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ABSTRACT

The theory for the non-linear, turbulent response in a system with intrinsic stochasticity is considered. It is argued that perturbative Eulerian theories, such as the Direct Interaction Approximation (DIA) are inherently unsuited to describe such a system. The exponentiation property that characterizes stochasticity appears in the Lagrangian picture and cannot even be defined in the Eulerian representation. An approximation for stochastic systems—the Normal Stochastic Approximation—is developed and states that the perturbed orbit functions (Lagrangian fluctuations) behave as normally distributed random variables. This is independent of the Eulerian statistics and, in fact, we treat the Eulerian fluctuations as fixed. A simple model problem (appropriate for the electron response in the drift wave) is subjected to a series of computer experiments. To within numerical noise the results are in agreement with the Normal Stochastic Approximation. The predictions of the DIA for this model show substantial qualitative and quantitative departures from the observations.

I. INTRODUCTION

The phenomenon of intrinsic stochasticity has received considerable attention in recent years [1]. To a large extent this has been motivated by its presumed relevance to anomalous transport in tokamaks [2]. Indeed, stochasticity for the electrons is difficult to avoid, since at the fluctuation levels needed to account for the observed anomalies, the island overlap or Chirikov condition is strongly satisfied. Thus the tokamak transport problem suggests a new branch of turbulence theory–namely that of incorporating orbit stochasticity into the non-linear wave dynamics in a self-consistent way. The stochasticity studies referenced above [1] do not address this problem but concentrate on specific orbit properties, exponentiation, Kolmogorov entropy, phase space maps and related phenomenon which have exhibited an incredible variety of structure and pathological behavior. In fact, the occurrence of so much structure in the orbits alone has suggested a kind of futility in attempting a theory for the self-consistent wave dynamics. To the contrary, the basic result of this paper is that stochasticity, in spite of its myriad complex and bewildering properties, actually *simplifies* that part of the theory that we need to know to calculate the non-linear electron response and the drift wave dynamics. This claim has been made previously [3], in an abbreviated form. Here we detail the theory and validate the assertion with a series of numerical experiments.

Although the problem we address has quite general implications for turbulence theories of stochastic systems, we use a specific system appropriate for the drift wave response. This is chosen both for its practical applications, and its theoretical convenience. The relevant phase space is everywhere stochastic. It is not complicated by division into integrable and stochastic regions which would require separate treatment. From a formal integration of the kinetic equation we will motivate an approximation based on the properties of the stochastic orbits. In essence, for the purpose of computing the fluctuations associated with the waves, the orbit perturbation can be treated as a normally distributed random variable with variance $< \delta x^2 >= 2Dt$. Hence, we use the term "Normal Stochastic Approximation" to describe this method. We then have the single parameter, D, (only weakly dependent on the wave spectrum) as the manifestation of the non-linear response. The result is simply a broadened resonance response. We must emphasize, however, that this is not equivalent to Dupree's theory [4] since the underlying arguments differ and the implications of our results go beyond the basic effect. Resonance broadening here, is tied up inextricably with stochasticity. It is not a separate effect added to the usual panoply of non-linear processes, wave-wave coupling, non-linear Landau damping, etc. Rather, for flucuations on the scale of the waves, the entire hierarchy of non-linear processes dissolves into the single complicated phenomenon of stochasticity. The parameter D is then the embodiment of all these processes which have combined to produce radial diffusion. In short, for the wave response, there is resonance broadening, nothing else. This is precisely what the computer experiments described below show. To within the numerical nose in the present experiments, there are no observable discrepancies with the Normal Stochastic Approximation.

II. DISCUSSION OF THE MODEL AND STATISTICAL PROCEDURES

Consider the simple model system with Hamiltonian, $H(J,\theta,t) = H_0(J) + \tilde{H}(J,\theta,t)$, and perturbing Hamiltonian in the form

$$\tilde{H}(J,\theta,t) = \sum_{m,n} H_{mn}(J)\exp[im\theta - i\omega_{mn}t]. \tag{1}$$

We assume H_{00} is not present, so that $\int_0^{2\pi} d\theta\tilde{H} = 0$. The equilibrium frequencies are given by $\omega_0(J) = \partial H_0/\partial J$. Shear, or dependence of the frequency on J, occurs in general and is necessary to produce the phase space islands associated with individual resonances. For small perturbations, δJ, about some equilibrium values, J_0, the frequency can be linearized, $\omega_0(J_0+\delta J) = \omega_0(J_0) + \frac{\partial\omega_0}{\partial J_0}\delta J$. The measure of this approximation is the ratio of the correlation length, J_c, (in units of action) to the global scale $J_s \equiv |d\ln\omega_0/dJ|^{-1}$, which is generally small in practice. For $J_c/J_s \ll 1$, we can, without loss of generality take $H_0 = \frac{1}{2}J^2$ in our model, since ω_0 can always be absorbed into ω_{mn}.

The distribution function $f(J,\theta,t)$ is described by the equation

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial J}\frac{\partial f}{\partial \theta} - \frac{\partial H}{\partial \theta}\frac{\partial f}{\partial J} = 0, \tag{2}$$

which, according to the interpretation of f is either the Vlasov or Klimontovich equation. Finally, we have a condition (such as Poisson's equation) relating f and $\tilde{H}$, $\tilde{H}(J,\theta,t) = \int dJ'\,d\theta'\,K(J,J';\theta,\theta')f(J',\theta',t)$ which will make Eq. (2) non-linear if $\tilde{H}$ is eliminated in favor of f. This gives a closed system evolving f self-consistently.

In spite of its simplicity there are many practical problems that can be reduced to essentially this model: electron driftwave turbulence ($\underline{E} \times \underline{B}$), stochastic magnetic field fluctuations, ion cyclotron resonance, velocity space Langmuir turbulence, etc. The $\tilde{H}$ we use below is intended to represent the electron drift wave reponse.

The approach is to integrate Eq. (2) generally to obtain f as a non-linear functional of $\tilde{H}$. Self-consistency can then be enforced as a final step in determining the non-linear dynamics. In this paper we consider only the first step of solving for f, the turbulence response. However, since the results are not dependent on the details of $\tilde{H}$ (in a stochastic regime), the requirement of self-consistency can be invoked later without difficulty.

Equation (2) can be cast in an integral form which is a convenient starting point for the non-linear theory.

We define an angular average as $<F> = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, F$, and decompose f into average, $f_0(J) = <f>$, and fluctuating, $\tilde{f}(J,\theta,t) = f - f_0$, parts. The angular average of Eq. (2) is

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial J} < \frac{\partial \tilde{H}}{\partial \theta}\tilde{f} >, \tag{3}$$

and the fluctuating part is

$$\left(\frac{\partial}{\partial t} + \frac{\partial H}{\partial J}\frac{\partial}{\partial \theta} - \frac{\partial \tilde{H}}{\partial \theta}\frac{\partial}{\partial J}\right)\tilde{f} = \frac{\partial \tilde{H}}{\partial \theta}\frac{\partial \tilde{f}}{\partial J} - \frac{\partial}{\partial J} < \frac{\partial \tilde{H}}{\partial \theta}\tilde{f} > \tag{4}$$

Equation (4) can be formally solved for $\tilde{f}$ by the method of characteristics, yielding,

$$\tilde{f}(J,\theta,t) = \int_{-\infty}^{t} dt' \left(\frac{\partial \tilde{H}}{\partial \theta}\frac{\partial f_0}{\partial J} - \frac{\partial}{\partial J} < \frac{\partial \tilde{H}}{\partial \theta}\tilde{f} > \right)\Bigg|_{\substack{J = J + \delta J(J,\theta,t,t') \\ \theta = \theta + J(t-t') + \delta\theta(J,\theta,t,t')}} \tag{5}$$

where the perturbed orbits or characteristics are given by

$$\frac{d\delta J}{dt'} = -\frac{\partial \tilde{H}}{\partial \delta\theta}$$

$$\frac{d\delta\theta}{dt'} = \delta J + \frac{\partial \tilde{H}}{\partial \delta J}, \tag{6}$$

with boundary conditions $\delta\theta = \delta J = 0$ at $t' = t$. Equations (3), (5) and (6) (which are equivalent to the Eulerian form (2)) together with the self-consistency condition, form a closed system. The RHS of Eq. (3) is a turbulent collision operator, which is zero—determining f_0—in the steady state. Generally we will consider f_0 to be slowly varying, consistent with $J_c/J_s \ll 1$, and ignore Eq. (3). To the same approximation the term $\frac{\partial}{\partial J} < \frac{\partial \tilde{H}}{\partial \theta}\tilde{f} >$ in Eq. (5) may be neglected. Our considerations are then reduced to Eqs. (5) and (6).

Equation (5) expresses the Eulerian perturbation $\tilde{f}$ in terms of the perturbed Hamiltonian, $\tilde{H}$, also Eulerian, and the characteristics δJ, $\delta\theta$. The characteristics are essentially a Lagrangian representation of the perturbations, $\tilde{H}$, determined by integration of (6). Thus Eqs. (5) and (6) form a mixed Eulerian–Lagrangian representation.

