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Exact Nonlinear Travelling Solutions of the Vlasov-Maxwell System

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Exact time dependent solutions of a system of charged particles with their self-consistent electromagnetic fields are obtained by solving the fully nonlinear Vlasov-Maxwell set of equations.

Exact nonlinear solutions to coupled field theories are generally rare. For the Vlasov-Maxwell system, two kinds of time-independent solutions have been reported; 1) the electrostatic solutions of the Bernstein-Green-Kruskal (BGK) type,[1] and 2) the magnetic solution of the type obtained by Pfirsch,[2] Laval-Pellat and Vuillemin,[3] and Marx.[4] In this letter, we present a time-dependent fully electromagnetic solution to the Vlasov-Maxwell equations which describes a system of charged particles interacting with their self-consistent fields.

We consider an electron-ion plasma embedded in a constant external magnetic field $\mathbf{B}_{\text{ext}} = \hat{e}_z B_0$. Assuming the field quantities to vary only in the x-direction ($\partial/\partial y = 0 = \partial/\partial z$), the relevant equations describing the system can be written as

$$\frac{\partial f_e}{\partial t} + v_x \frac{\partial f_e}{\partial x} - \frac{e}{m_e}\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times (\mathbf{B} + \hat{e}_z B_0)\right] \cdot \frac{\partial f_e}{\partial \mathbf{v}} = 0 \tag{1a}$$

$$\frac{\partial f_i}{\partial t} + v_x \frac{\partial f_i}{\partial x} + \frac{e}{m_i}\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times (\mathbf{B} + \hat{e}_z B_0)\right] \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 \tag{1b}$$

$$\frac{\partial E_x}{\partial x} = 4\pi\rho = -4\pi e \int d\mathbf{v}(f_e - f_i) \tag{2}$$

$$\frac{\partial B_x}{\partial x} = 0 \tag{3}$$

$$\frac{\partial}{\partial x}(\hat{e}_x \times \mathbf{E}) + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4}$$

$$\frac{\partial}{\partial x}(\hat{e}_z \times \mathbf{B}) - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} = -\frac{4\pi e}{c}\int d\mathbf{v}\ \mathbf{v}(f_e - f_i) \tag{5}$$

where Eqs. (1a) and (1b) are the Vlasov equations for the evolution of the electron and ion distribution functions f_e and f_i in the presence of the external field B_0 and the self-consistently generated fields $\mathbf{E}$ and $\mathbf{B}$ which obey Maxwell's equations (2-5). In Eqs. (1-5), $-e$ is the charge of the electron, m_e and m_i are the electron and ion masses, and ρ and $\mathbf{J}$ are the plasma charge and current densities.

The search for the solution begins by proposing displaced Maxwellian distributions for the particles

$$f_{e,i} = \frac{n_0}{\pi^{3/2} v_{e,i}^3} \exp\left[-\frac{(\mathbf{v} - \mathbf{u}_{e,i})^2}{v_{e,i}^2}\right] g_{e,i}(x - Ut) \tag{6}$$

where n_0 is a measure of the ambient density, $v_{e,i}$ and $\mathbf{u}_{e,i}$ are respectively the thermal speeds and the drift velocities, and the entire space-time dependence is contained in the factor $g_{e,i}$ which depends only on the variables $\eta = x - Ut$. Making use of Eq. (6), one obtains (explicit arguments will be dropped) the expression for the charge and current densities,

$$\rho = -en_0(g_e - g_i) \ , \tag{7}$$

$$\mathbf{J} = -en_0(\mathbf{u}_e g_e - \mathbf{u}_i g_1) \ , \tag{8}$$

which implies that the choice

$$g_e = g_i = g \quad , \quad E_x = 0 = B_x \tag{9}$$

trivially satisfies Eqs. (2) and (3). Using Eqs. (6-9) and remembering that $\partial/\partial t = -U d/d\eta$, $\partial/\partial x = d/d\eta$, the remaining nontrivial components of Maxwell's equations take the form

$$\frac{d}{d\eta}\left[E_y - \frac{U}{c}B_z\right] = 0 \tag{10}$$

$$\frac{d}{d\eta}\left[E_z - \frac{U}{c}B_y\right] = 0 \tag{11}$$

$$\frac{dB_z}{d\eta} = +\frac{4\pi e}{c} n_0 g \left(u_y^e - u_y^i\right) + \frac{U}{c}\frac{dE_y}{d\eta} \tag{12}$$

$$\frac{dB_y}{d\eta} = -\frac{4\pi e}{c} n_0 g \left(u_z^e - u_z^i\right) + \frac{U}{c}\frac{dE_z}{d\eta} \; . \tag{13}$$

