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Wave Energy Flow Conservation for Propagation in Inhomogeneous Vlasov-Maxwell Equilibria

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Abstract

Wave energy flow conservation is demonstrated for Hermitian differential operators that arise in the Vlasov-Maxwell theory for propagation perpendicular to a magnetic field. The energy flow can be related to the bilinear concomitant, for a solution and its complex conjugate, by using the Lagrange identity of the operator. This bilinear form obeys a conservation law and is shown to describe the usual WKB energy flow for asymptotically homogeneous regions. The additivity and uniqueness of the energy flow expression is discussed for a general superposition of waves with real and complex wave numbers. Furthermore, a global energy conservation theorem is demonstrated for an inhomogeneity in one-dimension and generalized reflection and transmission coefficients are thereby obtained.

I. Introduction

In the Vlasov-Maxwell theory wave energy conservation is commonly associated with Hermitian wave operators. Such operators arise, when particle resonances are not present or neglected, e.g., for wave propagation perpendicular to the magnetic field. In the present paper we wish to examine the conservation of wave energy flow for Hermitian operators. Such operators satisfy relations known as Lagrange identities. The purpose is to show how the general structure of these identities determines both the conservation law and the asymptotic properties of the energy flow.

Lagrange identities are well known from the theory of ordinary differential equations[1] and in an integrated form are familiar as Green's formula in potential theory.[2] They establish a relation between an operator, its adjoint and a boundary form, which is commonly called the bilinear concomitant. More generally, Lagrange identities can be derived for vector systems of differential equations of arbitrary order, such as those that occur frequently in the Vlasov-Maxwell theory.[3–4] For these vector systems explicit Lagrange identities can be obtained, where the concomitant is in general, only uniquely determined up to the curl of a bilinear vector field. We note in comparison that in general, the electromagnetic energy flow in a vacuum can only be identified as the Poynting vector up to a curl, but this is sufficient to identify the total energy flow out of a closed surface.

The concomitant of a Hermitian operator is related to a conserved current of the system, which has previously been used to express energy flow conservation for simple systems.[5–6] Here we wish to emphasize the general structure of these relations and present a discussion independent of the actual form of the operator. An important issue in this context is the identification of the concomitant expression with the physical wave energy flow. We find that this can be achieved asymptotically (i.e., in the boundary regions where the system is spatially homogeneous), and for general vector systems one obtains a straightforward derivation of the WKB energy flow expression from the Lagrange identity. With the relationship between the concomitant and the energy flow one can demonstrate a global energy conservation theorem for these Hermitian systems. For solutions corresponding to general radiative boundary conditions we discuss the uniqueness and the additivity (i.e., sum over normal mode energies) of the asymptotic energy flow and obtain generalized reflection and transmission coefficients that satisfy global energy conservation in one dimension.

While the concomitant conservation follows most immediately from the Lagrange identity, it may be instructive to discuss an alternative derivation from a version of Noether's theorem.[7] For Hermitian operators one can find variational forms which play a role analogous to the action integral in field theories. However, while the most common action integrals depend only on the fields and their first derivatives, the present functionals include derivatives up to an arbitrarily high order. It is this generalization that leads to the concomitant expression. From the symmetries of the variational form one obtains in a straightforward way conservation laws. These are well known for certain wave Lagrangians describing wave propagation for weakly inhomogeneous media.[8] Using the Lagrange identity one can show that energy flow conservation follows generally for arbitrary inhomogeneities from a gauge invariance. Furthermore, translational invariance for time independent and homogeneous media yields energy and momentum-like conserved quantities that can be related to the concomitant expression.