In certain applications it is convenient to write Eq. (5) in terms of a propagator $\tilde{G}$,

$$\tilde{f}(J,\theta,t) = \int_{-\infty}^{t} dt' \int dJ'\, d\theta'\, \tilde{G}(J,\theta,t;J',\theta',t') \left(\frac{\partial \hat{H}}{\partial \theta'}\frac{\partial f_0}{\partial J'} - \frac{\partial}{\partial J'} < \frac{\partial \hat{H}}{\partial \theta}\hat{f} > \right). \tag{7}$$

The propagator can be expressed directly in terms of the Lagrangian characteristics, $\tilde{G}(J,\theta,t;J',\theta',t') = \delta(J' - J - \delta J(J,\theta,t,t'))\delta(\theta' - \theta - J(t-t') - \delta\theta(J,\theta,t,t'))$, which implies the integral form,

$$\tilde{G}(J,\theta,t;J',\theta',t') = \frac{1}{(2\pi)^2}\int dk\, dm\, \exp[ik(J'-J) - ik\delta J + im(\theta - \theta') - imJ(t-t') - im\delta\theta]. \tag{8}$$

Alternately, $\tilde{G}$ can be determined from the differential equation

$$(\frac{\partial}{\partial t} + \frac{\partial H}{\partial J}\frac{\partial}{\partial \theta} - \frac{\partial \hat{H}}{\partial \theta}\frac{\partial}{\partial J})\hat{G}(J,\theta,t;J',\theta',t) = 0, \tag{9}$$

with boundary condition $\hat{G}(J,\theta,t;J',\theta',t) = \delta(J-J')\delta(t-t')$.

We define all statistical averages in terms of the Liouville density and, accordingly, regard f as the Klimontovich distribution. This is completely general. It encompasses the more restricted statistical averages, and can be utilized directly in the two-point, two-time theory for the calculation of fluctuations. All functions now depend parametrically on the particle variables J_i, θ_i, at time t in addition to the usual independent variables J, θ. For performing the statistical averages the notation can be abbreviated by writing $f = f(\underline{X}t; \underline{X}_1, \underline{X}_2, \cdots \underline{X}_N)$, where $\underline{X} = (J,\theta)$ is the phase space variable and $\underline{X}_i = (J_i, \theta_i)$ is the parametric variable corresponding to the particle phase space position at time t. The Liouville density, Γ, depends on the particle variables, $\Gamma = \Gamma(\underline{X}_1, \underline{X}_2, \cdots \underline{X}_N; t)$, and gives the joint probability that, at time t, particle 1 is at $\underline{X}_1$, particle 2 at $\underline{X}_2$, etc. The statistical average of any function F is given by

$$\{F\} = \overline{F}(\underline{X},t) = \int d\underline{X}_1 d\underline{X}_2 \cdots d\underline{X}_N\, F(\underline{X},t;\underline{X}_1,\underline{X}_2,\cdots \underline{X}_N)\,\Gamma(\underline{X}_1,\underline{X}_2,\cdots \underline{X}_N;t). \tag{10}$$

This is the complete formal statistical average, and does not restrict $\overline{F}$ in any way. In particular, $\overline{F}$ may be an arbitrarily complex function of J, θ. Formulated this way there is no distinction, statistically, between the various scales of F.

When the physics on the various scales differ (or in some cases when the physics is independent of scale [5]), it is desireable to have a statistical procedure which recognizes the different scales. To this end we decompose the perturbation, $\tilde{H}$, as follows

$$\tilde{H}(\underline{X}) = \sum_{\underline{k}=\underline{k}_1}^{\underline{k}_m} H_{\underline{k}}\, e^{i\underline{k}\cdot\underline{X}} + \tilde{H}_0,$$

where $\tilde{H}_0$ is the fine scale (microscale) fluctuation, corresponding to $|\underline{k}| > |\underline{k}_m|$. The identity,

$$1 = \int dH_{\underline{k}_1} dH_{\underline{k}_2} \cdots dH_{\underline{k}_m} \prod_{\underline{k}=\underline{k}_1}^{\underline{k}_m} \delta(H_{\underline{k}} - \int d\underline{X}\, e^{-i\underline{k}\cdot\underline{X}} \int d\underline{X}' K(\underline{X},\underline{X}') \times f(\underline{X}',t;\underline{X}_1,\underline{X}_2\cdots \underline{X}_N)),$$

is then inserted in the integrand of (10) and, upon reversing the integrations, the average becomes

$$\{F\} = \int dH_{\underline{k}_1} dH_{\underline{k}_2} \cdots dH_{\underline{k}_m} \int d\underline{X}_1 d\underline{X}_2 \cdots d\underline{X}_N\, F(\underline{X},t;\underline{X}_1 \cdots \underline{X}_N)$$

$$\times \Gamma(\underline{X}_1,\cdots \underline{X}_N,t) \prod_{\underline{k}=\underline{k}_1}^{\underline{k}_M} \delta(H_{\underline{k}} - \int d\underline{X}\, d\underline{X}'\, e^{-i\underline{k}\cdot\underline{X}} K(\underline{X},\underline{X}') f(\underline{X}',t;\underline{X}_1,\cdots \underline{X}_N)).$$

The integration over the particle coordinates in (11) is now restricted by the constraint that the fluctuations from $\underline{k}_1$ to $\underline{k}_m$ are all fixed. This is the microscale average. Denoting this by $[\,]_0$, Eq. (11) can be written,

$$\{F\} = \int dH_{\underline{k}_1}\, dH_{\underline{k}_2} \cdots dH_{\underline{k}_m}\, [F]_0(\underline{X}, t; H_{\underline{k}_1} \cdots H_{\underline{k}_m}) \Gamma'(H_{\underline{k}_1}, \cdots H_{\underline{k}_m}; t); \tag{12}$$

where the remainder of the ensemble is parametrized by the Fourier coefficients, and Γ' is their associated probability density. Now, we can repeat the procedure and average over some subset of the $H_{\underline{k}}$'s, fixing the remainder, and so on. This will define a hierarchy of scales and associated averages, so that $\{F\}$ can be written

$$\{F\} = \left[\cdots \left[[F]_0\right]_1 \cdots\right]_n. \tag{13}$$

The complete average is now performed as a sequence over ever increasing scales, where the long scale fluctuations are fixed while the average over the fine scale is performed. In the present paper, we will distinguish only two scales, so that, $\{F\} = \left[[F_0]\right]_1$.

In many cases, this sequential averaging procdure is of very little advantage. For example, suppose there is only one relevant scale and the fine scale average, $[\,]_0$, has a negligible effect. Then $[F]_0$ will depend sensitively on the perturbations $H_{\underline{k}}$ and the coarse scale average, $[\,]_1$, will be just as difficult as the full average, $\{\,\}$.

The utility of the decomposition, (13), is evident in problems with disparate scales. Then the fine scale average, $[\,]_0$, can eliminate significant information and simplify the long scale dynamics. Thus, in the stochasticity problem of present concern, the average propagator $[\tilde{G}]_0$ depends on the $H_{\underline{k}}$ only through D, Eq. (22), which is essentially constant over the realizations of interest, and the averages $[]_0$ and $[]_1$ can be done independently. This simplifies enormously the calculation of the long scale fluctuations.

The present paper considers only the first step, the average $[\,]_0$, for a stochastic regime, and as such does not require the sequential averaging procedure. The discussion here is intended to show the connection to the larger picture, and to demonstrate that since the long scale fluctuations are rigorously fixed, this step in the calculation can be performed before requiring self-consistency on the long scale.

III. NORMAL STOCHASTIC APPROXIMATION: HEURISTIC THEORY

In a typical plasma turbulence problem, the source of the turbulence is instability, localized in $\underline{k}$, ω space (m, n in our model). There may be many modes or m and n values present, but the fractional spectral width $\Delta m/m\ \Delta n/n$ is generally small, so that there is a single scale characterizing the Eulerian perturbations $\tilde{H}$. Eulerian theories, like the DIA [6,7], are designed for a single scale and tacitly assume that the Lagrangian Perturbations have the same scale. However, the defining characteristic of stochasticity is the exponentiation of neighboring trajectories. In this case, the perturbed orbit function, $\delta\theta$, is both *secular* with respect to $t - t'$ and *unstable* with regard to the initial conditions, θ, J. Not only do the deviations from the unperturbed orbits grow in time, but initially neighboring particles suffer radically different large perturbations. In fact, after a time τ_c, the correlation time, a very small element, $m\Delta\theta \ll 2\pi$, gets mapped to cover the full range 2π. This has been called the *mixing* property by Zaslavskii and Chirikov [8]. Its implications for the Lagrangian orbit function $\delta\theta$ are shown in Fig. 1. A plot like Fig. 1 for larger τ would show larger excursions in the amplitude of $\delta\theta$ *and* more rapid fluctuations as a function of θ. In short, it is a consequence of stochasticity that the Lagrangian perturbations develop much finer spatial scales than their Eulerian counterparts. We will exploit this disparity of scales with the microscale average to develop the Normal Stochastic Approximation.

