is done by using a Gardner type restacking argument.[8] We also obtain the monotonicity-isotropy condition in the purely electrostatic context. The slight generalization to equilibria with current free magnetic fields is presented.

Section III utilizes a variational formalism based on Hamilton-Jacobi theory.[9,10] This formulation can be viewed as a combined Eulerian and Lagrangian variable description. It appears presently to be limited for practical reasons to equilibria of one spatial dimension. However, it might turn out to be useful for obtaining the free energy within the context of kinetic guiding-center theories.[9,10] Section III is designed to be read independently of Secs. II & IV.

In Sec. IV we begin with the Lagrangian variable description of Low[11] and then use Noether's theorem (or equivalently the Legendre transform) in order to obtain the energy. The expression obtained is expressed in terms of the Lagrangian displacement and its time derivative. Requiring that these perturbations arise from a generating function leads to the desired free energy expression, Eq. (68). Formally this expression is valid for all equilibria.

In Sec. V we use the results of Sec. IV to treat examples. These include electrostatic equilibria with electrostatic perturbations, homogeneous equilibria with electrostatic and then with electromagnetic perturbations, followed by the case of general equilibria. It is here that we draw our conclusions detailed above.

II. Eulerian Description

Consider now a strictly Eulerian description, i.e., one where particle motions are not monitored. The state of the system is given by the phase space density of species ν, $f_\nu(\mathbf{x}, \mathbf{v}, t)$, and the fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$. We will state the results obtainable by free energy arguments that are accessible by this description and show how these results arise naturally from the noncanonical Hamiltonian formalism. In essence the results presented are a review of material contained in Ref. 3, although a more complete interpretation of the results is given. Also a variational principle for magnetized current free equilibrium and subsequent stability analysis is given, which is new. This section is included for completeness and for later comparison with the methods of Secs. III and IV.

The Eulerian description can be thought of as arising from an underlying particle description by the elimination of particle labeling information. On the particle level the equations of motion have the Hamiltonian form one would expect of Newton's second law, while on the Eulerian level the equations have the noncanonical form (for review see Ref. 7), which is given by

$$\frac{\partial \chi^i}{\partial t}(z,t) = \left[\chi^i, H\right] = J^{ij}\,\frac{\delta H}{\delta \chi^j}. \tag{1}$$

Here χ^i denotes the field components (e.g. f_ν, $\mathbf{E}$, and $\mathbf{B}$) defined on some "spatial" domain z (e.g. the phase space $(\mathbf{x},\mathbf{v})$) and temporal domain t. The expression $\delta H/\delta\chi^j$ is the functional derivative of the Hamiltonian functional $H[\chi]$ and the quantity $J^{ij}(\chi)$ is the cosymplectic operator, which possesses the necessary properties to make the bilinear operator $[\ \ ,\ \ ]$ a Poisson bracket. Unlike conventional Hamiltonian field theories, brackets for media fields possess a special class of constants called Casimir invariants that commute with any functional F, i.e.,

$$[F,C] = \frac{\delta F}{\delta \chi^j}\,J^{ij}\,\frac{\delta C}{\delta \chi^j} = 0. \tag{2}$$

This implies that $\delta C/\delta\chi^j$ must lie in the null space of J^{ij}, thus one can find Casimir invariants by analyzing J^{ij}. These invariants are special in that they are constant for any Hamiltonian, whereas constants such as angular momentum, etc. depend upon the specific form of H. For now we will restrict ourselves to the use of Casimir invariants and H to obtain Liapunov functionals for stability, but note that addition of momenta correspond to frame changes. Evidently equilibria are critical points of $F \equiv H + C$. This is apparent from Eq. (1) since

$$\frac{\partial \chi^i}{\delta t} = \left[\chi^i, H + C\right] = J^{ij}\,\frac{\delta F}{\delta \chi^j} = 0. \tag{3}$$

Usually in Hamiltonian systems equilibria are critical points of H, but in the Eulerian description such points correspond to the "vacuum" state and are thus uninteresting. One requires the C's to constrain the equilibrium away from such minimum energy states. It was shown in Ref. 2 that F is the free energy, which serves as a Liapunov functional for stability if $\delta F/\delta\chi^i = 0$ is the equilibrium of interest and

$$\delta^2 F = \int \frac{1}{2} \frac{\delta^2 F}{\delta\chi^i \delta\chi^j} \delta\chi^i \delta\chi^j dz$$

is definite. The quadratic functional $\delta^2 F$ corresponds physically to the free energy accessible to the system upon perturbation away from equilibrium. This functional is important because it determines the existence of negative energy modes. Such modes exist when $\delta^2 F$ is indefinite in all frames and there is linear stability.

First we consider one-dimensional Vlasov-Poisson equilibria. In this case the energy functional and Casimir invariants are given by

$$\begin{aligned} H\left[f_\nu\right] &= \sum_\nu \int \frac{m_\nu v^2}{2} f_\nu(x,v) dx\, dv \\ &+ \sum_\nu \int e_\nu f_\nu(x,v) \Phi\left(x; f_\nu\right) dx\, dv \qquad (4) \\ C^{(\nu)}\left[f_\nu\right] &= \int \mathcal{F}^{(\nu)}\left(f_\nu\right) dx\, dv, \qquad (5) \end{aligned}$$

where the dynamical variables are f_ν, ν indicating species, and Φ is a shorthand notation for the Green's function solution of Poisson's equation. The constants $C^{(\nu)}\left[f_\nu\right]$ physically corresponds to the conservation of phase space volume (c.f. Ref. 3). Variation of $F = H + \sum_\nu C^{(\nu)}$ yields

$$\delta F = \sum_\nu \int \delta f_\nu \left[\mathcal{E}_\nu + \mathcal{F}^{(\nu)\prime}\left(f_\nu\right)\right] dx\, dv = 0, \qquad (6)$$

where $\mathcal{E}_\nu = \frac{1}{2} m_\nu v^2 + e_\nu \Phi$. Solution of Eq. (6) for all δf_ν requires $\mathcal{E}_\nu + \mathcal{F}^{(\nu)\prime}\left(f_\nu\right) = 0$, an equation that is solvable for an equilibrium distribution $f_\nu^{(0)}$, provided $\mathcal{F}^{(\nu)\prime}$ is monotonic. If we assume so, then we obtain monotonic equilibria, $f_\nu^{(0)}\left(\mathcal{E}_\nu\right)$. In a strictly Eulerian model these are the only equilibria for this system that are obtainable from a variational principle, although nonmonotonic $f_\nu^{(0)}\left(\mathcal{E}_\nu\right)$ are also equilibria. In Ref. 3 nonmonotonic equilibria were obtained by adding a passively advected Eulerian tracer field, which was assumed not to "slip" relative to the dynamics of f_ν. This artifice amounts to the inclusion of Lagrangian variable information.

Now we consider the second variation, but for convenience we restrict to a single species of charge e. One obtains the same expression for the perturbed free energy in both the monotonic and nonmonotonic cases. Varying Eq. (4) once more (and dividing by 2) yields

$$\delta^2 F = \frac{1}{2}\int \mathcal{F}''\left(f^{(0)}\right)(\delta f)^2 dx\, dv + \frac{1}{8\pi}\int (\delta E)^2 dx, \tag{7}$$

where $(\delta E)^2$ is shorthand for the second variation of the second term of Eq. (4). We will neglect this term since its apparent stabilizing effect is mitigated by the fact that we can choose δf so as not to produce a charge perturbation (c.f. Ref. 3). This is general enough for our purposes. Differentiating the equilibrium relation we obtain

$$\mathcal{F}'' \frac{\partial f^{(0)}}{\partial \mathcal{E}} = -1, \tag{8}$$

which upon substitution into (7) yields

$$\delta^2 F = -\frac{1}{2}\int \frac{(\delta f)^2}{\left(\partial f^{(0)}/\partial \mathcal{E}\right)}\, dx\, dv. \tag{9}$$

Thus we have stability if $\partial f^{(0)}/\partial \mathcal{E} < 0$ and indefiniteness if $f^{(0)}$ is nonmonotonic. In the later case the Penrose criterion may predict spectral stability, in which case we have negative energy modes.

What physically is the meaning of the expression of Eq. (9)? To answer this consider a distribution function with a single maximum at $\mathcal{E}_*$. Suppose the phase space is divided up into cells labeled by various energies, $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_i$, etc. Recall that the Liouville constraint can be succinctly stated as follows: particles can be moved around in phase space such that the number of cells with a given number of particles remains fixed. We are free to position any cell at any phase space point, i.e., at any energy. Let us investigate the energy change that results from the interchange of particles between a cell $\mathcal{E}_i$, where $\mathcal{E}_i > \mathcal{E}_*$, and its neighbors.

If we take $f_i - f_{i+1}$ particles from cell $\mathcal{E}_i$ and add them to cell $\mathcal{E}_{i+1}$ we maintain the Liouville constraint and obtain an energy change of $\Delta\mathcal{E}\,(f_i - f_{i+1})$, which is a positive quantity. A similar exchange between $\mathcal{E}_i$ and $\mathcal{E}_{i-1}$ yields another positive energy change, $\Delta\mathcal{E}\,(f_{i-1} - f_i)$. This occurs for all exchanges where $\partial f^{(0)}/\partial\mathcal{E} < 0$. Alternatively, exchanges in the vicinity of an energy $\mathcal{E}_1$, where $\mathcal{E}_1 < \mathcal{E}^*$, yield a decrease in energy, e.g., taking $f_2 - f_1$ particles from $\mathcal{E}_2$ and placing them in $\mathcal{E}_1$ yields, $-\Delta\mathcal{E}\,(f_2 - f_1)$, a negative energy perturbation. Physically the sign of this energy perturbation arises because particles have been slowed down. We can only do this and observe the Liouville constraint if $\partial f^{(0)}/\partial\mathcal{E} > 0$. In fact it is simple to see the origin of $\delta^2 F$ as given by Eq. (9). Estimating the change in energy ΔE, assuming small $\Delta\mathcal{E}$, yields

$$\Delta E \;=\; -\Delta\mathcal{E}\left[f^{(0)}(\mathcal{E}+\Delta\mathcal{E}) - f^{(0)}(\mathcal{E})\right], \tag{10}$$

but

$$\delta f \;=\; f^{(0)}(\mathcal{E}+\Delta\mathcal{E}) - f^{(0)}(\mathcal{E}) \approx \Delta\mathcal{E}\,\frac{\partial f^{(0)}}{\partial\mathcal{E}}, \tag{11}$$

so finally we obtain

$$\Delta E \;=\; -(\Delta\mathcal{E})^2\,\frac{\partial f^{(0)}}{\partial\mathcal{E}} = -\frac{(\delta f)^2}{\partial f^{(0)}/\partial\mathcal{E}}. \tag{12}$$

Summing over many such exchanges we obtain Eq. (9).