II. Exact Conservation Relations

In the Vlasov-Maxwell theory the propagation of waves with frequency ω and electric field $\mathbf{E}(\mathbf{x})e^{-i\omega t}$ is governed by vector systems,

$$\boldsymbol{L} \cdot \mathbf{E} = 0, \tag{1}$$

where the operator $\boldsymbol{L}$ in general represents an integral operator, and under not very restrictive approximations can be approximated by differential operators of arbitrarily high order. In the following we will assume that the operator $\boldsymbol{L}$ satisfies a Lagrange identity and from this structure discuss some consequences for the wave energy flow in inhomogeneous media. The Lagrange identity for $\boldsymbol{L}$ and its adjoint $\boldsymbol{L}^*$ is a relation of the form,

$$\phi^* \cdot (\boldsymbol{L} \cdot \psi) - (\boldsymbol{L}^\dagger \cdot \phi)^* \cdot \psi = i\nabla \cdot \mathbf{J}(\phi^*, \psi) \tag{2}$$

that holds for arbitrary vector fields ϕ and ψ. The vector $i\mathbf{J}(\phi^*, \psi)$ is bilinear with respect to the arguments and is known as the concomitant. The star denotes the complex conjugate and the factor i has been inserted for convenience. In the Appendix we derive Lagrange identities for vector systems of differential equations of arbitrary order. The concomitant here is in general only determined up to the curl of a bilinear vector field, $\nabla \times \mathbf{b}(\phi^*, \psi)$. This ambiguity however is unimportant for one-dimensional problems and for global conservation relations as discussed in Sec. III.

From the Lagrange identity (2) there follows by Gauss' theorem the Green's formula,

$$\int_V d^3\mathbf{x} \left[\phi^* \cdot (\boldsymbol{L} \cdot \psi) - (\boldsymbol{L}^\dagger \cdot \phi)^* \cdot \psi\right] = -\int_{\partial V} d\mathbf{A} \cdot \mathbf{J}(\phi^*, \psi), \tag{3}$$

for any volume V with surface ∂V. If the whole plasma is contained in V and ϕ, ψ are subject to boundary conditions where the surface term vanishes, Eq. (3) expresses the general definition of the adjoint operator $\boldsymbol{L}^\dagger$.

Let us now choose ψ to be a solution of Eq. (1) and ϕ to satisfy the adjoint equation,

$$\boldsymbol{L}^\dagger \cdot \phi = 0. \tag{4}$$

Then, according to Eq. (2), $\mathbf{J}(\phi^*, \psi)$ represents a conserved current, whose divergence is zero. This relation becomes especially useful for Hermitian operators, where $\boldsymbol{L} = \boldsymbol{L}^\dagger$, and $\phi = \psi$ is a solution of the same physical system. The relation

$$\nabla \cdot \mathbf{J}(\psi^*, \psi) = 0 \tag{5}$$

becomes a conservation law for any solution and its complex conjugate. We will see that $\mathbf{J}(\psi^*, \psi)$ can be identified asymptotically with the energy flux of the system. For non-Hermitian operators we define the energy flux by an expression of the same form, however, with $\boldsymbol{L}$ replaced by its Hermitian part $\boldsymbol{L}_H$,

$$\nabla \cdot \mathbf{J}_H(\psi^*, \psi) = -i \left[\psi^* \cdot (\boldsymbol{L}_H \cdot \psi) - (\boldsymbol{L}_H \cdot \psi)^* \cdot \psi\right]. \tag{6}$$

Writing $\boldsymbol{L} = \boldsymbol{L}_H + i\boldsymbol{L}_A$ with an anti-Hermitian part $i\boldsymbol{L}_A$ and choosing ψ again as a solution of Eq. (1), one obtains from Eq. (6),

$$\nabla \cdot \mathbf{J}_H(\psi^*, \psi) = -\left[\psi^* \cdot \boldsymbol{L}_A \cdot \psi + (\boldsymbol{L}_A \cdot \psi)^* \cdot \psi\right]. \tag{7}$$

The right-hand side no longer vanishes, and thus describes the dissipation of energy associated with $\boldsymbol{L}_A$.