To be more explicit, we use the form (1) for the perturbation and compute the m'th Fourier component of Eq. (5). There results

$$f_{m'}(J,t) = \sum_{m,n} e^{-i\omega_{mn}t} \int_0^\infty d\tau\, e^{i(\omega_{mn}-mJ)\tau} < e^{i(m-m')\theta + im\delta\theta(J,\theta;\tau)} im\, H_{mn}(J+\delta J) > . \tag{14}$$

We focus on values of m and m' typical of the normal modes. That is, we assume the spectrum of fluctuations is dominated by the normal modes, and compute here only these Fourier components. Of course, the stochastic mapping produces much higher m, presumably incoherent fluctuations, but these belong to the microscale and require a separate treatment. We also assume that all of phase space (J,θ) is stochastic, without isolated integrable curves. In this case $\delta\theta$ is secular with respect to time and the exponent cannot be expanded as a perturbation series. Rather, we evaluate (14) asymptotically using the properties of $\delta\theta$ implied by stochasticity. Furthermore, for simplicity, we treat the case of constant H_{mn} (the results are readily generalized) so that we must evaluate

$$A_{mm'} = < \exp(i(m-m')\theta + im\delta\theta(J,\theta,\tau)) > . \tag{15}$$

For small τ, $\delta\theta \ll 1$, and $A_{mm'} \simeq \delta_{m,m'}$. For large τ/τ_c, the θ dependence of $\delta\theta$ is the dominant variation in the integrand, and $A_{mm'}$ can be computed as a series of integrals over small intervals $2\pi/M$. Thus, letting $\theta_j = 2\pi j/M$,

$$A_{mm'} = \sum_{j=1}^{M} \frac{1}{2\pi} \int_0^{2\pi/M} d\theta\, \exp[i(m-m')(\theta_j+\theta) + im\delta\theta(\theta_j+\theta,\tau)] \tag{16}$$

$$= \sum_{j=1}^{M} \frac{e^{i(m-m')\theta_j}}{2\pi} \int_0^{2\pi/M} d\theta\, e^{im\delta\theta(\theta_j+\theta,\tau)}[1 + i(m-m')\theta + \cdots].$$

The essential feature of stochasticity is that an M can be chosen such that $\delta\theta$ undergoes many oscillations with respect to θ in period $2\pi/M$ but $(m-m')/M$ is small. Thus, the integral above will approach an average, independent of θ_j, and the sum gives a Kronecker delta, yielding,

$$A_{mm'} \simeq \delta_{mm'} < e^{im\delta\theta} >, \tag{17}$$

with an error of order $\big((m-m')/M\big) < e^{im\delta\theta} >$. For stochastic orbits, M increases with τ, such that $M \to \infty$, so that (17) is asymptotically exact in the limit of large τ/τ_c. The disparity between the wavelength and the spatial scale of the Lagrangian orbit functions is the key element here. By restricting m values, however, a separate microscale average was not required.

The basic assertion of the Normal Stochastic Approximation can now be stated. The microscale average, $A_{mm'} \to [A_{mm'}]_0$, is equivalent to averaging with $\delta\theta$ as a normally distributed random variable. With δJ a diffusion or Wiener process $[\delta J^2]_0 = 2D\tau$, $\delta\theta$ is the integrated Wiener process or $[\delta\theta^2]_0 = \frac{2}{3}D\tau^3$, and therefore

$$[A_{mm'}]_0 = \delta_{m,m'}[< e^{im\delta\theta} >]_0 = \delta_{m,m'}e^{-\frac{1}{2}m^2[\delta\theta^2]_0} = \delta_{m,m'}e^{-\frac{1}{3}m^2D\tau^3}. \tag{18}$$

More formally, the relation between the moments, $[\delta\theta^n]_0$ of $\delta\theta$, is normal,

$$[\delta\theta^{2p}]_0 = \frac{(2p)!}{p!2p}[\delta\theta^2]_0^p.$$

We now show how the stochastic properties of the orbits lead to this result. The perturbed orbit function is expanded in a series with τ and J dependent coefficients

$$\delta\theta(J,\theta;\tau) = \sum_{\ell}\theta_\ell(J,\tau)e^{i\ell\theta}. \tag{19}$$

Since $\delta\theta$ has zero mean, the $\ell = 0$ term is not present. Using (19) yields,

$$< e^{im\delta\theta} >= \sum_{n=0}^{\infty}\frac{(im)^n}{n!}\frac{1}{2\pi}\int < d\theta(\sum_{\ell_1}\delta_{\ell_1}e^{i\ell_1\theta})\cdots(\sum_{\ell_n}\theta_{\ell_n}e^{i\ell_n\theta}). \tag{20}$$

The θ integral yields a selection rule for the ℓ sums, $\ell_1+\ell_2+\cdots+\ell_n = 0$. The cumulant expansion results by assuming a hierarchy where the dominant contribution arises from combining the ℓ_n in pairs, etc. Whether or not a hierarchy exists, Eq. (20) can always be expressed in terms of this rearrangement. Thus the pairwise combinations, requiring $n = 2p$, can be made $(2p)!/p!2^p$ ways, and this leads to

$$\begin{aligned} < e^{im\delta\theta} >_{(2)} &= \sum_{p=0}^{\infty}\frac{(-m^2)^p}{(2p)!}\frac{(2p)!}{p!2^p}(\sum_{\ell_1}|\theta_{\ell_1}|^2)\cdots(\sum_{\ell_p}|\theta_{\ell_p}|^2) \\ &= \exp(-\frac{1}{2}m^2 < \delta\theta^2 >), \end{aligned} \tag{21}$$

where $<\delta\theta^2>=\sum_\ell |\theta_\ell|^2$. The pairwise combinations are equivalent to a normal distribution.

The next order may be computed in a similar manner in terms of the triple moment

$$<\delta\theta^3>\equiv\sum_{\ell_1,\ell_2}\theta_{\ell_1}\theta_{\ell_2}\theta_{-\ell_1-\ell_2}.$$

Here, however, as in all the higher cumulants, there is dependence on the phases of the Fourier coefficients, which implies sensitivity of other variables, like J and τ. The type of behavior indicated in Fig. (1) also occurs with respect to J in a stochastic situation. This implies rapid oscillatory behavior of the triplet (and higher cumulants) with respect to J. Thus integration over J tends to annihilate the higher order cumulants, leaving only the pairs. It is this phase independence of the pair combinations which makes the normal distribution dominant. The essential point about stochasticity is that the scale of oscillations in J decreases rapidly with τ, so that the *microscale* interval goes to zero for $\tau\gg\tau_c$.

The foregoing is basically a duplication of the central limit theorem, in a physical context. There is, in effect, a random phase principle acting when stochasticity is present. We emphasize that it is the phases of the Fourier coefficients of the perturbed orbit functions or the Lagrangian representation, *not the Eulerian wave phases*, which are relevant here.

It remains to compute the diffusion coefficient from the orbit equations (6), by using the microscale average,

$$D=[\delta J^2]_0/2t=\frac{1}{2t}\int_0^t dt_1\int_0^t dt_2[\tilde{H}(\theta+Jt_1+\delta\theta(t_1),t_1)\cdot\tilde{H}(\theta+Jt_2+\delta\theta(t_2),t_2)]_0.$$

Substituting for $\tilde{H}$, letting $t_2=t_1-\tau$, and, to order $\tau_c/t\ll 1$, extending the τ integral to infinity this becomes,

$$D=\frac{1}{t}\int_0^t dt_1\int_0^\infty d\tau\sum_{\substack{m,n\\ m',n'}}H_{mn}H_{m'n'}(-1)mm'\exp(i(m+m')\theta$$

$$+i((m+m')J-(\omega_{mn}+\omega_{m'n'})t_1+i(\omega_{m'n'}-mJ)\tau)$$

$$\times[\exp(im\,\delta\theta(t_1)+im\,\delta\theta(t_1-\tau))]_0.$$

To order τ_c/t, we can assume $t_1\gg\tau_c$, then, provided τ_c^{-1} exceeds the frequency separation of modes—which is essentially the Chirikov overlap condition—the t_1 oscillations reduce the double sum to the diagonal $m'=-m$, $n'=-n$, leaving

$$D=\frac{1}{t}\int_0^t dt_1\int_0^\infty d\tau\sum_{m,n}m^2|H_{mn}|^2e^{i(mJ-\omega_{mn})\tau}[\exp(im\delta\theta'(\tau))]_0,$$

where $\delta\theta'(\tau) = \delta\theta(t_1) - \delta\theta(t_1 - \tau)$. This is the usual characteristic but with fields evaluated at position $J = J + \delta J(t_1), \theta = \theta + Jt_1 + \delta\theta(t_1)$ for $\tau = 0$. The exponential factor is then the same form as above, and we get

$$D = \sum_{m,n} m^2 |H_{mn}|^2 \int_0^\infty d\tau \exp[i(mJ - \omega_{mn})\tau - \frac{1}{3}m^2 D\tau^3], \quad (22)$$

or the usual Fokker-Planck form with a broadened resonance instead of $\pi\delta(mJ - \omega_{mn})$.

To summarize, we have formulated the theory from the mixed Eulerian-Lagrangian representation, Eqs. (5) and (6), together with a microscale average. In a stochastic regime, the Lagrangian orbit functions, δJ, $\delta\theta$, tend to become normally distributed under the microscale average, with variances, $[\delta J^2]_0 = 2D\tau$, $[\delta J\, \delta\theta]_0 = D\tau^2$, $[\delta\theta^2]_0 = \frac{2}{3}D\tau^3$. The Normal Stochastic Approximation (for this model) can then be succinctly expressed as the rule

$$[\tilde{H}(\theta + Jt + \delta\theta(t))]_0 = \int d\delta\theta \frac{\exp(-\frac{\delta\theta^2}{\frac{4}{3}Dt^3})}{\sqrt{4\pi Dt^3/3}} \tilde{H}(\theta + Jt + \delta\theta). \quad (23)$$

This can be readily generalized to include δJ variations. Using rule (23), the Eulerian responses in Eq. (5) can be immediately averaged to obtain Eq. (18).

The Normal Stochastic Approximation is the leading order in an expansion for the distribution of Lagrangian orbit perturbations. It does not depend on the smallness of the Eulerian field amplitudes, H_{mn}, which must be *above* some threshold to give stochasticity. The small parameter is kurtosis, of the distribution, measuring nearness to normal. This contrasts the DIA which maximizes normality of the Eulerian field fluctuations, $\tilde{H}$, and as shown below, cannot properly account for the secularities, scale disparity, and distribution of the Lagrangian fluctuations which characterize stochasticity.