It is evident that something peculiar happens at the maximum, $\mathcal{E}_*$. To begin with, since this is an absolute maximum one cannot add particles to this cell without violating the Liouville constraint, since there is only one cell with $f(\mathcal{E}_*)$ particles and none with more. But this cannot be the entire reason for the singularity, since we still have divergence if $\delta f\,(\mathcal{E}_*)$ is negative, corresponding to a subtraction of particles from this cell. Also if $f(\mathcal{E}_*)$ were only a relative maximum we would still get divergence even though the Liouville constraint can be satisfied. The problem arises because of nonanalytic behavior of the energy upon δf at this point, if $\delta f\,(\mathcal{E}_*) \neq 0$. This is evident since we must expand Eq. (11) to higher order to get a nonzero contribution. We obtain

$$\delta f\left(\mathcal{E}_*\right) = \frac{\partial^2 f^{(0)}(\mathcal{E}_*)}{\partial \mathcal{E}^2}\frac{(\Delta\mathcal{E})^2}{2} \tag{13}$$

which yields the following expression for the energy change:

$$\Delta E = \Delta\mathcal{E}\delta f = \frac{|\delta f|^{3/2}}{\sqrt{\frac{1}{2}\left|\partial^2 f^{(0)}/\partial\mathcal{E}^2\right|}} \tag{14}$$

unless $\delta f\left(\mathcal{E}_*\right) = 0$. It remains to address the question of accessibility, i.e., is it possible for $\delta f\left(\mathcal{E}_*\right) \neq 0$ to occur during the course of the dynamics. From Eq. (41) of Ref. 3 we see that $\delta f\left(\mathcal{E}_*\right) \neq 0$ implies that the tracer field slips with respect to f. If the Vlasov equation becomes singular in time, this is the likely place of trouble. The situation is much like that in ideal magnetohydrodynamics where kink or ideal perturbation are required to vanish on rational surfaces. Such trial functions are inserted into δW. In the same spirit we will consider only "ideal" perturbations, i.e., such that $\delta f\left(\mathcal{E}_*\right) = 0$. In fact, if we choose $\delta f = \left[G, f^{(0)}\right]$ where $[\ \ ,\ \ ]$ is the usual Poisson bracket and G is an arbitrary function, then the Casimir constraints of Eq. (5) are maintained to first order. This follows since

$$\begin{aligned}\delta C = \int \mathcal{F}'\left(f^{(0)}\right)\delta f\, dx\, dv &= \int \mathcal{F}'\left(f^{(0)}\right)\left[G, f^{(0)}\right] dx\, dv\\ &= \int [G,\mathcal{F}]\, dx\, dv = 0.\end{aligned}$$

The perturbation can be written out as

$$\begin{aligned}\delta f = \left[G, f^{(0)}\right] &= \frac{1}{m}\left[\frac{\partial G}{\partial x}\frac{\partial f^{(0)}}{\partial v} - \frac{\partial G}{\partial v}\frac{\partial f^{(0)}}{\partial x}\right]\\ &= \left[v\frac{\partial G}{\partial x} - \frac{e}{m}\frac{\partial \Phi}{\partial x}\frac{\partial G}{\partial v}\right]\frac{\partial f^{(0)}}{\partial \mathcal{E}},\end{aligned}$$

whence we obtain

$$\delta^2 F = -\frac{1}{2}\int [\mathcal{E}, G]^2 \frac{\partial f^{(0)}}{\partial \mathcal{E}}\, dx\, dv. \tag{15}$$

In the case of homogeneous equilibria this becomes

$$\delta^2 F = -\frac{1}{2m}\int \left(\frac{\partial G}{\partial x}\right)^2 v\,\frac{\partial f^{(0)}}{\partial v}\,dx\,dv.$$

Our conclusion with respect to one-dimensional Vlasov-Poisson is that all nonmonotonic distribution functions possess either linear instability or negative energy modes.

Now consider three-dimensional Vlasov-Poisson equilibria, putting aside the question of existence. The procedure carried out above can be formally mimicked in this case yielding Eq. (6), which means that here $f^{(0)}$ has to be isotropic. Taking the velocity and space gradients of the equilibrium relation yields the following:

$$\mathcal{F}''\,\frac{\partial f^{(0)}}{\partial \mathbf{v}} = -m\mathbf{v}, \qquad \mathcal{F}''\,\frac{\partial f^{(0)}}{\partial \mathbf{x}} = -e\,\frac{\partial \Phi}{\partial \mathbf{x}}, \tag{16}$$

which imply

$$\mathcal{F}'' = -\frac{[G,\mathcal{E}]}{[G,f^{(0)}]}.$$

Now assuming, as in the one-dimensional case, $\delta f = \left[G, f^{(0)}\right]$ we obtain upon substitution

$$\delta^2 F = -\frac{1}{2}\int [G,\mathcal{E}]^2\,\frac{\partial f^{(0)}}{\partial \mathcal{E}}\,d^3x\,d^3v + \frac{1}{8\pi}\int (\delta E)^2 d^3x. \tag{17}$$

Again, as in the one-dimensional case the crucial quantity is the first integrand. In the case of homogeneous equilibria Eq. (17) reduces to

$$\delta^2 F = -\frac{1}{2m}\int \left(\frac{\partial G}{\partial \mathbf{x}}\cdot \mathbf{v}\right)\left(\frac{\partial G}{\partial \mathbf{x}}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right) d^3x\,d^3v + \int \frac{(\delta E)^2}{8\pi}\,d^3x.$$

If we assume $G \sim e^{i\mathbf{k}\cdot\mathbf{x}}$ + c.c., then we have positive definiteness if and only if

$$(\mathbf{k}\cdot\mathbf{v})\left(\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right) < 0, \tag{18}$$

for all $\mathbf{k}$, $\mathbf{x}$, and $\mathbf{v}$. This monotonicity-isotropy condition is quite general and not only valid for homogeneous and isotropic $f^{(0)}$; it will emerge again in both Secs. III and IV and will be proven there for general $f^{(0)}(\mathbf{x},\mathbf{v})$.

If $f^{(0)}$ is a nonmonotonic function, then inequality (18) can be violated. This is clear since we can pick $\mathbf{k}$ in the direction in velocity space where there is nonmonotonicity, then defining $v_k = \hat{\mathbf{k}} \cdot \mathbf{v}$ we obtain

$$(\mathbf{k} \cdot \mathbf{v})\mathbf{k} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} = k^2 v_k^2 \frac{\partial f^{(0)}}{\partial \mathcal{E}_k},$$

where $\mathcal{E}_k \equiv \frac{1}{2} v_k^2$. Thus condition (18) essentially contains the condition of Eq. (15), and one can relocate the particles to obtain a lower energy state when it is violated.

If we assume $f^{(0)}$ is monotonic decreasing, then inequality (18) is violated if and only if $f^{(0)}$ is anisotropic. To see this suppose there is a point $\mathbf{v}$ in velocity space where there is anisotropy; without loss of generality we assume

$$\frac{1}{v_x} \frac{\partial f^{(0)}}{\partial v_x} > \frac{1}{v_y} \frac{\partial f^{(0)}}{\partial v_y},$$

where the expressions on both sides are negative. Let

$$R \equiv \left[\frac{1}{v_x} \frac{\partial f^{(0)}}{\partial v_x} \right] \bigg/ \left[\frac{1}{v_y} \frac{\partial f^{(0)}}{\partial v_y} \right],$$

then evidently $1 > R > 0$. Choosing $\mathbf{k} = (1/v_x, \delta/v_y, 0)$ we obtain

$$\mathbf{k} \cdot \mathbf{v} \mathbf{k} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} = \frac{1}{v_y} \frac{\partial f^{(0)}}{\partial v_y} (R + \delta)(1 + \delta)$$

which is positive for $-1 < \delta < -R$. Physically these negative energy anisotropic perturbations can be explained in a manner similar to that given above for nonmonotonic $f^{(0)}(\mathcal{E})$. Given any anisotropic distribution one can relocate particles so as to approach isotropy, in a manner consistent with the constraints where the resulting energy change is negative.

There is one other class of equilibria that is accessible to the strict Eulerian description. In addition to the "Liouville" Casimir invariant the Maxwell-Vlasov equations possess the following:

$$C_E = \int U \left(\nabla \cdot \mathbf{E} - 4\pi \int e f \, d^3 v \right) d^3 x \tag{19}$$

$$C_B = \int \psi \nabla \cdot \mathbf{B} \, \mathrm{d}^3 \mathrm{x}, \tag{20}$$

where U and ψ are arbitrary functions of $\mathbf{x}$. It is of course well known that if these quantities are initially zero then they remain zero, but adding them to the free energy allows us to obtain slightly more general equilibria. In the perturbed free energy these quantities should be satisfied to first order. Including Eqs. (19) and (20) we have the following free energy functional:

$$\begin{aligned} F &= \int \frac{mv^2}{2} f(\mathbf{x}, \mathbf{v}) d^3x\, d^3v + \frac{1}{8\pi} \int \left(E^2 + B^2\right) d^3x + \int \mathcal{F}\, d^3x\, d^3v \\ &- \frac{1}{4\pi} \int U \left(\nabla \cdot E - 4\pi \int ef\, d^3v\right) d^3x + \frac{1}{4\pi} \int \chi \nabla \cdot \mathbf{B}\, d^3x, \end{aligned} \tag{21}$$

where the first two terms correspond to the $(0,0)$ component of the Maxwell-Vlasov energy-momentum tensor (c.f. Ref. 10). Again for simplicity we have assumed a single species. Varying F yields

$$\begin{aligned} \delta F &= \int \delta f \left[\frac{mv^2}{2} + eU + \mathcal{F}'(f)\right] d^3x\, d^3v \\ &+ \frac{1}{4\pi} \int \left[\delta\mathbf{E} \cdot (\mathbf{E} + \nabla U) + \delta\mathbf{B} \cdot (\mathbf{B} - \nabla\chi)\right] d^3x, \end{aligned} \tag{22}$$

which implies the equilibrium relations

$$f^{(0)}(\mathcal{E}), \qquad \mathbf{E}^{(0)} = -\nabla U = -\nabla\Phi^{(0)}, \qquad \mathbf{B}^{(0)} = \nabla\chi^{(0)}. \tag{23}$$

The equilibrium electric fields are electrostatic and the magnetic field is derivable from a potential, which means there is no plasma current since $\nabla \times \mathbf{B}^{(0)} = \nabla \times \nabla\chi^{(0)} = 0$. Taking the second variation of (22) and assuming δf is ideal, we obtain

$$\delta^2 F = -\frac{1}{2} \int [G, \mathcal{E}]^2 \frac{\partial f^{(0)}}{\partial \mathcal{E}}\, d^3x\, d^3v + \frac{1}{8\pi} \int \left[(\delta E)^2 + (\delta B)^2\right] d^3x. \tag{24}$$

Equation (24), with the appropriate choice of G results in the same necessary and sufficient condition for positive definites as before, i.e., that of Eq. (18). Let us now turn to the Hamilton-Jacobi description.