Equations (10) and (11) immediately yield

$$E_y = \frac{U}{c} B_z + E_0 \equiv \frac{U}{c}[B_z + B_0] \tag{14}$$

and

$$E_z = -\frac{U}{c} B_y \tag{15}$$

which guides us to the interpretation that U must be a drift in the x-direction, and in the moving frame, the effective electric field remains zero. Making use of Eqs. (6), (14) and (15),

$$\mathbf{u}_{e,i} = \hat{e}_z u_z^{e,i} + \hat{e}_y \, u_y^{e,i} + \hat{e}_x U \tag{16}$$

and some straightforward algebra, the Vlasov equations reduce to $[T_{e,i} = 0.5 \, m_{e,i} v_{e,i}^2$ is the temperature]

$$\frac{1}{g_e}\frac{dg_e}{d\eta} = \frac{e}{cT_e}\left[B_y u_z^e - (B_z + B_0) u_y^e\right] \tag{17}$$

$$\frac{1}{g_i}\frac{dg_i}{d\eta} = \frac{e}{cT_i}\left[i B_y u_z^i + (B_z + B_0) u_y^i\right] \tag{18}$$

which can satisfy the constraint $g_e = g_i = g$ only if

$$\frac{u_z^e}{T_e} = -\frac{u_z^i}{T_i} \quad , \quad \frac{u_y^e}{T_e} = -\frac{u_y^i}{T_i} \tag{19}$$

leading to the single Vlasov Equation $\left[\overline{B}_{y(z)} = (e u_{z(y)}^e / cT_e) B_{y(z)}\right]$

$$\frac{1}{g}\frac{dg}{d\eta} = \overline{B}_y - \overline{B}_z \tag{20}$$

to be solved in conjunction with

$$\frac{d}{d\eta}\overline{B}_z = +\frac{2}{\delta^2}\left(\frac{u_y^e}{u^e}\right)^2 g \tag{21}$$

$$\frac{d}{d\eta}\overline{B}_y = -\frac{2}{\delta^2}\left(\frac{u_z^e}{u^e}\right)^2 g \tag{22}$$

which are Eqs. (12) and (13) rewritten in the context of Eqs. (14), (15) and (19) where

$$\delta^2 = \frac{c^2}{\omega_e^2}\frac{v_e^2}{u^{e^2}(1+\tau)}(1 - U^2/c^2) \tag{23}$$

with $\omega_e^2 = (4\pi n_0 e^2/m_e)$, $u^{e^2} = u_z^{e^2} + u_y^{e^2}$ and $\tau = T_i/T_e$ as the ion-electron temperature ratio. The set of Eqs. (20-22) allows the following self-consistent solutions

$$g = \mathrm{sech}^2 \frac{\mathrm{x} - \mathrm{Ut}}{\delta} , \tag{24}$$

$$B_y = -\frac{u_z^e}{u^e}\left(\frac{cT_e}{eu_e}\frac{2}{\delta}\right) \tanh \frac{x - Ut}{\delta} , \tag{25}$$

$$B_z = -B_0 + \frac{u_y^e}{u^e}\left(\frac{cT_e}{eu_e}\frac{2}{\delta}\right) \tanh \frac{x - Ut}{\delta} , \tag{26}$$

$$E_y = \frac{U}{c}(B_z + B_0) , \tag{27}$$

and

$$E_z = -\frac{U}{c}B_y , \tag{28}$$

which are travelling pulses of width δ. The width of the pulse (or the shock front if δ is very small) is a strong function of the plasma properties (density, temperature and the drift speeds) and can vary over a large range. For $B_0 = 0$, the electromagnetic fields are purely transverse ($E \cdot B = 0$) and become more and more light-like as $U \rightarrow c$. However, in this relativistic limit, we must use the relativistic correct form for the Vlasov equation ($\mathbf{p}$ is the momentum),

$$\frac{\partial f(\mathbf{x},\mathbf{p},t)}{\partial t} + v \cdot \frac{\partial f(\mathbf{x},\mathbf{p},t)}{\partial \mathbf{x}} + q\left[\mathbf{E} + \frac{\mathbf{v}\times B}{c}\right] \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \tag{29}$$

with the velocity-momentum relation

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}}{m}\left[1 + \frac{p^2}{m^2c^2}\right]^{-1/2} \tag{30}$$

which introduces some complication in the calculation of the plasma current. This difficulty can be readily overcome by assuming that the flow speed U can be relativistic while the temperature of the plasma remains nonrelativistic. In that case γ can be simply replaced by $\gamma_0 = (1-\frac{U^2}{c^2})^{-1/2} = (1+p_U^2/m^2c^2)^{1/2}$ (where $p_U = \gamma_0 mU$) and the entire analysis remains valid provided the plasma frequency ω_e is replaced by the relativistic form $\omega_{er} = (4\pi n_0 e^2/\gamma_0 m)^{1/2}$. It is also convenient to rewrite Eq. (23) as

$$\delta^2 = \frac{1}{\gamma_0^2}\frac{c^2}{\omega_{er}^2}\frac{v_e^2}{u_e^2(1+\tau)} \ . \tag{31}$$

In this paper, we have presented exact travelling pulse-like solutions of the fully nonlinear Vlasov-Maxwell system. Expressions for the self-consistent fields and the particle distribution function are explicitly displayed in Eqs. (24-28). Those solutions should find widespread applications in understanding the phenomena taking place in the laboratory as well as in the astrophysical plasmas; they should also serve as reference solutions for nonlinear numerical calculations.

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