IV. FORMAL THEORY: COMPARISON WITH DIA

We now use a common framework within which both the DIA and the Normal Stochastic Approximation can be simply expressed and reduced to a few explicit assumptions. The object of present concern is the propagator, $\tilde{G}$, and the different forms proposed for the ensemble average of it. Of course the averaging procedures differ markedly between the two theories. In the DIA the complete statistical average is performed at the outset and one attempts to truncate the chain of statistically averaged equations. In the Normal Stochastic Approximation, the statistical averages are perforcmd sequentially, to utilize the disparity of scales, and become an essential part of the argument justifying the approximation. For comparing the different results for the propagator in the present section, however, the statistical distinctions are secondary and will not be emphasized. We consider the DIA first.

The starting point is Eq. (9) for the propagator. For $\hat{H}$ independent of J, and defining $\tilde{E}(\theta, t) = -\partial\hat{H}/\partial\theta$, this becomes,

$$(\frac{\partial}{\partial t} + J\frac{\partial}{\partial\theta} + \hat{E}(\theta, t)\frac{\partial}{\partial J})\hat{G}(J, \theta, t; J', \theta', t') = 0, \quad (24)$$

with $\hat{G}(J, \theta, t; J', \theta', t) = \delta(J - J')\delta(\theta - \theta')$. The procedure of the DIA, according to Orzag and Kraichman [6], is as follows. The propagator is developed as a powers series in the Eulerian fluctuation amplitude $\hat{E}$. This is done by using the unperturbed propagator, G_0, obeying $(\frac{\partial}{\partial t} + J\frac{\partial}{\partial\theta})G_0 = 0$, to construct the integral equation

$$\tilde{G}(J, \theta, t; J', \theta', t') = \int_{t'}^{t} dt'' \int dJ''\, d\theta''\, G_0(J, \theta, t; J'', \theta'', t'') \times \tilde{E}(\theta'', t'')\frac{\partial}{\partial J''}\tilde{G}(J'', \theta'', t''; J', \theta', t'), \quad (25)$$

which is then iterated in the obvious way. The average is then performed, retaining only pair correlations*, and the reduced series resumed to give $\{\tilde{G}\} = G^{\mathrm{DIA}}$. This can be done simply with diagrammatic methods and leads to an integral equation for the DIA propagator

$$(\frac{\partial}{\partial t} + J\frac{\partial}{\partial\theta})G^{\mathrm{DIA}}(J, \theta, t; J', \theta', t') - \frac{\partial}{\partial J}\int_{t'}^{t} dt''\, dJ''\, d\theta''\, G^{\mathrm{DIA}}(J, \theta, t; J'', \theta'', t'')\{\tilde{E}(\theta, t)\hat{E}(\theta'', t'')\}$$
$$\times \frac{\partial}{\partial J''}G^{\mathrm{DIA}}(J'', \theta'', t''; J', \theta', t') = 0. \quad (26)$$

Thus, the DIA makes two assumptions. First, the perturbation expansion of $\tilde{G}$ in powers of the Eulerian amplitude $\tilde{E}$ is assumed to converge. Second, a statistical assumption, is that pair correlations of the $\hat{E}$ dominate, or in other words, that the Eulerian fluctuations are normally distributed. While the second assumption is plausible, the convergence assumption is crucial and highly dubious in a stochastic regime. Since the propagator is just a formal expression of the orbit this is equivalent to assuming convergence of the orbit expansion in powers of the Eulerian fields. But the resonant behavior underlying stochasticity is *not* analytic, even for an isolated resonance. For sufficiently small amplitude, well below the stochasticity threshold, the phase space volume occupied by the resonances is small, and the perturbation expansion converges in the remaining, dominant, volume of phase space. Here the DIA would presumably be appropriate. For this reason, even though certain select terms are summed to all orders, the DIA is inherently a perturbative theory. Furthermore, the physical effect represented by the non-linear (broadening) term in Eq. (26), correspond to the third order, two wave, one particle, perturbative process of Compton scattering.

Equation (26) can be reduced to a diffusion form, commonly used in applications [9] where the operand of G^{DIA} depends only on θ. In this case, one needs only $\overline{G}^{\mathrm{DIA}}(\theta, t; \theta', t'; J) = \int dJ'\, G^{\mathrm{DIA}}(J, \theta, t; J', \theta', t')$. Noting that G^{DIA} depends rapidly on the difference variable, $J - J'$, but weakly on the sum, $\frac{1}{2}(J + J')$,

*Actually certain *crossing diagrams* are also neglected in the resummation.

integration over J' in Eq. (26) gives

$$\left(\frac{\partial}{\partial t} + J\frac{\partial}{\partial \theta}\right)\overline{G}^{\rm DIA}(J) = \frac{\partial}{\partial J}\int_{t'}^{t} dt''\, d\theta''\, d\delta J'\, G^{\rm DIA}(J, J'')\{\hat{E}\hat{E}\}$$
$$\times \frac{\partial}{\partial J''}\, G^{\rm DIA}(J, J'' + \delta J')$$
$$= \frac{\partial}{\partial J}\int_{t'}^{t} dt''\, d\theta''\, d\delta J''\, G^{\rm DIA}(J, J + \delta J'')\{\hat{E}\hat{E}\}$$
$$\times \frac{\partial}{\partial J}\,\overline{G}^{\rm DIA}(J + \delta J'')$$
$$= \frac{\partial}{\partial J}\int_{t'}^{t} dt''\, d\theta''\,\overline{G}^{\rm DIA}(\theta, t; \theta'', t''; J)\{\tilde{E}(\theta, t)\tilde{E}(\theta'', t'')\}$$
$$\times \frac{\partial}{\partial J}\,\overline{G}^{\rm DIA}(\theta'', t''; \theta', t'; J).$$

This can be transformed in t and θ, assuming a quasistationary fluctuation spectrum and causal propagators to finally give

$$-i(\omega - mJ)\overline{G}^{\rm DIA}_{m,\omega}(J) - \frac{\partial}{\partial J}\sum_{m',\omega'}\overline{G}^{\rm DIA}_{m-m',\omega-\omega'}(J)\{|E|^2\}_{m',\omega'}\frac{\partial}{\partial J}\,\overline{G}^{\rm DIA}_{m,\omega}(J) = 1. \quad (27)$$

Equation (27) is in the form of a diffusion equation for $\overline{G}^{\rm DIA}_{m,\omega}$, where the diffusion coefficient,

$$D_{m,\omega} = \sum_{m',\omega'}\overline{G}^{\rm DIA}_{m-m',\omega-\omega'}\{|E|^2\}_{m',\omega'}, \quad (28)$$

depends on the frequency and wave number. Apart from the broadening of the resonance function $\overline{G}$, Eq. (28) has the form of diffusion resulting from Compton scattering in the standard weak turbulence theory. It corresponds to diffusion arising from the beat wave resonance, $\omega - \omega' = (m - m')J$. The resonance width is not of practical importance for the evaluation of $D_{m,\omega}$. The resonance function is invariably narrow (in m', ω') compared to the spectrum, $\{|E|^2\}_{m',\omega'}$, so that one may take $\overline{G}^{\rm DIA}_{m-m',\omega-\omega'}(J) \simeq \pi\delta[\omega - \omega' - (m - m')J]$ in Eq. (28), and pass from the sum over m' to an integral. Treatment of the sum as an integral is legitimized by the resonance width of $\overline{G}^{\rm DIA}$.

The DIA departs from the standard weak turbulence theory by including $D_{m,\omega}$ in the equation for the propagator, and accordingly deleting some of the standard terms appearing in the equations for the one particle distribution function and the spectrum. This is the familiar *renormalization*, basically a rearrangement of the perturbation series. In a strict sense it requires that the series converge. If the series does not converge, the procedure may still be valid in an asymptotic sense, but becomes largely an article of faith. Some aspects of the renormalization are then more reasonable than others. For example, the elimination of certain terms from the standard perturbation theory remains plausible because it does not require infinite summation. This part of the procedure was the main concern in the renormalization of Quantum Field Theory, where certain divergent diagrams had to be eliminated. In the Quantum theory the deleted diagrams never appear explicitly, but, when summed to all orders, serve to redefine the mass and charge. One can never observe, separately, the renormalized terms. By contrast in the DIA, these terms—requiring infinite summation—give an explicit form for the broadening of the propagator. This form is the issue of the present paper and observable in our numerical experiments.