III. The Hamilton-Jacobi Description — A Combined Eulerian-Lagrangian Formulation

In Refs. 9 and 10 it was shown how to derive the Maxwell-Vlasov theory from a variational principle by employing Hamilton-Jacobi theory. For completeness we will briefly review this here, before proceeding to the question of negative energy perturbations in this context.

The Lagrangian is given by

$$
\begin{aligned}
L = & - \sum_{\nu} \int d^3x\, d^3\alpha\, \phi_\nu \left[\frac{\partial S_\nu}{\partial t} + e_\nu \Phi + \frac{1}{2m_\nu} \left(\frac{\partial S_\nu}{\partial \mathbf{x}} - \frac{e_\nu}{c} \mathbf{A} \right)^2 \right] \\
& + \frac{1}{8\pi} \int d^3x \left(E^2 - B^2 \right),
\end{aligned}
\tag{25}
$$

where the functions $S_\nu(\mathbf{x}, \boldsymbol{\alpha}, t)$ are Hamilton-Jacobi functions for the particles of species ν. The quantities $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \partial S_\nu / \partial \boldsymbol{\alpha}$ are constants of motion. The densities, ϕ_ν, can be expressed in terms of the distribution function f_ν by

$$
\phi_\nu = f_\nu(\boldsymbol{\alpha}, \boldsymbol{\beta}) \left\| \frac{\partial^2 S_\nu}{\partial x_i \partial \alpha_\mu} \right\| , \tag{26}
$$

where

$$
D_\nu = \left\| \frac{\partial^2 S_\nu}{\partial x_i \partial \alpha_\mu} \right\| \tag{27}
$$

is the so-called Van Vleck determinant. Variation of the action, $\int L\, dt$, with respect to ϕ_ν, S_ν and the potentials Φ and $\mathbf{A}$ yield equations equivalent to the Maxwell-Vlasov system.

We expand the Lagrangian by assuming

$$
\begin{aligned}
\phi_\nu = \phi_\nu^{(0)} + \delta\phi_\nu, \qquad & S_\nu = S_\nu^{(0)} + \delta S_\nu, \\
\Phi = \Phi^{(0)} + \delta\Phi, \qquad & \mathbf{A} = \mathbf{A}^{(0)} + \delta\mathbf{A},
\end{aligned}
\tag{28}
$$

where the superscript 0 denotes unperturbed quantities and the δ indicates perturbation. Equations for the perturbed quantities are obtained by expanding Eq. (25) to second order

$$
L = L^{(0)} + \delta L + \delta^2 L.
$$

The term $L^{(0)}$ describes the unperturbed system. When varying with respect to unperturbed quantities, the following holds:

$$\delta \int_{t_1}^{t_2} L^{(0)} dt = 0. \tag{29}$$

One can view Eqs. (28) to be a trial function ansatz where the "0" variables are assumed to be equilibrium quantities. If they are assumed to be known Eq. (29) is manifestly satisfied; if not, Eq. (29) yields equilibrium equations. In the following we assume the equilibrium is at hand and then search for neighboring solutions by varying with respect to the perturbed quantities.

Now at first order we obtain no new information. Since δL is just the first variation of $L^{(0)}$, $\delta \int \delta L \, dt = 0$ is automatic because of Eq. (29).

Proceeding, we are then left with the second order Lagrangian

$$\begin{aligned}\delta^2 L = & - \sum_\nu \int d^3x \, d^3\alpha \Bigg\{ \delta\phi_\nu \left[\frac{\partial \delta S_\nu}{\partial t} + e_\nu \delta\Phi + \frac{1}{m_\nu} \left(\frac{\partial S_\nu^{(0)}}{\partial \mathbf{x}} - \frac{e_\nu}{c} \mathbf{A}^{(0)} \right) \cdot \left(\frac{\partial \delta S_\nu}{\partial \mathbf{x}} - \frac{e_\nu}{c} \delta \dot{\mathbf{A}}_\nu \right) \right] \\ & + \frac{1}{2m_\nu} \phi_\nu^{(0)} \left(\frac{\partial \delta S_\nu}{\partial \mathbf{x}} - \frac{e_\nu}{c} \delta \mathbf{A}_\nu \right)^2 \Bigg\} + \frac{1}{8\pi} \int d^3x \left(\delta E^2 - \delta B^2 \right).\end{aligned} \tag{30}$$

Variation of (30) with respect to $\delta\phi_\nu$, δS_ν, $\delta\Phi$, and $\delta\mathbf{A}$, i.e.,

$$\delta \int_{t_1}^{t_2} \delta^2 L \, dt = 0$$

yields the correct first order equations for these perturbed quantities.

From $\delta^2 L$ one can in the usual way find the expression for the energy from the canonical energy density, $\delta^2 \Theta_{00}$. This is equivalent to Legendre transformation. We obtain

$$\begin{aligned}\delta^2 \mathcal{E} = \int d^3x \delta^2 \Theta_{00} = & \sum_\nu \int d^3x \, d^3\alpha \left[e_\nu \delta\Phi \delta\phi_\nu + \frac{m_\nu}{2} \left(\delta v_\nu^2 \, \phi_\nu^{(0)} + 2\delta \mathbf{v}_\nu \cdot \mathbf{v}_\nu^{(0)} \delta\phi_\nu \right) \right] \\ & - \int d^3x \left[\frac{1}{4\pi c} \delta \mathbf{E} \cdot \delta \dot{\mathbf{A}} + \frac{1}{8\pi} \left(\delta E^2 - \delta B^2 \right) \right].\end{aligned} \tag{31}$$

Here we have used the abbreviations

$$\mathbf{v}_\nu^{(0)} = \frac{1}{m_\nu} \left(\frac{\partial S_\nu^{(0)}}{\partial \mathbf{x}} - \frac{e_\nu}{c} \mathbf{A}^{(0)} \right) \tag{32}$$

$$\delta \mathbf{v}_\nu = \frac{1}{m_\nu}\left(\frac{\partial \delta S_\nu}{\partial \mathbf{x}} - \frac{e_\nu}{c}\,\delta \mathbf{A}\right). \tag{33}$$

It holds furthermore that

$$\frac{1}{c}\,\delta\dot{\mathbf{A}} = -\delta\mathbf{E} - \nabla\delta\Phi \tag{34}$$

$$\sum_\nu \int d^3\alpha e_\nu \delta\phi_\nu = \delta\rho, \tag{35}$$

where $\delta\rho$ is the perturbed charge density. Upon making use of Eqs. (34) and (35), the energy becomes

$$\delta^2\mathcal{E} = \sum_\nu \int d^3x\, d^3\alpha\, \frac{m_\nu}{2}\left(\delta v_\nu^2 \phi_\nu^{(0)} + 2\delta\mathbf{v}_\nu\cdot\mathbf{v}_\nu^{(0)}\delta\phi_\nu\right) + \frac{1}{8\pi}\int d^3x\left(\delta E^2 + \delta B^2\right). \tag{36}$$

Recall that connection with the distribution function is made by Eq. (26), which relates ϕ_ν, S_ν, and f_ν. There is some subtlety in expanding this equation to relate the perturbed quantities. We assume the perturbations are turned on at time $t = -\infty$ and rise adiabatically. Since the distribution functions $f_\nu(\boldsymbol{\alpha},\boldsymbol{\beta})$ are only functions of the constants of motion, the functional form remains unaltered for such an adiabatic turn on. Since at $t = -\infty$ one has $f_\nu(\boldsymbol{\alpha},\boldsymbol{\beta}) = f_\nu^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and since the f_ν are constant along the particle orbits this must hold for all times. It is the meaning of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ that changes as the perturbations rise. This observation simplifies the perturbation of Eq. (26). We obtain

$$\phi_\nu^{(0)} = f_\nu^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta}) D_\nu^{(0)} \tag{37}$$

$$\delta\phi_\nu = f_\nu^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta})\delta D_\nu + \frac{\partial f_\nu^{(0)}}{\partial\boldsymbol{\beta}}\cdot\frac{\partial\delta S_\nu}{\partial\boldsymbol{\alpha}} D_\nu^{(0)}, \tag{38}$$

where the D_ν, given by Eq. (27), can be written as

$$D_\nu = \nabla\frac{\partial S_\nu}{\partial\alpha_1}\cdot\left(\nabla\frac{\partial S_\nu}{\partial\alpha_2}\times\nabla\frac{\partial S_\nu}{\partial\alpha_3}\right).$$

Thus to zero and first order

$$D_\nu^{(0)} = \nabla\frac{\partial S_\nu^{(0)}}{\partial\alpha_1}\cdot\left(\nabla\frac{\partial S_\nu^{(0)}}{\partial\alpha_2}\times\nabla\frac{\partial S_\nu^{(0)}}{\partial\alpha_3}\right) \tag{39}$$

$$\delta D_\nu = \sum_{\substack{i,k,l=1\\ \text{cyclic}}}^{3} \nabla\frac{\partial\delta S_\nu}{\partial\alpha_i}\cdot\left(\nabla\frac{\partial S_\nu^{(0)}}{\partial\alpha_l}\times\nabla\frac{\partial S_\nu^{(0)}}{\partial\alpha_k}\right). \tag{40}$$

There is some freedom in choosing $S_\nu^{(0)}$, but in order for $\delta^2\mathcal{E}$ to be a conserved quantity it is required that $\partial S_\nu^{(0)}/\partial\mathbf{x}$ be time independent. This will be the case if one assumes a variable separated form

$$S_\nu^{(0)}(\mathbf{x},\boldsymbol{\alpha},t) = -E^{(0)}t + \hat{S}_\nu^{(0)}(\mathbf{x},\boldsymbol{\alpha}),$$

where $E^{(0)} = E^{(0)}(\boldsymbol{\alpha})$. Note also that $f_\nu^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta})$ must be time independent.

In expression (31) δS_ν, $\delta\mathbf{A}$, and $\delta\dot{\mathbf{A}}$ can be chosen independently, whereas $\delta\Phi$ is bound by the constraint $\nabla\cdot\delta\mathbf{E} = 4\pi\delta\rho$. The field quantities $\delta\mathbf{A}$ and $\delta\dot{\mathbf{A}}$ are independent of the particle quantity δS_ν, because Maxwell's equations allow for the production of a displacement current that makes a given particle-field configuration consistent. Fortunately, $\delta\Phi$ and $\delta\dot{\mathbf{A}}$ do not enter the phase space ($\int d^3x\, d^3\alpha$) contribution to $\delta^2\mathcal{E}$. It is thus convenient to replace the $\delta\Phi$ and $\delta\dot{\mathbf{A}}$ variations by a $\delta\mathbf{E}$ variation that is subject to the Poisson equation constraint. The equivalence of these variations follows because $\delta\rho$ is linearly related to the particle perturbations (but not identical to, since phase space information has been integrated out) and because $\delta^2\mathcal{E}$ is bilinear in the perturbations.

In light of the above, the positive semi-definite electric field energy contribution can be considered independently. We incorporate the Poisson constraint by using a Lagrange multiplier $\delta U(\mathbf{x})$ as follows:

$$\delta\int\left[\frac{\delta E^2}{2} - \delta U(\mathbf{x})(\nabla\cdot\delta\mathbf{E} - 4\pi\delta\rho)\right] = 0. \tag{41}$$

(Note that this procedure is in essence equivalent to that of Sec. II.) Equation (41) yields

$$\delta\mathbf{E} = -\nabla\delta U. \tag{42}$$

In terms of initial conditions Eq. (42) means

$$\delta\dot{\mathbf{A}} = 0, \qquad \delta\phi = \delta U(\mathbf{x})$$

with

$$\nabla^2\delta U(\mathbf{x}) = -4\pi\delta\rho.$$

The minimum electric field energy is therefore achieved for $\delta\rho \equiv 0$.

Subsequently, we also wish to write our energy expressions in terms of physically recognizable quantities. This requires transformation from the variable $\boldsymbol{\alpha}$ to the velocity variable $\mathbf{v}$, related to the unperturbed state. The following replacements must be made:

$$\mathbf{v} \equiv \mathbf{v}_\nu^{(0)} = \frac{1}{m_\nu}\left(\frac{\partial S_\nu^{(0)}}{\partial \mathbf{x}} - \frac{e_\nu}{c}\mathbf{A}^{(0)}\right) \tag{43}$$

$$D_\nu^{(0)} d^3\alpha = m_\nu^3 d^3 v. \tag{44}$$

Also, it is desired to write $\delta\mathbf{v}_\nu$ as a function of $\mathbf{x}$ and $\mathbf{v}$. To accomplish this we write δS_ν as a function of $\mathbf{x}$ and $\mathbf{v}$ and use Eq. (33). To complete this transformation requires δD_ν. If we define $\mathbf{P} = \mathbf{p} + \delta\mathbf{p}$, then the following expressions are exact:

$$\begin{aligned}
d^3x\, d^3P &= d^3x\, d^3\alpha D_\nu = d^3x\, d^3\alpha \left(D_\nu^{(0)} + \delta D_\nu\right) \\
dP_i &= dp_i + d\delta p_i = dp_i + d\left.\frac{\partial \delta S_\nu}{\partial x_i}\right|_\alpha \\
dx_i' &= dx_i,
\end{aligned}$$

where we have used $\delta\mathbf{p} = \partial\delta S_\nu/\partial\mathbf{x}|_\alpha$. To first order

$$\begin{aligned}
d^3x\, d^3P &= \left(1 + \frac{\partial}{\partial \mathbf{p}}\cdot\left(\left.\frac{\partial\delta S_\nu}{\partial \mathbf{x}}\right|_\alpha\right)\right) d^3x\, d^3p \\
&= \left(D_\nu^{(0)} + \delta D_\nu\right) d^3x\, d^3\alpha \\
&= \left(1 + \frac{\delta D_\nu}{D_\nu^{(0)}}\right) d^3x\, d^3p
\end{aligned} \tag{45}$$

and therefore

$$\delta D_\nu = -D_\nu^{(0)} \frac{\partial}{\partial \mathbf{p}}\cdot\left(\left.\frac{\partial\delta S\left(\mathbf{x}, \boldsymbol{\alpha}(\mathbf{x},\mathbf{p}), t\right)}{\partial \mathbf{x}}\right|_\alpha\right), \tag{46}$$

Finally we transform from the variable $\mathbf{p}$ to $\mathbf{v}$ using, $\mathbf{p} = m_\nu\mathbf{v} + \frac{e_\nu}{c}\mathbf{A}^{(0)}$.

We are now in a position to rewrite the expression of Eq. (36) for $\delta^2\mathcal{E}$. Assuming for the moment that our equilibrium configuration is determined by the constants $\boldsymbol{\alpha}$ alone, i.e.,

$f_\nu^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta}) = f_\nu^{(0)}(\boldsymbol{\alpha}) = f_\nu^{(0)}(\mathbf{x},\mathbf{v})$, which is often the case, we obtain upon making use of Eqs. (37)–(40), (43), (44), and (46),

$$\begin{aligned}
\delta^2\mathcal{E} &= \sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} f_\nu^{(0)} \left[\left| \frac{\partial \delta S_\nu\left(\mathbf{x}, \boldsymbol{\alpha}\left(\mathbf{x}, m_\nu \mathbf{v} + \frac{e_\nu}{c}\mathbf{A}^{(0)}\right), t\right)}{\partial \mathbf{x}}\right|_\alpha \right. \\
&- \left. \frac{e_\nu}{c}\delta\mathbf{A}\right|^2 + 2\mathbf{v}\cdot\left(\left.\frac{\partial \delta S_\nu}{\partial \mathbf{x}}\right|_\alpha - \frac{e_\nu}{c}\delta\mathbf{A}\right)\frac{\partial}{\partial \mathbf{v}}\cdot\left.\frac{\partial \delta S_\nu}{\partial \mathbf{x}}\right|_\alpha \Bigg] \\
&+ \frac{1}{8\pi}\int d^3x \left(\delta E^2 + \delta B^2\right) \qquad (47)
\end{aligned}$$

where we have absorbed a factor of m_ν^3 into the definition of $f_\nu^{(0)}$.

If one drops the assumption $f_\nu^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta}) = f_\nu^{(0)}(\boldsymbol{\alpha})$ in Eq. (36) the term $D_\nu^{(0)}\partial f^{(0)}/\partial\boldsymbol{\beta}\cdot \partial S_\nu/\partial\boldsymbol{\alpha}$ must be taken into account. To write this term in terms of $\mathbf{x}$ and $\mathbf{v}$ we must know the dependence of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on these variables.

Now consider a simple example, that of a one-dimensional electrostatic equilibrium. In this case one has

$$p = \frac{\partial S^{(0)}}{\partial x} = \sqrt{2m\left(\alpha - e\Phi^{(0)}(x)\right.}. \qquad (48)$$

Setting $\mathbf{A}^{(0)} = \delta\mathbf{A} = 0$ and restricting to one-dimension and single species, Eq. (47) becomes

$$\begin{aligned}
\delta^2\mathcal{E} &= \int \frac{dx\, dv}{2m} f^{(0)} \left[\left(\frac{\partial \delta S}{\partial x}\right)^2 + 2v\,\frac{\partial \delta S}{\partial x}\,\frac{\partial^2 \delta S}{\partial v \partial x}\right] \\
&+ \frac{1}{8\pi}\int dx\, \delta E^2. \qquad (49)
\end{aligned}$$

We wish to compare this expression with that obtained in Secs. II and IV. Thus we map from the function $\delta S(x,\alpha)$ to a generating function $G(x,p)$. To this end we expand $p = \partial S(x,\alpha)/\partial x$ about $x = x^{(0)} + \delta x$ and obtain

$$\delta p = \frac{\partial \delta S(x^{(0)},\alpha)}{\partial x^{(0)}} + \delta x\,\frac{\partial^2 S^{(0)}(x^{(0)},\alpha)}{\partial x^{(0)2}} \equiv -\frac{\partial G}{\partial x^{(0)}}(x^{(0)},p). \qquad (50)$$

Define $\hat{G}$ by $\frac{\partial \delta S(x,\alpha)}{\partial x} = -\frac{\partial \hat{G}(x,p)}{\partial x}$. Upon dropping the superscript (0) and making use of the derivative of Eq. (48), Eq. (50) becomes

$$\frac{\partial \hat{G}}{\partial x} = \frac{\partial G}{\partial x} + \frac{eE^{(0)}(x)}{v}\frac{\partial G}{\partial p}.$$

Replacing δS by $\hat{G}$, Eq. (49) can be written as

$$\delta^2\mathcal{E} = -\frac{1}{2}\int dx\, dv\, \frac{\partial f^{(0)}}{\partial \mathcal{E}}\left[\frac{mv^2}{2}, \hat{G}\right]^2 + \frac{1}{8\pi}\int \delta E^2\, dx,$$

and we can replace $\hat{G}$ by G; noting that $\left[\frac{mv^2}{2}, \hat{G}\right] = [\mathcal{E}, G]$, yields

$$\delta^2\mathcal{E} = -\frac{1}{2}\int dx\, dv\, \frac{\partial f^{(0)}}{\partial \mathcal{E}}[\mathcal{E}, G]^2 + \frac{1}{8\pi}\int \delta E^2\, dx.$$

This expression is in agreement with Eq. (15) of Sec. II and the one-dimensional restriction of Eq. (73). Unfortunately the generalization of this result to three-dimensions is hampered by the fact that there does not in general exist a $\hat{G}$ such that

$$\left.\frac{\partial \delta S(\mathbf{x}, \boldsymbol{\alpha})}{\partial \mathbf{x}}\right|_{\alpha} = \left.\frac{\partial \hat{G}(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}\right|_{p},$$

where $\boldsymbol{\alpha}(\mathbf{x}, \mathbf{p})$. This apparent shortcoming is overcome by the method of Sec. IV.