The Normal Stochastic Approximation begins with the integral form for the propagator, Eq. (8). This is averaged over the microscale directly assuming normality of the Lagrangian fluctuations. We use the result that $[\exp(ik\delta J + im\delta\theta)]_0 = \exp(-\frac{1}{2}k^2[\delta J^2]_0 - km[\delta J\,\delta\theta]_0 - \frac{1}{2}m^2[\delta\theta^2]_0)$ for normal δJ and $\delta\theta$. This is shown as in Sec. III, by writing the series, averaging and resumming. Note, however, that no convergence assumption is required since the exponential function is analytic. This then gives, from Eq. (8),

$$G^{\mathrm{NSA}}(J,\theta,t';J',\theta',t') = \frac{1}{(2\pi)^2}\int dk\,dm\,\exp(ik(J-J') + im(\theta-\theta') + imJ(t-t') - \frac{1}{2}k^2[\delta J^2]_0 - km[\delta J\,\delta\theta]_0 - \frac{1}{2}m^2[\delta\theta^2]_0). \quad (29)$$

The variance $[\delta J^2]_0$, $[\delta\theta^2]_0$ and $[\delta J\,\delta\theta]_0$ can be computed under the same approximation, as indicated above, and are independent of J and θ. Thus, integration over J' gives

$$\overline{G}^{\mathrm{NSA}}(\theta,t;\theta',t';J) = \frac{1}{2\pi}\int dm\exp(im(\theta-\theta') - imJ(t-t') - \frac{1}{2}m^2[\delta\theta^2]_0).$$

Finally, this can be transformed, and using the expresison for $[\delta\theta^2]_0$ from Eqs. (18) and (22), is equivalent to

$$-i(\omega - mJ)\overline{G}^{\mathrm{NSA}}_{m,\omega}(J) - \frac{\partial}{\partial J}\sum_{m',\omega'}\overline{G}^{\mathrm{NSA}}_{m',\omega'}(J)[|E|^2_{m',\omega'}]_0\frac{\partial}{\partial J}\overline{G}^{\mathrm{NSA}}_{m,\omega}(J) = 1. \quad (30)$$

This is to be compared with Eq. (27) for the DIA. It is a diffusion equation with $D = \sum_{m',\omega'}\overline{G}^{\mathrm{NSA}}_{m',\omega'}(J)[|E|^2_{m',\omega'}]_0$ independent of frequency and wave number, corresponding to diffusion arising from the primary wave–particle resonance $\omega' = m'J$. Thus the two approximations differ in physical content on the processes causing diffusion in the propagator.

To summarize the DIA assumes normality of the Eulerian field fluctuations and convergence of the series expansion for the propagator in powers of the Eulerian amplitudes. The latter assumption is untenable in a stochastic regime. The Normal Stochastic Approximation assumes normality of the Lagrangian fluctuations and does not require a convergence assumption. In practical terms the two theories predict different forms for the broadening of the propagator. The regimes of expected validity are also distinct. Thus the DIA is basically an amplitude expansion, presumably valid below the stochasticity threshold. The Normal Stochastic Approximation is essentially a multiple scale expansion valid for amplitudes above the stochasticity threshold.

V. NUMERICAL EXPERIMENT

From a spectrum of known fields, chosen to be in a stochastic regime, one can numerically compute a family of orbits to generate the Lagrangian fluctuations $\delta\theta, \delta J$, and evaluate the expression $I_{mn'}$ of Eq. (15). This is compared to the theoretical predictions of this quantity from the Normal Stochastic Approximation and the DIA.

Parameters are chosen to replicate the electron response for the drift wave. The m's correspond to poloidal mode numbers, typically in the several hundred range. The actual mode frequency for the drift wave is small, and $\omega_{mn} \equiv nq$. Resonances then occur at the action locations, $J = nq/m$. The m spectrum ranges from $M - \Delta m/2$ to $M + \Delta m/2$, whereas the n spectrum is sufficiently large that boundaries are not encountered in the simulation. To estimate the resonance spacing we take q order unity, consider the region near $J = 1$, and take the n values from $M - \Delta m/2$ to $M + \Delta m/2$ which fill this region. There are then Δm^2 resonances from $J = 1 - \Delta m/M$ to $J = 1 + \Delta m/M$ for an *average* resonance spacing of $\Delta J_R \simeq 1/M\Delta m$. The island width of the individual resonances is $\Delta J_I \simeq \sqrt{H_{mn}}$. A crude etimate of the island overlap criterion is then

$$\left(\frac{\Delta J_I}{\Delta J_R}\right)^4 \simeq |H_{mn}|^2 m^4 \Delta m^4 > 1. \tag{31}$$

The diffusion coefficient can be estimated (see below) as $D \simeq |H_{mn}|^2 M^2 \Delta m$, so the correlation time is expressed as

$$\tau_c^{-3} \simeq |H_{mn}|^2 M^4 \Delta m. \tag{32}$$

The value M has been chosen from practical considerations. Parameters H_{mn} and Δm are determined from Eqs. (31) and (32) by requiring strong inequality in (31) and $\tau_c > 1$ in (32) so that the decay in time of $< \exp(im\delta\theta) >$ is observable on the unit time scale of our model. These two conditions require $\Delta m > 1$, so that spectral width is important.

These considerations can become invalid near low order resonances ($\frac{m}{n}$ approximating a low order ration or integer) where the resonances are not evenly spaced but cluster together and leave gaps. For example near $J = 1$ and $q = 1$, the resonances cluster in groups of Δm modes, the groups being spaced by $1/m$ in J but within the cluster resonance spacing is $1/m^2$. Then, if $\Delta m/m \ll 1$, there will obviously be large gaps. To avoid this difficulty, we have used an irrational J (Clearly q can be scaled out of the problem so we take $q = 1$, without loss of generality). In addition the resonances were tabulated numerically and compared to the island widths to be sure that no such gaps occurred.

Although the integration of Eqs. (6) is straightforward, it can be time consuming numerically because of the accuracy required to resolve the fine scale structure. Many of the relevant orbit properties can be studied using a simplified model where Eqs. (6) reduce to a simple mapping. The model is obtained by making the further simplification of an infinite n spectrum with constant amplitude and phase. Specifically, we take

$$\hat{H}(J,\theta,t) = \sum_{m=M_1}^{M_2} \sum_{n=-\infty}^{+\infty} H_m \cos[(mJ - n)t + m\theta + \varphi_m].$$

The identity, $\sum_{n=-\omega}^{+\infty} e^{inqt} = 2\pi \sum_{\ell=-\infty}^{+\infty} \delta(t - 2\pi\ell)$, is used to express the force as a series of impulses, Eqs. (6) are then integrated to give the mapping

$$\Delta J_{\ell+1} = \delta J_\ell + 2\pi \sum_m mH_m \sin[2\pi\ell mJ + m(\theta + \delta\theta_\ell) + \varphi_m] \tag{33}$$

$$\delta\theta_{\ell+1} = \delta\theta_\ell + 2\pi\delta J_\ell,$$

where the ℓth step occurs at time $2\pi\ell$. The orbits were computed using both the mapping, Eqs. (33), and differential form of Eqs. (6). Although the results are virtually indistinguishable and in good agreement with the Normal Stochastic Approximation, the mapping is somewhat degenerate in that Eqs. (22) and (28) become identical. The differential form (6) is necessary to resolve the differences with the DIA discussed in the preceding section.

With the mapping, orbits for 3000 particles distributed uniformly in θ at $J = 1 + 12/\pi$ are computed. For these experiments, the m spectrum extended form 250 to 500, with the amplitudes constant at $H_m = 10^{-8}$ and phases, φ_m, chosen at random, but, of course, fixed throught the integration.

With the differential form, orbits were computed for 8400 particles distributed uniformly in θ at $J = 1.4$. (The same results were obtained with 25600 particles). The m spectrum ranged from 90 to 111, while the n spectrum ran from 132 to 153 with amplitudes $H_{mn} = 3 \times 10^{-5}$ and phases, φ_{mn}, also random. Particles remained within the resonance region for several correlation times. The inevitable computer truncation errors play a natural coarse-graining role as the orbits become excessively fine scaled.

The results are displayed in Figs. (2)-(6). For comparison purposes the data are plotted in normalized form relative to the diffusion coefficient, D, evaluated from Eq. (22), for each model. The broadened resonance allows the n sum to be done as an integral since it is wide on the scale of the resonance spacing but narrow compared to the spectral width in n. There is an additional factor of $\frac{1}{2}$ owing to the cosine form used in the model. Thus

$$D = \frac{\pi}{2} \sum_{m=M+1}^{M-2} m^2 H^2_{m,mJ/q} \simeq \frac{\pi}{6} H^2_{m,mJ/q}(M_2^3 - M_1^3). \tag{34}$$

The observed variances, $< \delta J^2 > /2D$ and $< \delta\theta^2 > 3/2D$, plotted in Figs. (2) and (3) show the expected t and t^3 time dependencies respectively, for both the mapping and differential models. Actually, there is a slight discrepancy in the short time behavior, $t < \tau_c$, of $< \delta\theta^2 >$ for the differential model. It shows up more clearly in a plot of $(< \delta\theta^2 > 3/2D)^{1/2}$ where the initial slope is somewhat larger than one. This may be due to the relatively small number of modes in the differential model since the short time behavior is dependent more on the random initial phases than the randomness in the Lagrangian phases when stochasticity develops for $t > \tau_c$.

A typical correlation function, $I_{mn} =< \exp(im\delta\theta) >$, where n is the toroidal mode multiplied, is shown in Fig. (4). These exhibit the same behavior for both models and all m. They decay down to noise with roughly a t^3 exponent, while the off diagonal terms $A_{mm'}$, for $m \neq m'$, (not shown), start at zero and build up to noise. The noise is produced by the finite number, N, of particles, and no smoothing has been done to remove it. By varying N we have verified that the level scales like $N^{-1/2}$ as consistent with white noise.

The characteristic form for the correlation function in both theories is the same, namely,

$$I_{mn} = \exp(-\frac{1}{3} m^2 D_{mn} \tau^3).$$

The degree to which this describes the observations is indicated in Fig. (5), (for the differential model).

The theories differ in the forms claimed for D_{mn}. The Normal Stochastic Approximation gives $D_{mn} = D$, or simply the diffusion coefficient, independent of m and n, whereas the DIA has D_{mn} as given by Eq. (20) with $\omega = nq$. These predictions are compared by determing the observed D_{mn}, fitting the form (35) to the numerical data using least squares to compute the slope as indicated in Fig. (15). The results are shown in Fig. (6) where D_{mn}/D is plotted as a function of m, using the parameters of the differential equation model.