To conclude this section we note that $\delta^2\mathcal{E}$ is gauge invariant. If we let $\mathbf{A}^{(0)} \to \mathbf{A}^{(0)} + \nabla\psi^{(0)}$ and $\delta\mathbf{A} \to \delta\mathbf{A} + \nabla\delta\psi$, then

$$G_\nu(\mathbf{v}, \mathbf{x}) \to G_\nu\left(\mathbf{v} + \frac{e_\nu}{m_\nu c}\nabla\psi^{(0)}, x\right) - \frac{e_\nu}{c}\delta\psi. \tag{51}$$

With these substitutions it is evident that Eqs. (47) and (50) do not change.

IV. Lagrangian Description

It is well known that the Maxwell-Vlasŏv equations possess an action principle when the media is represented in terms of Lagrange variables. This is because the usual concept of a field is replaced by a continuum of particles, which of course are governed by Newton's second law with the Lorentz force. We will not review this here, but refer the reader to Refs. 11, 12, and 13. The main contribution of this section is a procedure for obtaining the general second order perturbed free energy. To our knowledge this quantity has not previously appeared in print.

Recall that in order to uniquely label a particle in phase space two continuum labels are required. This is because more than one particle can occupy a configuration space point. Thus we suppose particle orbits are given by

$$\mathbf{x}\left(\mathbf{x}_0, \mathbf{v}_0, t\right), \tag{52}$$

where $\mathbf{x}\left(\mathbf{x}_0, \mathbf{v}_0, 0\right) = \mathbf{x}_0$ and $\dot{\mathbf{x}}\left(\mathbf{x}_0, \mathbf{v}_0, 0\right) = \mathbf{v}_0$. We expand about an assumed known reference trajectory according to

$$\mathbf{x} = \mathbf{x}^{(0)}\left(\mathbf{x}_0, \mathbf{v}_0, t\right) + \delta\mathbf{x}\left(\mathbf{x}_0, \mathbf{v}_0, t\right) \tag{53}$$

with corresponding field perturbations

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^{(0)}(\mathbf{x}) + \delta\mathbf{E}(\mathbf{x}, t) = -\nabla\Phi^{(0)} - \frac{1}{c}\frac{\partial \mathbf{A}^{(0)}}{\partial t} - \nabla\delta\Phi - \frac{1}{c}\frac{\partial \delta\mathbf{A}}{\partial t} \\ \mathbf{B} &= \mathbf{B}^{(0)}(\mathbf{x}) + \delta\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}^{(0)} + \nabla \times \delta\mathbf{A}. \end{aligned} \tag{54}$$

We assume that the reference orbit gives rise to macroscopic quantities that are stationary in time. Expanding in the smallness of $\delta\mathbf{x}$, $\delta\mathbf{B}$, and $\delta\mathbf{E}$ one obtains the second order action

$$\begin{aligned} \delta^2 S &= \int \delta^2 L\, dt = \int dt \int d^3x_0 d^3v_0 f_0\left(\mathbf{x}_0, \mathbf{v}_0\right) \Bigg\{ \frac{m}{2}\left(\frac{\partial \delta\mathbf{x}}{\delta t}\right)^2 \\ &+ e\Bigg[-\delta\mathbf{x}\cdot\frac{\partial\delta\Phi}{\partial\mathbf{x}} - \frac{1}{2}\frac{\partial^2\Phi^{(0)}}{\partial x_i \partial x_j}\delta x_i \delta x_j + \frac{1}{c}\frac{\partial\delta\mathbf{x}}{\partial t}\cdot\delta\mathbf{A} + \frac{1}{c}\frac{\partial\delta x_i}{\partial t}\frac{\partial A_i^{(0)}}{\partial x_j}\delta x_j \\ &+ \frac{1}{c}\frac{\partial x_i}{\partial t}\frac{\partial\delta A_i}{\partial x_j}\delta x_j + \frac{1}{2c}\frac{\partial x_i}{\partial t}\frac{\partial^2 A_i^{(0)}}{\partial x_j \partial x_l}\delta x_j \delta x_l\Bigg]\Bigg\} + \frac{1}{8\pi}\int\left[\delta E^2 - \delta B^2\right] d^3x. \end{aligned} \tag{55}$$

In general, the integrand of Eq. (55) contains explicit time dependence that arises because the field quantities are evaluated on the reference trajectory, $\mathbf{x}^{(0)}$. This occurs if we are interested in nontrivial equilibrium, i.e., ones for which there is particle motion. Because of this the action $\delta^2 S$ is a complicated object. In general one cannot obtain it explicitly since explicit expressions for the reference trajectories do not exist. This complication is avoided by referencing the perturbation with respect to the reference trajectory at time t, rather than its position at time $t = 0$. Calling $\dot{\mathbf{x}}^{(0)} \equiv \mathbf{v}$ and dropping the (0) superscript, $\mathbf{x}^{(0)} \rightarrow \mathbf{x}$, we write the perturbation as follows:

$$\delta\mathbf{x}'(\mathbf{x}, \mathbf{v}, t) \equiv \delta\mathbf{x}\,(\mathbf{x}_0, \mathbf{v}_0, t)\,. \tag{56}$$

The map between $(\mathbf{x}, \mathbf{v})$ and $(\mathbf{x}_0, \mathbf{v}_0)$ formally exists since the Jacobian, $\partial(\mathbf{x}, \mathbf{v})/\partial\,(\mathbf{x}_0, \mathbf{v}_0)$, is unity. Also we have

$$\frac{\partial \delta x}{\partial t} = D\delta x',$$

where

$$D \;\equiv\; \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \mathbf{a}^{(0)}\cdot\frac{\partial}{\partial \mathbf{v}}$$

and

$$\mathbf{a}^{(0)} \;=\; \frac{e}{m}\left(\mathbf{E}^{(0)} + \frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right),$$

and finally $f_0\,(\mathbf{x}_0, \mathbf{v}_0)\,d^3x_0 d^3v_0 = f^{(0)}(\mathbf{x}, \mathbf{v}) d^3x\, d^3v$. Here $f^{(0)}(\mathbf{x}, \mathbf{v})$ is the equilibrium distribution function, which is assumed to be time independent. With these substitutions the Lagrangian becomes (dropping the prime on $\delta\mathbf{x}$)

$$\begin{aligned}
\delta^2 L \;=\; & \int d^3x\, d^3v\, f^{(0)}(\mathbf{x}, \mathbf{v})\Bigg\{\frac{m}{2}\,(D\delta\mathbf{x})^2 + e\Bigg[-\delta\mathbf{x}\cdot\nabla\delta\Phi - \frac{1}{2}\frac{\partial^2\Phi^{(0)}}{\partial x_i \partial x_j}\,\delta x_i \delta x_j \\
& + \frac{1}{c}\delta\mathbf{A}\cdot D\delta\mathbf{x} + \frac{1}{c}\,\delta x_j\,\frac{\partial A_i^{(0)}}{\partial x_j}\,D\delta x_i + \frac{v_i}{c}\,\delta x_j\,\frac{\partial \delta A_i}{\partial x_j} + \frac{v_i}{2c}\frac{\partial^2 A_i^{(0)}}{\partial x_j \partial x_l}\,\delta x_j \delta x_l\Bigg]\Bigg\} \\
& + \frac{1}{8\pi}\int\left[\delta E^2 - \delta B^2\right] d^3x,
\end{aligned} \tag{57}$$

which now has no explicit time dependence. Moreover, this expression is gauge invariant and produces the correct linearized equations of motion upon variation.[11]

The transformation performed above is quite desirable since it allows one to obtain the conserved second order energy. This can be achieved by either Legendre transforming or equivalently by calculating the $(0,0)$ component of the canonical energy-momentum tensor, both of which yield

$$\delta^2 H = \int \frac{\delta\delta^2 L}{\delta\delta\dot{\mathbf{x}}} \cdot \delta\dot{\mathbf{x}}\, d^3x\, d^3v + \int \frac{\delta\delta^2 L}{\delta\delta\dot{\mathbf{A}}} \cdot \delta\dot{\mathbf{A}}\, d^3x - \delta^2 L. \tag{58}$$

The canonical momentum density conjugate to $\delta\mathbf{x}$ is given by

$$\delta\boldsymbol{\pi} = \frac{\delta\delta^2 L}{\delta\delta\dot{\mathbf{x}}} = f^{(0)}(\mathbf{x},\mathbf{v}) \left[mD\delta\mathbf{x} + \frac{e}{c}\,\delta\mathbf{A} + \frac{e}{c}\,\delta\mathbf{x}\cdot\nabla\mathbf{A}^{(0)} \right], \tag{59}$$

but we will find it useful to use the perturbed particle momentum defined by $\delta\mathbf{p} = mD\delta\mathbf{x} + \frac{e}{c}\,\delta\mathbf{A} + \frac{e}{c}\,\delta\mathbf{x}\cdot\nabla\mathbf{A}^{(0)}$. Performing the operations indicated in Eq. (58), making use of the linearized equations and Eqs. (54), and writing the result in terms of $\delta\dot{\mathbf{x}} = \partial\delta\mathbf{x}/\partial t$ yields the following:

$$\begin{aligned}
\delta^2 H \;=\; & \int d^3x\, d^3v\, f^{(0)} \Bigg\{ \frac{m}{2}\left[|\delta\dot{\mathbf{x}}|^2 - |d\delta\mathbf{x}|^2 \right] + \frac{e}{2}\,\delta x_i\,\delta x_j \frac{\partial^2\Phi^{(0)}}{\partial x_i\,\partial x_j} \\
& - \frac{e}{c}\left[\delta x_i\,\delta x_j\,\frac{v_k}{2}\,\frac{\partial^2 A_k^{(0)}}{\partial x_i\,\partial x_j} + (d\delta x_i)\delta x_j\,\frac{\partial A_i^{(0)}}{\partial x_j} + v_k\delta x_i\,\frac{\partial\delta A_k}{\partial x_i} + \delta A_i\, d\delta x_i \right] \Bigg\} \\
& + \frac{1}{8\pi}\int d^3x \left[\delta E^2 + \delta B^2 \right],
\end{aligned} \tag{60}$$

which is equivalent to

$$\begin{aligned}
\delta^2 H \;=\; & \int d^3x\, d^3v\, f^{(0)}(\mathbf{x},\mathbf{v}) \Bigg\{ \frac{m}{2}\left[|\delta\dot{\mathbf{x}}|^2 - |d\delta\mathbf{x}|^2 \right] \\
& + \frac{e}{2}\left[-2\delta\mathbf{x}\cdot\left(\frac{\mathbf{v}\times\delta\mathbf{B}}{c}\right) + \left(\frac{\delta\mathbf{x}\times\mathbf{B}^{(0)}}{c}\right)\cdot d\delta\mathbf{x} - \delta\mathbf{x}\cdot(\delta\mathbf{x}\cdot\nabla)\left(\mathbf{E}^{(0)} + \frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right) \right] \Bigg\} \\
& + \frac{1}{8\pi}\int d^3x \left[\delta E^2 + \delta B^2 \right],
\end{aligned} \tag{61}$$

where we have used the shorthand $d \equiv \mathbf{v}\cdot\nabla + \mathbf{a}^{(0)}\cdot\frac{\partial}{\partial\mathbf{v}}$.