The observations show clearly that D_{mn} is independent of m, consistent with the Normal Stochastic Approximation. The quantitative agreement is good although there appears to be a slight systematic error. We interpret this as an error in the variance $< \delta\theta^2 >$ as noted earlier and not a deviation from normal in the distribution of δJ. This interpretation was confirmed by directly measuring the Kurtosis, $<< \delta J^4 >> \equiv (< \delta J^4 > -3 < \delta J^2 >^2)/3 < \delta J^2 >^2$, which was found to be small, of order .1, independent of time. This measurement can be improved by reducing the noise level (increasing the number of particles) so that the short time behavior contibutes less to the value of the fitted slope. In the present experiments the noise level was reached after a few correlation times so that discrepancies for $\tau < \tau_c$ (which are expected) effect the determination of D_{mn}.

In contrast, the DIA predictions have some characteristic features that are noticeably absent in the data. First, there is the dependence of the correlation function on n and the related resonance function dependence on the frequency difference $\omega_{mn} - \omega_{m'n'}$. These are natural features of the Eulerian representation; (arising from the beat wave resonance) but difficult to understand in the Lagrangian picture, where a secular dependence of $\delta\theta$ on t, in Eq. (14), would be implied. This would be a rather obvious feature in the numerics and nothing of the kind was observed. Second, even for fixed n, there is a strong m dependence of D_{mn} in the DIA, and this is not observed either. The magnitude of this discrepancy is illustrated in Fig. (5) where D_{mn}/D has been plotted versus m for $n = 130$. It should be emphasized that D_{mn} is observed in an exponent. In fact, the DIA would predict no observable decay of I_{mn} during the course of the experiment for most values of m.

To summarize, the diffusion coefficient determined from the observed variances, $< \delta\theta^2 >$, $< \delta J^2 >$ and the correlation functions, I_{mn}, was the same in all cases with the value, Eq. (34), as predicted by the Normal Stochastic Approximation. The discrepancies with the DIA are evident, amounting to orders of magnitude in a measured exponent. These observations strongly support the view that the physical process responsible for the resonance broadening in a stochastic regime is diffusion, not Compton scattering.

VI. CONCLUSION

The main points of this paper follow from an examination of the implications of intrinsic stochasticity for collisionless turbulence theory. Most of the consequences stem from stochasticity's most striking property, namely the pathological, complex, fine scale, phase space structures that can result from relatively simple, long scale, perturbations. The fine scale structures result from integrating the long scale perturbations along a particle orbit and thus suggest, at least in part, a Lagrangian representation. The fine scale structures cannot be obtained by a perturbative treatment of the orbit (implicit in any perturbative Eulerian theory). This casts doubt on the applicability of theories like the DIA, for a stochastic system. The basic features that distinguish a stochastic system are thus the inherent nonanalyticity of the orbits and the inevitable multiplicity of scales.

The theory described in this paper exploits these features by using a Lagrangian representation and a statistical procedure of averaging sequentially over increasing scales. The dynamics of the long scale fluctuations (associated with the waves or instabilities) can be simplified by averaging over the fine scales to give, in effect, a diffusing orbit response. We have termed this limit the Normal Stochastic Approximation. It is essentially a multiple scale theory, appropriate for amplitudes above the stochasticity threshold. Below the stochasticity threshold one would expect Eulerian amplitude expansion theories, like the DIA, to be valid. Not surprisingly, the two theories give different predictions with different underlying physical causes.

A qualifying comment is in order here. There is actually a third regime at still higher amplitude when the individual islands overlap the entire resonant region. One then gets a slowly modulated single island, no evident stochasticity [8]. Trapping and diffusion phenomenon appear together without one being clearly dominant. This regime was observed numerically for a different model but not, as yet, studied extensively.

The situation is somewhat reminiscent of that in fluid turbulence, where the DIA has had its most notable success. At moderate Reynold's number, where the relevant phenomenon are limited to a single scale, the Eulerian DIA works very well. However, as the Reynold's number gets larger, the scales multiply and discrepancies appear. The DIA does not give the Kolmogorov, $k^{-5/3}$ spectrum for intertial range turbulence [10]. The difficulty is that the Eulerian DIA treats all scales identically. Any interacting triplet of fluctuations irreversibly creates or destroys correlations. The theory does not allow for the small scales to be convected without being destroyed by the large scales as required for the Kolmogorov spectrum.

Attempts to remedy these deficiencies have utilized a Lagrangian representation [11], and physical arguments to treat the large scales differently [12]. Fluid turbulence is, of course, quite different from the kinetic theory turbulence problem addressed in this paper. The comparison here is intended simply to underscore the limits of the DIA for dealing with problems involving disparate scales. We are echoing Kraichnan's seventeen year old refrain [11] by making a plea for Lagrangian theories.

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Figure Captions

<u>Fig. 1.</u> Characteristic orbit functions in stochastic case: For $t - t' > \tau_c$, where τ_c is the correlation time and approximate Kolmogorov entropy. For large $t - t'$, the oscillations have larger amplitude and finer scale.

<u>Fig. 2 & 3.</u> Observed variances for the mapping (o) and differential (x) models compared to the theoretical value (solid line).

<u>Fig. 4.</u> Typical correlation function I_{mn} versus time showing characteristic decay to noise.

<u>Fig. 5.</u> Plot of $(\ln |I_{mn}|)^{1/3}$ to display characteristic time dependence of correlation function. The linear fit to the initial decay, from which D_{mn} is determined, is also shown. The form of Eq. (35) is seen to be a reasonable description of the decay until the noise level is reached.

<u>Fig. 6.</u> Comparison of the decay parameter, D_{mn} in the correlation function. The predictions of the DIA are plotted for fixed $n = 130$.