Since the equations of motion are second order in time, the variable $\delta\dot{\mathbf{x}}$ is independent of $\delta\mathbf{x}$. This independence would be manifest if we rewrote Eq. (61) in terms of the particle momentum conjugate to $\delta\mathbf{x}$. As in Sec. III we desire to restrict our choice of $\delta\mathbf{x}$ and $\delta\dot{\mathbf{x}}$ in such a way that these quantities are dynamically accessible; that is they must arise from infinitesimal canonical transformation. Suppose the total dynamics, $\mathbf{x} = \mathbf{x}^{(0)} + \delta\mathbf{x}$, arises from a mixed variable generating function $F(\mathbf{P}, \mathbf{x})$, which we suppose is near identity

$$F(\mathbf{P}, \mathbf{x}) \;=\; \mathbf{P}\cdot\mathbf{x} + G(\mathbf{P}, \mathbf{x}). \tag{62}$$

The perturbations are then generated by

$$\delta\mathbf{x} \;=\; \frac{\partial G}{\partial \mathbf{p}}(\mathbf{p}, \mathbf{x}) \tag{63}$$

$$\delta\mathbf{p} \;=\; -\frac{\partial G}{\partial \mathbf{x}}(\mathbf{p}, \mathbf{x}), \tag{64}$$

where as usual the infinitesimal transformations are not of the mixed type, i.e., we can replace $\mathbf{P}$ by $\mathbf{p}$ to first order. Perturbations $\delta\mathbf{x}$ and $\delta\mathbf{p}$ that are obtained from a generating function, as in Eqs. (63) and (64), can be viewed as arising out of the infinite past. In the case of instability this happens for an infinitesimal perturbation at $t = -\infty$. In the case of a linearly stable system, one can imagine an external perturbation to the force law that adiabatically generates $\delta\mathbf{x}$ and $\delta\mathbf{p}$. To relate $\delta\mathbf{p}$ and $\delta\dot{\mathbf{x}}$, note that $\mathbf{p} = mD\,\mathbf{x} + \frac{e}{c}\mathbf{A}$, and thus

$$\delta\mathbf{p} \;=\; mD\delta\mathbf{x} + \frac{e}{c}\delta\mathbf{A} = -\frac{\partial G}{\partial \mathbf{x}}\bigg|_{\mathbf{p}} \tag{65}$$

or using $\delta\dot{\mathbf{x}} = D\delta\mathbf{x} - d\delta\mathbf{x}$ we get

$$\delta\dot{\mathbf{x}} \;=\; D\delta\mathbf{x} - d\delta\mathbf{x} = -\frac{1}{m}\frac{\partial G}{\partial \mathbf{x}}\bigg|_{\mathbf{p}} - \frac{e}{mc}\delta\mathbf{A} - d\frac{\partial G}{\partial \mathbf{p}}\bigg|_{\mathbf{x}}. \tag{66}$$

Writing $G(\mathbf{p}, \mathbf{x}) = G\left(m\mathbf{v} + \frac{e\mathbf{A}^{(0)}}{c}, \mathbf{x}\right)$ this becomes

$$\delta\dot{\mathbf{x}} \;=\; -\frac{1}{m}\frac{\partial G}{\partial \mathbf{x}}\bigg|_{\mathbf{v}} + \frac{e}{m^2c}\frac{\partial G}{\partial v_i}\bigg|_{\mathbf{x}}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}} - \frac{e}{mc}\delta\mathbf{A} - \frac{1}{m}d\frac{\partial G}{\partial \mathbf{v}}\bigg|_{\mathbf{x}}. \tag{67}$$

This expression for $\delta\dot{\mathbf{x}}$ is gauge invariant [c.f. Eq. (51)], and the same can be said for $\delta\mathbf{x}$. Thus when we substitute Eqs. (63) and (67) into Eq. (60) or (61), we obtain an expression for the second order gauge invariant energy, where the particle degrees of freedom are contained entirely in G. Gauge invariance follows from Eq. (61), but we choose to write our energy expression in terms of the potentials, as follows:

$$
\begin{aligned}
\delta^2 H &= \sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} f_\nu^{(0)}(\mathbf{x},\mathbf{v}) \Bigg\{ \left| \frac{\partial G_\nu}{\partial \mathbf{x}} - \frac{e_\nu}{m_\nu c} \frac{\partial G_\nu}{\partial v_i} \frac{\partial A_i^{(0)}}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu c}\, \delta \mathbf{A} \right|^2 \\
&+ 2d \frac{\partial G_\nu}{\partial \mathbf{v}} \cdot \left[\frac{\partial G_\nu}{\partial \mathbf{x}} - \frac{e_\nu}{m_\nu c} \frac{\partial G_\nu}{\partial v_i} \frac{\partial A_i^{(0)}}{\partial \mathbf{x}} + \frac{e_\nu}{c}\, \delta \mathbf{A} \right] + \frac{e_\nu}{m_\nu} \frac{\partial G_\nu}{\partial v_i} \frac{\partial G_\nu}{\partial v_j} \frac{\partial^2 \Phi^{(0)}}{\partial x_i \partial x_j} \\
&- \frac{e_\nu}{m_\nu c} \left[\frac{\partial G_\nu}{\partial v_i} \frac{\partial G_\nu}{\partial v_j} v_k \frac{\partial^2 A_k^{(0)}}{\partial x_i \partial x_j} + 2 \frac{\partial G_\nu}{\partial v_j}\, d\, \frac{\partial G_\nu}{\partial v_i} \frac{\partial A_i^{(0)}}{\partial x_j} \right] - \frac{2e_\nu}{c} \left[\frac{\partial G_\nu}{\partial v_i} v_k \frac{\partial \delta A_k}{\partial x_i} \right. \\
&+ \left. \delta A_i\, d\, \frac{\partial G_\nu}{\partial v_i} \right] \Bigg\} + \frac{1}{8\pi} \int \left[\delta E^2 + \delta B^2 \right] d^3x. \qquad (68)
\end{aligned}
$$

Here we have generalized the result by adding the species index ν.

Equation (68) is a complicated expression that can be written in many ways by integration by parts and neglect of surface terms. In Sec. V we will examine it in greater detail in special cases. For now we restrict to a single species and neglect the equilibrium field $\mathbf{A}^{(0)}$ and, as well, the perturbation $\delta\mathbf{A}$. In Sec. V it is shown that spatial localization of the perturbation from equilibrium renders the positive energy contribution, arising from the fields, negligible. Without the magnetic field, the energy expression of Eq. (61) becomes

$$
\begin{aligned}
\delta^2 H &= \int \frac{f^{(0)}}{2m} \left\{ \left| \frac{\partial G}{\partial \mathbf{x}} \right|^2 + 2d \frac{\partial G}{\partial \mathbf{v}} \cdot \frac{\partial G}{\partial \mathbf{x}} + \frac{e}{m} \frac{\partial G}{\partial \mathbf{v}} \cdot \frac{\partial^2 \Phi^{(0)}}{\partial \mathbf{x} \partial \mathbf{x}} \cdot \frac{\partial G}{\partial \mathbf{v}} \right\} d^3x\, d^3v \\
&\quad + \frac{1}{8\pi} \int \delta E^2 d^3x, \qquad (69)
\end{aligned}
$$

where the operator d reduces to

$$
d = \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{e}{m} \frac{\partial \Phi^{(0)}}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{v}} = -\left[\mathcal{E}, \cdot \right]. \qquad (70)
$$

In the next section [Sec. VA] we see that positive definiteness of Eq. (69) depends upon the monotonicity-isotropy condition of Sec. II. Assuming $f^{(0)}(\mathcal{E})$, where $\mathcal{E} = mv^2/2 + e\Phi^{(0)}$ is the particle energy, a sequence of integrations by parts and neglect of surface terms results in the following:

$$\delta^2 H = -\frac{1}{2}\int \frac{\partial f^{(0)}}{\partial \mathcal{E}}\,[\mathcal{E}, G]^2\, d^3x\, d^3v + \frac{1}{8\pi}\int \delta E^2 d^3x. \tag{71}$$

As shown in Sec. II, the expression on the r.h.s. of Eq. (71) is positive definite if and only if inequality (18) is satisfied. This velocity-space criterion is the crucial thing in the general case.

V. Examples

Now we consider several examples, beginning with Eq. (68), the energy expression of Sec. IV.

A. Electrostatic Equilibrium — Electrostatic Perturbations

For simplicity we consider a single species with an equilibrium characterized by $f^{(0)}(\mathcal{E})$. Equation (69), with the neglect of the electrostatic energy term, (indicated by prime) can be written out as follows:

$$\begin{aligned}\delta^2 H' &= \int \frac{d^3x\, d^3v}{2m} f^{(0)}(\mathcal{E})\left\{ \left|\frac{\partial G}{\partial \mathbf{x}}\right|^2 + 2\,\frac{\partial G}{\partial x_k} v_i \frac{\partial}{\partial x_i}\frac{\partial G}{\partial v_k}\right.\\ &\quad \left. - \frac{2e}{m}\frac{\partial G}{\partial x_k}\frac{\partial^2 G}{\partial v_i \partial v_k}\frac{\partial \Phi^{(0)}}{\partial x_i} + \frac{e}{m}\frac{\partial G}{\partial v_k}\frac{\partial G}{\partial v_i}\frac{\partial^2 \Phi^{(0)}}{\partial x_k \partial x_i}\right\}.\end{aligned} \tag{72}$$

Integrating half of the second term of Eq. (72) by parts in v_k and the other half by parts in x_i yields

$$\begin{aligned}\delta^2 H' &= \int \frac{d^3x\, d^3v}{2m}\left\{ m\mathbf{v}\cdot\frac{\partial G}{\partial \mathbf{x}}\,[\mathcal{E}, G]\,\frac{\partial f^{(0)}}{\partial \mathcal{E}} + \frac{e}{m} f^{(0)}\frac{\partial G}{\partial v_k}\frac{\partial}{\partial x_k}\left(\frac{\partial G}{\partial v_i}\frac{\partial \Phi^{(0)}}{\partial x_i}\right)\right.\\ &\quad \left. - \frac{e}{m}\frac{\partial G}{\partial x_k}\frac{\partial \Phi^{(0)}}{\partial x_i} f^{(0)}\frac{\partial^2 G}{\partial v_i \partial v_k}\right\}.\end{aligned}$$

Integrating the second term by parts in x_k and the third by parts in v_k yields the desired result

$$\delta^2 H = -\frac{1}{2}\int d^3x\, d^3v\, \frac{\partial f^{(0)}}{\partial \mathcal{E}}\,[\mathcal{E}, G]^2 + \int \frac{\delta E^2}{8\pi}\, d^3x, \tag{73}$$

where we have added back the electrostatic energy term. This is consistent with the results of Secs. II and III.

B. Homogeneous Equilibria — Electrostatic Perturbations

For this first example we suppose $\mathbf{E}^{(0)} = 0$, $\mathbf{A}^{(0)} = 0$, $\partial f_\nu^{(0)}/\partial \mathbf{x} = 0$ and set $\delta \mathbf{A} = 0$. Equilibria of this type can be written as $f_\nu^{(0)}(\mathbf{v})$. In this case the integrand of $\delta^2 H$ is independent of $\mathbf{x}$, so it is natural to suppose that the spatial domain is a large periodic box of volume V. Perturbed quantities such as G_ν, which minimize $\delta^2 H$, are represented as follows:

$$G_\nu = \hat{G}_\nu(\mathbf{v}, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.} \tag{74}$$

Thus Eq. (68) for $\delta^2 H$ becomes (upon dropping the "caret")

$$\begin{aligned} \delta^2 H &= \frac{V}{2} \sum_\nu \int d^3v \, \frac{f_\nu^{(0)}}{2m_\nu} \left[k^2 \left| G_\nu \right|^2 + G_\nu^* \, \mathbf{k}\cdot\mathbf{v}\, \mathbf{k} \cdot \frac{\partial G_\nu}{\partial \mathbf{v}} + G_\nu \, \mathbf{k}\cdot\mathbf{v}\, \mathbf{k} \cdot \frac{\partial G_\nu^*}{\partial \mathbf{v}} \right] \\ &+ \frac{V}{16\pi} \left| \delta \mathbf{E} \right|^2 . \end{aligned} \tag{75}$$

Integrating by parts yields

$$\delta^2 H = -\frac{V}{2} \sum_\nu \int d^3v \, \frac{1}{2m_\nu} \left| G_\nu \right|^2 \mathbf{k}\cdot\mathbf{v}\, \mathbf{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} + \frac{V}{16\pi} \left| \delta \mathbf{E} \right|^2 . \tag{76}$$

If the frame of reference is chosen so that

$$\sum_\nu m_\nu \int \mathbf{v} f_\nu^{(0)} d^3v = 0, \tag{77}$$

$\delta^2 H$ will not contain a contribution from the center of mass kinetic energy. We assume this is the case.

Choosing a "particle perturbation"; that is, one for which $\delta \mathbf{E} = 0$, the stabilizing effect of the electrostatic energy is lost and $\delta^2 H$ can be made negative if

$$\mathbf{k}\cdot\mathbf{v}\, \mathbf{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} > 0 \tag{78}$$

for some $\mathbf{k}$ and $\mathbf{v}$. Note that again we have obtained the monotonicity-isotropy condition. As already shown in Sec. II any deviation of $f_\nu^{(0)}$ from being a monotonic function of v^2 results in the existence of negative energy perturbations. This includes all non-isotropic distributions.

C. Homogeneous Equilibria — Electromagnetic Perturbations

Now consider the question of whether, for the same unperturbed system as that treated in Sec. VB, a nonzero choice of $\delta\mathbf{A}$ could lead to lower energies. If so, the $\delta^2 H$ threshold for the transition to negative energies would be given by a condition that is less restrictive than Eq. (78). A somewhat lengthy calculation is presented in the Appendix, which shows that this is *not* the case; the condition remains unchanged.

D. General Maxwell-Vlasov Equilibria

We conclude with a sufficient criterion for the existence of negative energy modes in general Maxwell-Vlasov Equilibria. To this end we localize our perturbations G_ν, for one species ν, to intervals of size Δx, Δy, and Δz, each being small compared to the typical gyroradius of this species. All other G_ν's we set equal to zero. Furthermore, we take our perturbations, which are localized inside these intervals, to be proportional to $e^{i\mathbf{k}\cdot\mathbf{x}}$ with

$$k_x r_{L_\nu}, \quad k_y r_{L_\nu}, \quad k_z r_{L_\nu} \gg 1. \tag{79}$$

The right-hand side of Eq. (68) is then dominated by terms that are bilinear in $\partial G_\nu/\partial\mathbf{x}$ and $\partial/\partial\mathbf{x} \cdot \partial G_\nu/\partial\mathbf{v}$. These terms form the same expression as the exact one for the homogeneous magnetic field-free case with $\delta\mathbf{A} = 0$. We obtain again condition (78), now as a sufficient one, for the existence of negative energy perturbations. We draw therefore the conclusion that, for negative energy perturbations to exist in any Maxwell-Vlasov equilibrium it is sufficient that at least one particle species has an unperturbed distribution function in the vicinity of a single point, that deviates from a monotonic function of the energy. We require this to occur in a frame where the energy of the unperturbed system is minimized.

It seems likely to us that this sufficient condition for the existence of localized negative energy perturbations is also necessary, just as in the case of the field-free homogeneous plasma. Nevertheless, there does not exist an inhomogeneous plasma fulfilling the sufficient condition.

The strongly localized modes considered above possibly are not the most dangerous ones. It is therefore of interest to investigate the degree of localization required for negative modes to exist. Since the Vlasov equation is only valid for wavelengths larger than the Debye length, one must check to see if the localization hedges this validity. Preliminary calculations suggest that this is not the case. We will report on these calculations in the future.

Appendix

We consider an equilibrium with $\mathbf{E}^{(0)} = 0$, $\mathbf{A}^{(0)} = 0$, and $\partial f_\nu^{(0)}/\partial \mathbf{x} = 0$, but allow variations, $\delta\mathbf{A} \neq 0$, in the vector potential. By a sequence of extremizations we seek the minimum value of $\delta^2 H$. Assuming $\delta\mathbf{A} = \delta\hat{\mathbf{A}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}$ and $G_\nu = \hat{G}_\nu e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}$, Eq. (68) becomes (upon dropping the "caret" again)

$$\begin{aligned}
\delta^2 H &= \frac{V}{2}\sum_\nu \int d^3v\, \frac{f_\nu^{(0)}}{2m_\nu}\left[\left|i\mathbf{k}G_\nu - \frac{e_\nu}{c}\delta\mathbf{A}\right|^2 + \mathbf{v}\cdot\left(-i\mathbf{k}G_\nu^* - \frac{e_\nu}{c}\delta\mathbf{A}^*\right) i\mathbf{k}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}} + \text{c.c.}\right] \\
&+ \frac{V}{16\pi}\left[\,|\delta\mathbf{E}|^2 + k^2|\delta\mathbf{A}|^2 - |\mathbf{k}\cdot\delta\mathbf{A}|^2\right]. \qquad \text{(A1)}
\end{aligned}$$

The question of interest is whether the presence of $\delta\mathbf{A}$ leads to a condition less restrictive than (78) for the existence of negative energy perturbations. Manipulations paralleling those of the electrostatic case of Sec. VB result in the following expression for $\delta^2 H$:

$$\begin{aligned}
\delta^2 H &= \frac{V}{2}\sum_\nu \int d^3v\, \frac{1}{2m_\nu}\left[-\mathbf{k}\cdot\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}|G_\nu|^2 + f_\nu^{(0)}\frac{e_\nu^2}{c^2}|\delta\mathbf{A}|^2\right. \\
&- \left. i f_\nu^{(0)}\frac{e_\nu}{c}\left(\mathbf{k}\cdot\delta\mathbf{A}^* G_\nu + \mathbf{v}\cdot\delta\mathbf{A}^*\mathbf{k}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right) + \text{c.c.}\right] \\
&+ \frac{V}{16\pi}\left[\,|\delta\mathbf{E}|^2 + k^2|\delta\mathbf{A}|^2 - |\mathbf{k}\cdot\delta\mathbf{A}|^2\right]. \qquad \text{(A2)}
\end{aligned}$$

Minimizing $\delta^2 H$ with respect to $\delta\mathbf{A}^*$ yields

$$-\frac{V}{2}\sum_\nu \int d^3v\, \frac{e_\nu}{2m_\nu c} f_\nu^{(0)}\left[i\mathbf{k}G_\nu + i\mathbf{v}\,\mathbf{k}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}} - \frac{e_\nu}{c}\delta\mathbf{A}\right] + \frac{V}{16\pi}\left(k^2\delta\mathbf{A} - \mathbf{k}\,\mathbf{k}\cdot\delta\mathbf{A}\right) = 0. \qquad \text{(A3)}$$

We define

$$\sum_\nu \int d^3v\, \frac{e_\nu^2 f_\nu^{(0)}}{2m_\nu} \equiv \sum_\nu \frac{\omega_{p\nu}^2}{8\pi} \equiv \frac{\Omega_p^2}{8\pi}, \qquad \text{(A4)}$$

where $\omega_{p\nu}$ is the plasma frequency for species ν. With Eq. (A4), Eq. (A3) becomes

$$\begin{aligned}
&-\frac{iV}{2}\sum_\nu \int d^3v\, \frac{e_\nu}{2m_\nu c} f_\nu^{(0)}\left(\mathbf{k}\,G_\nu + \mathbf{v}\,\mathbf{k}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right) \\
&\qquad + \frac{V}{16\pi}\left[\left(\frac{\Omega_p^2}{c^2} + k^2\right)\delta\mathbf{A} - \mathbf{k}(\mathbf{k}\cdot\delta\mathbf{A})\right] = 0. \qquad \text{(A5)}
\end{aligned}$$

Combining the first two terms of (A5) and then integrating by parts yields

$$\frac{iV}{2}\sum_\nu \int d^3v \, \frac{e_\nu}{2m_\nu c} \, \mathbf{v} \, G_\nu \, \mathbf{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} + \frac{V}{16\pi}\left[\left(\frac{\Omega_p^2}{c^2} + k^2\right)\delta\mathbf{A} - \mathbf{k}\mathbf{k}\cdot\delta\mathbf{A}\right] \;=\; 0. \qquad \text{(A6)}$$

Multiplication by $\mathbf{k}$ yields an expression for $\mathbf{k}\cdot\delta\mathbf{A}$,

$$\frac{iV}{2}\sum_\nu \int d^3v \, \frac{e_\nu}{2mc} \, G_\nu \, \mathbf{k}\cdot\mathbf{v}\,\mathbf{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} + \frac{V}{16\pi}\frac{\Omega_p^2}{c^2}\mathbf{k}\cdot\delta\mathbf{A} \;=\; 0, \qquad \text{(A7)}$$

and the minimizing $\delta\mathbf{A}$ is then given by

$$\frac{iV}{2}\sum_\nu \int d^3v \, \frac{e_\nu}{2m_\nu c} \, G_\nu \, \mathbf{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \left[\mathbf{v} + \mathbf{k}\mathbf{k}\cdot\mathbf{v}\,\frac{c^2}{\Omega_p^2}\right] + \frac{V}{16\pi}\left(\frac{\Omega_p^2}{c^2} + k^2\right)\delta\mathbf{A} \;=\; 0. \qquad \text{(A8)}$$

We will use this expression subsequently.