Fig. 1

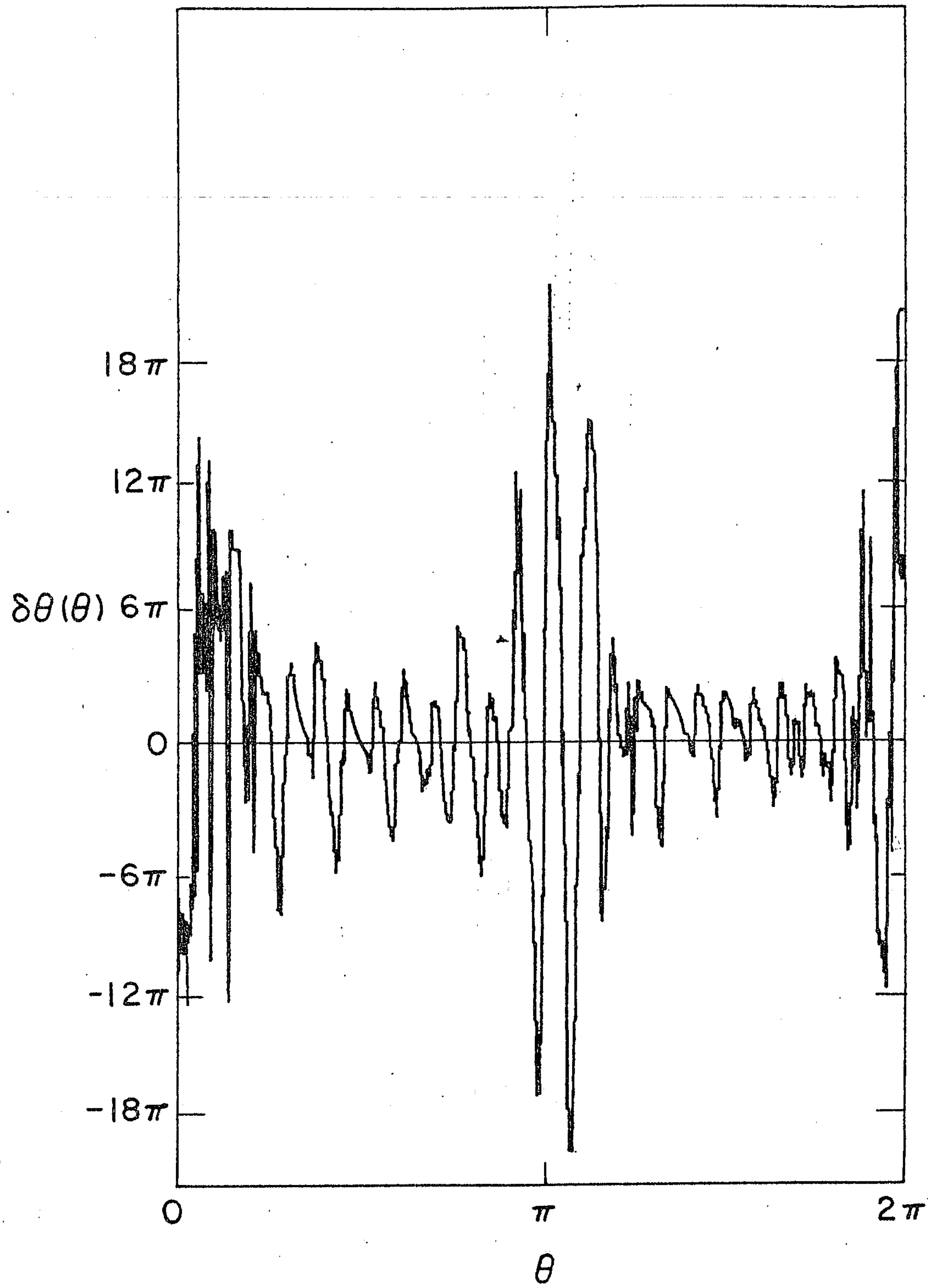

Fig. 2

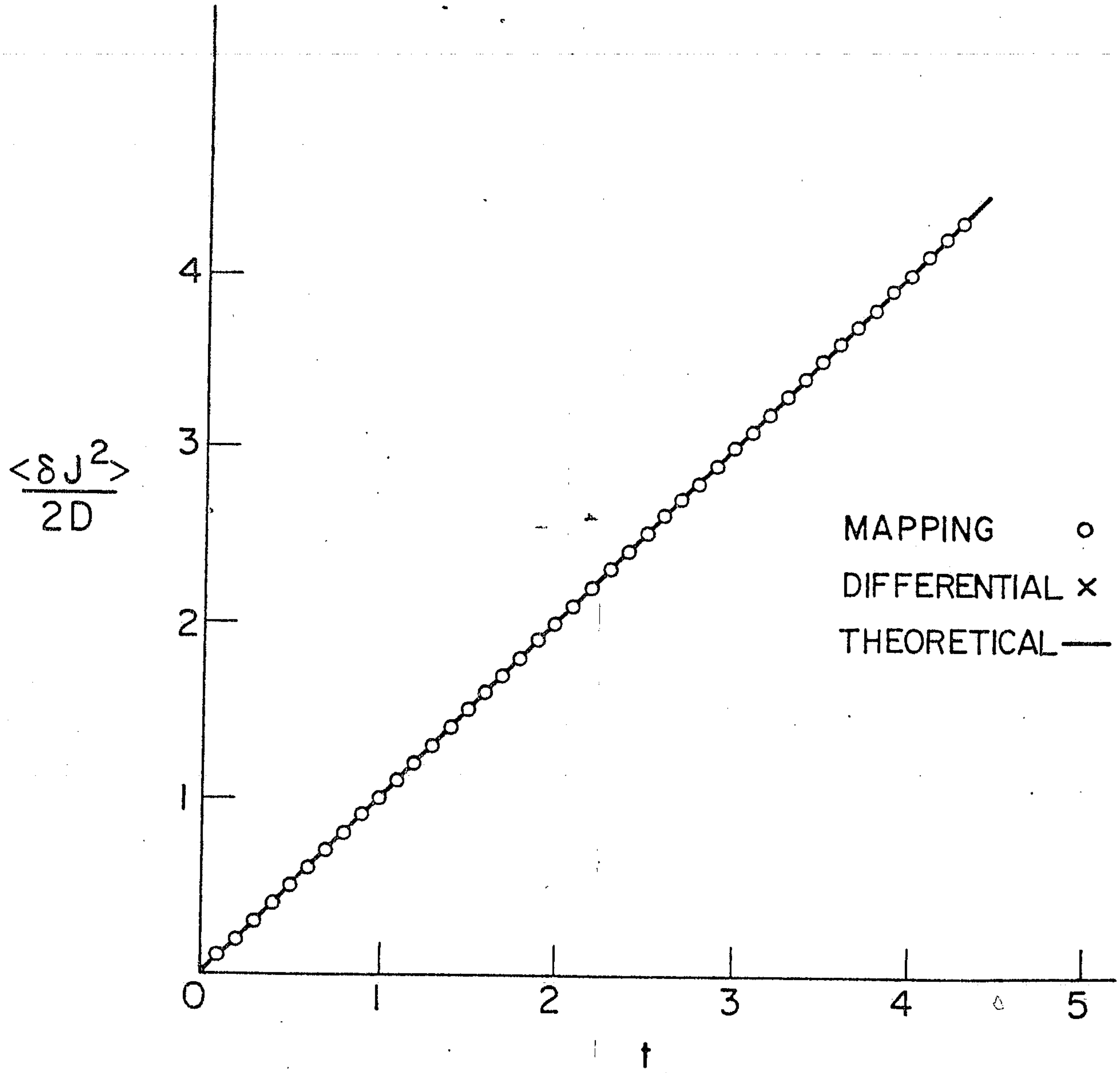

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Fig. 3

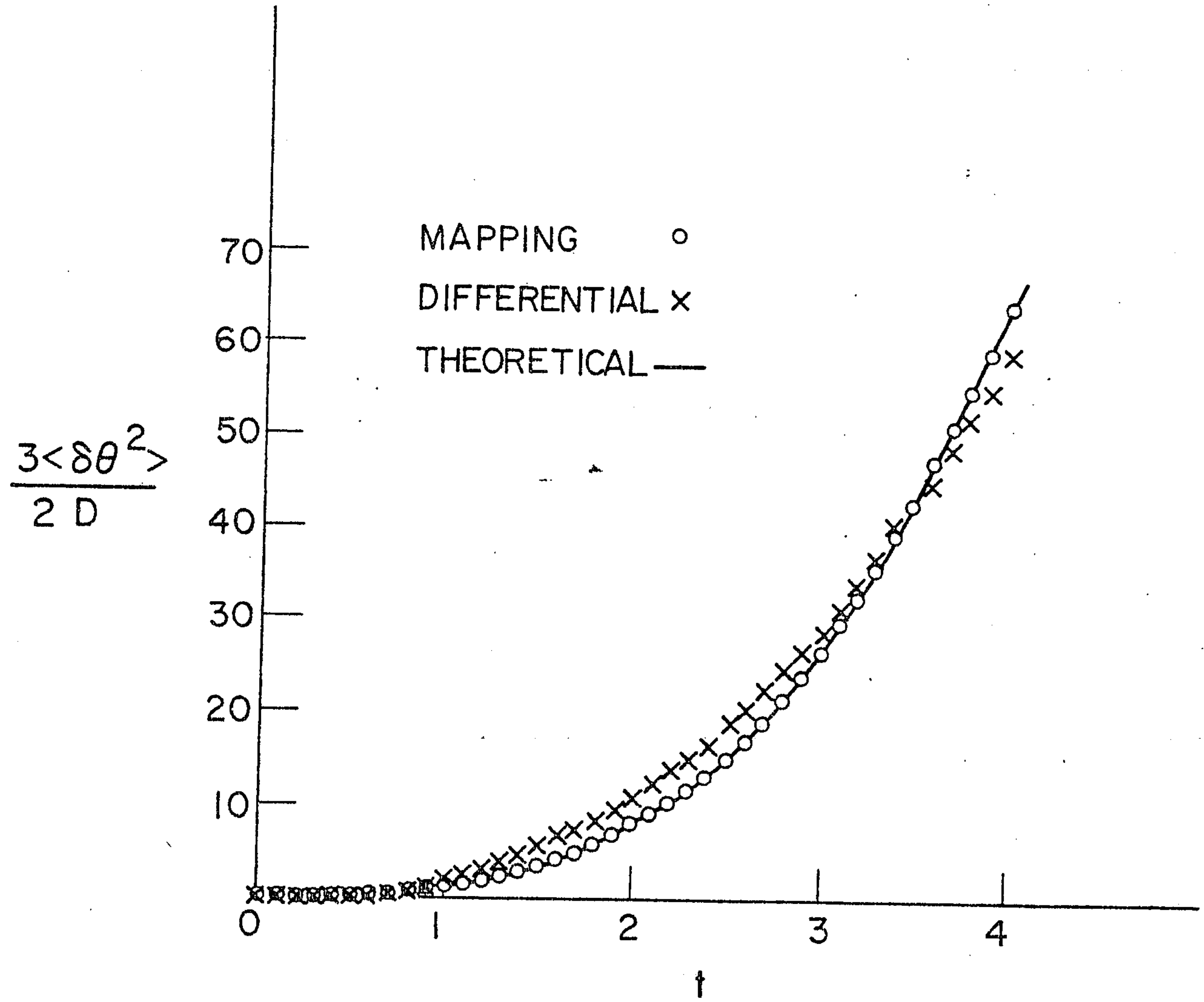
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Fig. 4