Symbolically we can write Eq. (A2) for $\delta^2 H$ in the form

$$\delta^2 H = \delta\mathbf{A}^* \cdot \underline{\underline{R}} \cdot \delta\mathbf{A} + \delta\mathbf{A}^* \cdot \mathbf{T} + \delta\mathbf{A}\cdot\mathbf{T}^* + S, \qquad \text{(A9)}$$

where we explicitly display the dependence on $\delta\mathbf{A}$ and $\delta\mathbf{A}^*$. Thus Eq. (A3) reads

$$\underline{\underline{R}} \cdot \delta\mathbf{A} + \mathbf{T} \;=\; 0. \qquad \text{(A10)}$$

Since $\underline{\underline{R}}$ is real and symmetric, the complex conjugate of (A10) is

$$\delta\mathbf{A}^* \cdot \underline{\underline{R}} + T^* \;=\; 0, \qquad \text{(A11)}$$

which corresponds to the minimization of (A9) with respect to $\delta\mathbf{A}$. If we solved (A10) and (A11) for $\delta\mathbf{A}$ and $\delta\mathbf{A}^*$, respectively, and inserted the result into (A9) we would obtain the "$\delta\mathbf{A}$" minimum of $\delta^2 H$. More conveniently, we multiply (A10) by $\delta\mathbf{A}^*$ and (A11) by $\delta\mathbf{A}$, and then eliminate the linear terms of (A9) to obtain

$$\delta^2 H_{\min A} \;=\; -\,\delta\mathbf{A}^* \cdot \underline{\underline{R}} \cdot \delta\mathbf{A} + S \qquad \text{(A12)}$$

Noting that

$$R_{ik} \;=\; \frac{V}{16\pi}\left[\left(\frac{\Omega_p^2}{c^2} + k^2\right)\delta_{ik} - k_i k_k\right], \qquad \text{(A13)}$$

and S is given from Eq. (A2), Eq. (A12) becomes

$$\begin{aligned}\delta^2 H_{\min A} = & - \frac{V}{16\pi}\left[\left(\frac{\Omega_p^2}{c^2}+k^2\right)|\delta\mathbf{A}|^2 - |\mathbf{k}\cdot\delta\mathbf{A}|^2\right] \\ & - \frac{V}{2}\sum_\nu \int d^3v\, \frac{1}{2m_\nu}|G_\nu|^2\,\mathbf{k}\cdot\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}},\end{aligned} \tag{A14}$$

where $\delta\mathbf{A}$ is shorthand for the expression resulting from Eq. (A8). Note that we have neglected the electrostatic term since its minimum is zero.

Since the first term of Eq. (A14) is always negative, we conclude that a sufficient criterion for the existence of negative energies $\delta^2 H$ is the same as the sufficient and necessary criterion for purely electrostatic perturbations, namely

$$\mathbf{k}\cdot\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} > 0 \tag{A15}$$

for some $\mathbf{k}$ and $\mathbf{v}$.

If inequality (A15) is nowhere fulfilled we can further seek the minimum of $\delta^2\mathcal{E}$ by varying with respect to G_ν. Since $\delta^2 H$ is now a purely bilinear expression in the functions G_ν, we require a normalization condition. It is most convenient to choose, as such, the following:

$$-\frac{V}{2}\sum_\nu \int d^3v\, \frac{1}{2m_\nu}|G_\nu|^2\,\mathbf{k}\cdot\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}. \tag{A16}$$

This is by assumption a positive definite quantity reminiscent of the kinetic energy norm for the usual MHD energy principle. Our variational principle can then be written as

$$-\sum_\nu \int d^3v\, G_\nu^* \mathbf{a}_\nu \cdot \underline{\underline{R}} \cdot \sum_{\nu'} \int d^3v' G_{\nu'}\mathbf{a}_{\nu'} + \lambda \sum_\nu \int |G_\nu|^2\, b_\nu d^3v, \tag{A17}$$

where λ is a Lagrange multiplier and the $\mathbf{a}_\nu$ and b_ν follow from Eqs. (A14), (A8), and (A16), respectively:

$$\mathbf{a}_\nu \equiv \sqrt{\frac{V}{16\pi}}\,\frac{8\pi}{\Omega_p^2/c^2+k^2}\,\frac{e_\nu}{2m_\nu c}\,\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\left(\mathbf{v}+\frac{c^2}{\Omega_p^2}\mathbf{k}\,\mathbf{k}\cdot\mathbf{v}\right) \tag{A18}$$

$$b_\nu \equiv -\frac{V}{2}\frac{1}{2m_\nu}\mathbf{k}\cdot\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}. \tag{A19}$$

Now, variation of (A17) with respect to G_ν^* yields

$$-\mathbf{a}_\nu \cdot \underline{\underline{R}} \cdot \sum_{\nu'} \int d^3v' G_{\nu'} \mathbf{a}_{\nu'} + \lambda G_\nu b_\nu = 0. \tag{A20}$$

Defining

$$\mathbf{y} \equiv \sum_\nu \int d^3v\, \mathbf{a}_\nu G_\nu,$$

Eq. (A20) can be compactly written as

$$\sum_\nu \int d^3v \, \frac{\mathbf{a}_\nu \mathbf{a}_\nu}{b_\nu} \cdot \underline{\underline{R}} \cdot \mathbf{y} = \lambda \mathbf{y}. \tag{A21}$$

This is an eigenvalue problem for the vector $\mathbf{y}$ with eigenvalue λ. Knowing λ, we can write the minimum $\delta^2 H$ as

$$\delta^2 H_{\text{min}} = (1 - \lambda_{\text{max}}) \sum_\nu \int d^3v \, |G_\nu|^2 \, b_\nu. \tag{A22}$$

This follows from Eqs. (A14) and (A17).

From Eq. (A22) we observe that if $\lambda_{\text{max}} > 1$ there can be negative energy perturbations without inequality (A15) being fulfilled, i.e., with $b_\nu < 0$. Hence, it is important to consider the eigenvalue problem in some detail. Using Eqs. (A13) and (A18) the eigenvalue equation (A21) becomes

$$\underline{\underline{M}} \cdot \left(\underline{\underline{I}} + \frac{c^2}{\Omega_p^2} \mathbf{k}\,\mathbf{k} \right) \cdot \mathbf{y} = \lambda \mathbf{y}, \tag{A23}$$

where the matrix $\underline{\underline{M}}$ is given by

$$\underline{\underline{M}} \;=\; -\sum_\nu \int d^3v \, \frac{1}{\Omega_p^2/c^2 + k^2} \, \frac{4\pi e_\nu^2}{m_\nu c^2} \, \frac{1}{\mathbf{k} \cdot \mathbf{v}} \, \mathbf{k} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \, \mathbf{v}\,\mathbf{v}. \tag{A24}$$

Since we are interested in the case where (A15) is not satisfied we specialize to isotropic distribution functions, $f_\nu^{(0)} = f_\nu^{(0)}(v^2/2)$. The matrix of Eq. (A24) becomes

$$\begin{aligned}
\underline{\underline{M}} \;&=\; -\sum_\nu \int d^3v \, \frac{1}{\Omega_p^2/c^2 + k^2} \, \frac{4\pi e_\nu^2}{m_\nu c^2} \, f_\nu^{(0)\prime} \mathbf{v}\,\mathbf{v} \\
&=\; -\sum_\nu \int d^3v \, \frac{1}{\Omega_p^2/c^2 + k^2} \, \frac{4\pi e_\nu^2}{m_\nu c^2} \, f_\nu^{(0)\prime} \, \frac{v^2}{3} \underline{\underline{I}} \\
&\equiv\; M \underline{\underline{I}},
\end{aligned} \tag{A25}$$

and Eq. (A23) is then simply

$$M\left(\underline{\underline{I}}+\frac{c^2}{\Omega_p^2}\mathbf{k}\mathbf{k}\right)\cdot\mathbf{y}=\lambda\mathbf{y}. \tag{A26}$$

The eigenvalues of Eq. (A26) are easily obtained

$$\lambda = M, M, M\left(1+\frac{c^2k^2}{\Omega_p^2}\right), \tag{A27}$$

and the existence of negative $\delta^2 H_{\rm min}$ occurs for

$$M\left(1+\frac{c^2k^2}{\Omega_p^2}\right) > 1. \tag{A28}$$

For general isotropic distribution functions

$$\begin{aligned}\int d^3v\, f_\nu^{(0)\prime}\left(\frac{v^2}{2}\right)\frac{v^2}{3} &= -4\pi\int v^2 dv\, f_\nu^{(0)} \\ &= -\int d^3v\, f_\nu^{(0)} \equiv -n_\nu^{(0)},\end{aligned} \tag{A29}$$

and therefore

$$M=\frac{\Omega_p^2/c^2}{\Omega_p^2/c^2+k^2}, \tag{A30}$$

and thus

$$M\left(1+\frac{c^2k^2}{\Omega_p^2}\right) = 1. \tag{A31}$$

Hence Eq. (A15) is necessary and sufficient for the existence of negative $\delta^2 H$, a result identical to the purely electrostatic case.

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