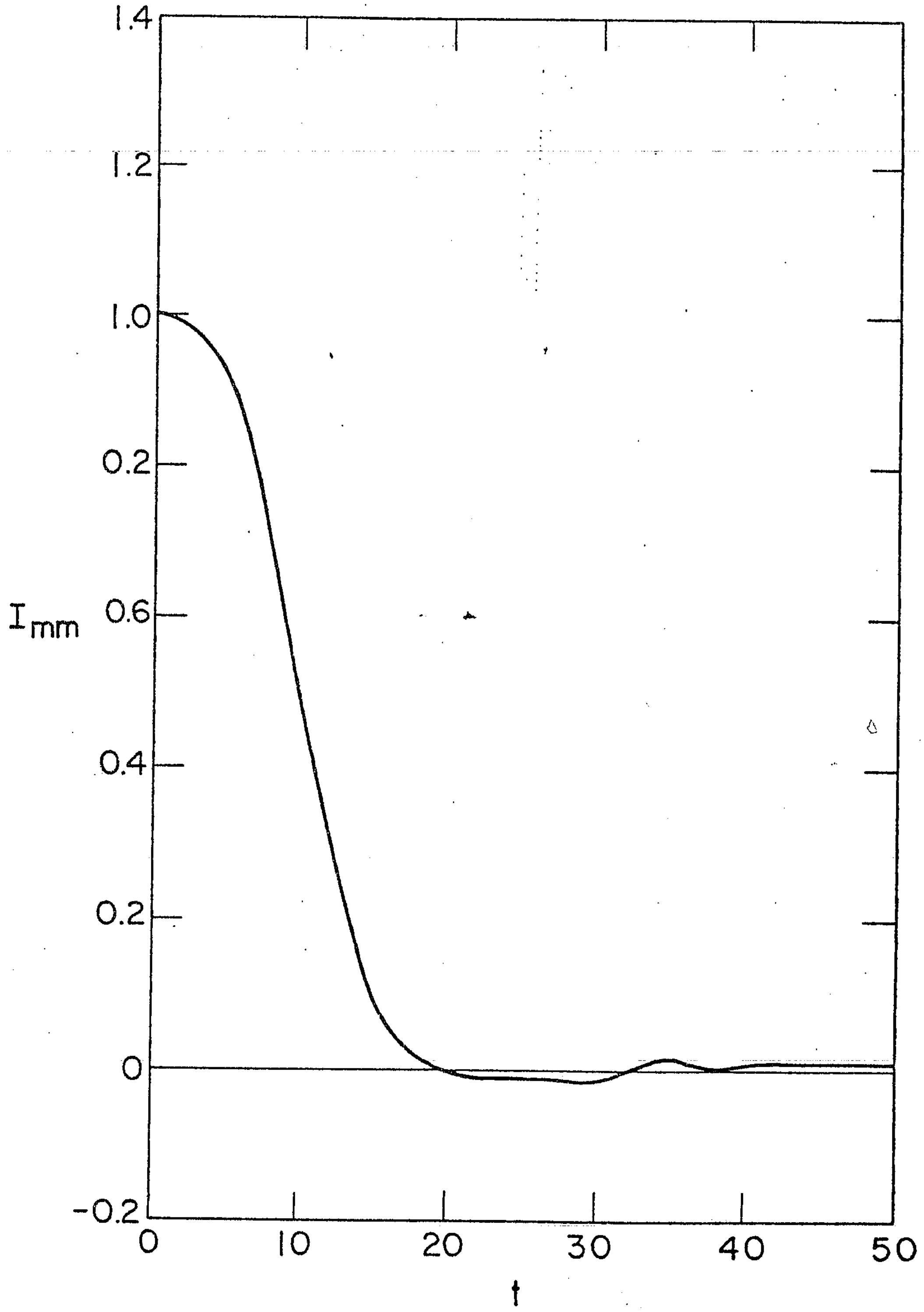


Fig. 5

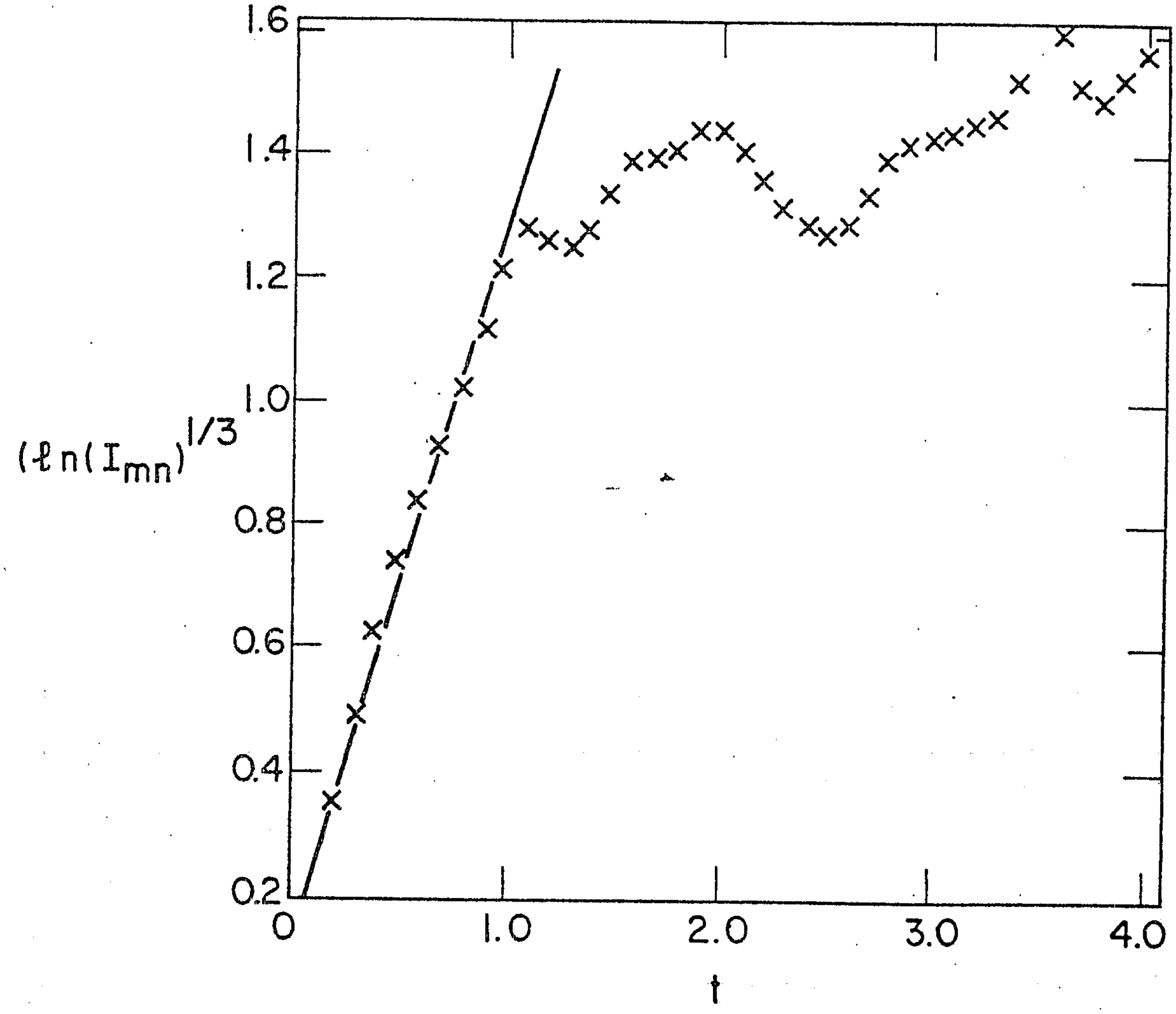
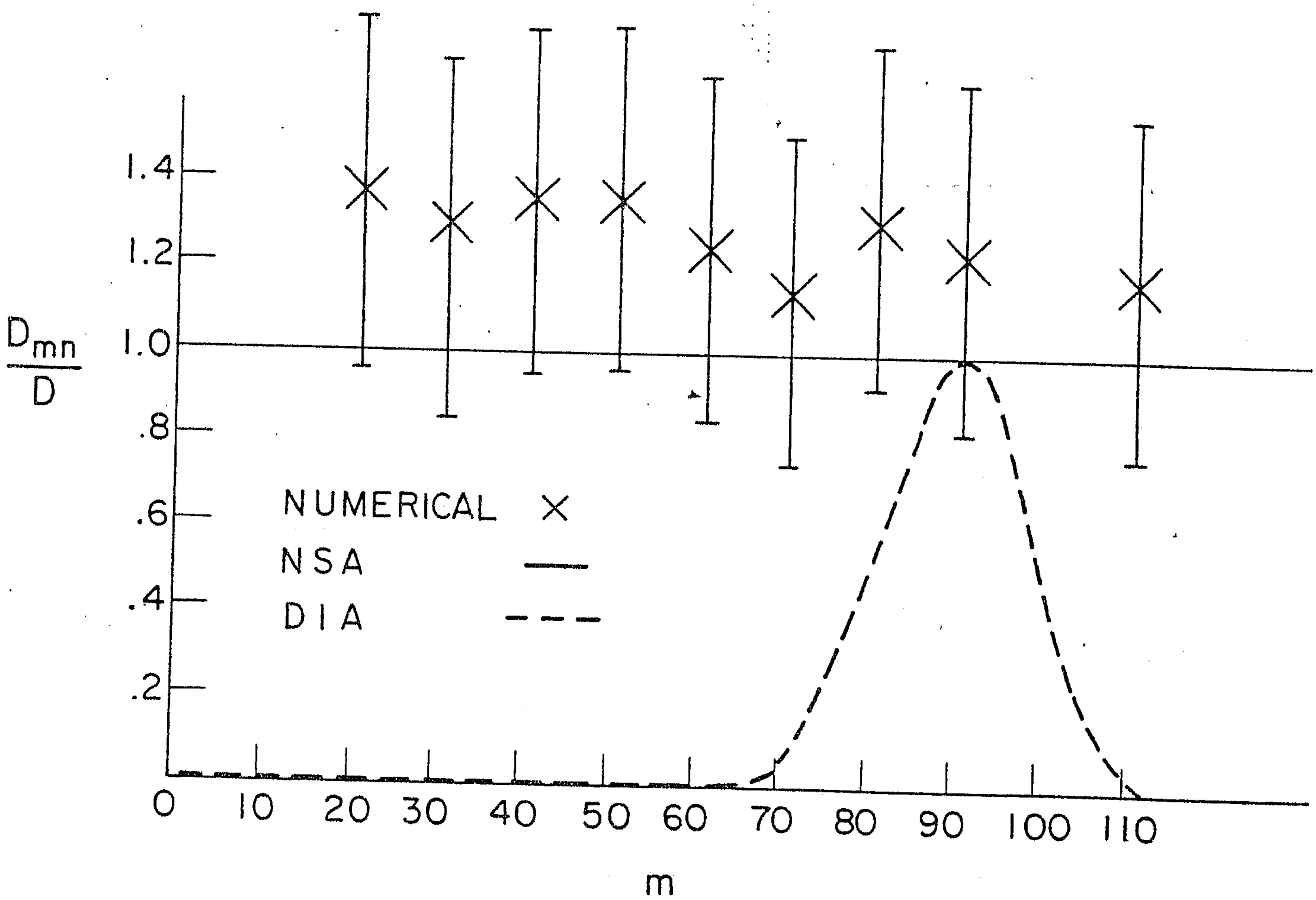


Fig. 6



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