III. Global Energy Conservation

We now evaluate the concomitant expression for Hermitian operators asymptotically as the system approaches infinity, where spatial homogeneity is assumed. In such a region the well-known expression for the WKB wave energy flow is obtained.[9] More generally, the uniqueness and the additivity of the asymptotic energy flow for a general superposition of waves, including propagating and evanescent waves, is discussed. For an inhomogeneity in one spatial dimension, a global energy conservation theorem is demonstrated and generalized reflection and transmission coefficients are obtained.

Let us consider an arbitrary inhomogeneous medium in a finite volume that is surrounded by spatially homogeneous regions. Integrating Eq. (5) over a volume whose surface ∂V lies entirely in the homogeneous medium, one obtains by Gauss' theorem a global conservation law for the asymptotic flux,

$$\int_{\partial V} d\mathbf{A} \cdot \mathbf{J}(\psi^*, \psi) = 0. \tag{8}$$

We now show that $\mathbf{J}(\psi^*, \psi)$ represents asymptotically the wave energy flow, so that Eq. (8) becomes a statement of global energy conservation.

Asymptotically, the solutions of Eq. (1) can be taken as a superposition of plane waves,

$$\psi = \hat{\psi} = e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{9}$$

For a homogeneous medium, the form of Eq. (9) is an exact solution and the operator equation (1) becomes an algebraic system,

$$\boldsymbol{D}(\mathbf{k}) \cdot \psi = 0. \tag{10}$$

Nontrivial solutions of (10) require

$$\Delta(\mathbf{k}) = \det |D_{ij}(\mathbf{k})| = 0,$$

a solubility condition that determines the possible values of one component of $\mathbf{k}$ in terms of the remaining components and the frequency ω. We also have to consider the adjoint system (4). In the Appendix it is shown that its corresponding algebraic form is given by,

$$\phi = \hat{\phi} e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\boldsymbol{D}^{\dagger}(\mathbf{k}^*) \cdot \phi = 0$$

$$\Delta^*(\mathbf{k}^*) = \det |D^{\dagger}_{ij}(\mathbf{k}^*)| = 0, \tag{11}$$

where the adjoint matrix is defined as $D^{\dagger}_{ij}(\mathbf{k}) = D^*_{ji}(\mathbf{k})$. For Hermitian operators Eqs. (10) and (11) are identical which implies

$$\boldsymbol{D}(\mathbf{k}) = \boldsymbol{D}^{\dagger}(\mathbf{k}^*). \tag{12}$$

With this relation one has $\Delta^*(\mathbf{k}^*) = \Delta(\mathbf{k})$ and consequently the possible wavevectors for Hermitian systems occur as complex-conjugate pairs, $\mathbf{k}$ and $\mathbf{k}^*$.

Let us now assume a general asymptotic solution,

$$\psi = \sum_n C_n \psi_n \tag{13}$$

with complex coefficients C_n and plane waves ψ_n with corresponding wavnumbers $\mathbf{k}_n$. The asymptotic expression for the bilinear concomitant then takes the form

$$\mathbf{J}(\psi^*, \psi) = \sum_{n,m} C^*_n C_m \mathbf{J}_{nm} \tag{14}$$

with $\mathbf{J}_{nm} \equiv \mathbf{J}(\psi^*_n, \psi_m)$. In the double sum (14) there occur spatially constant terms when $\mathbf{k}_m = \mathbf{k}^*_n$ and spatially periodic terms when $\mathbf{k}_m \neq \mathbf{k}^*_n$. We denote the solution (9) that depends on $\mathbf{k}^*_n$ by ψ_{n^*} and the constant terms in the expansion (14) that occur for complex conjugate wave vectors by $\mathbf{J}_m \equiv \mathbf{J}(\psi^*_{m^*}, \psi_m)$. The Lagrange identity determines the form of the vectors $\mathbf{J}_m$ in the following way. Setting in Eq. (2) $\phi = \psi_n$ and $\psi = \psi_m$, and using Eqs. (10)-(12) yields

$$\psi^*_n \cdot \boldsymbol{D}(\mathbf{k}_m) \cdot \psi_n - \left[\boldsymbol{D}^{\dagger}(\mathbf{k}^*_n) \cdot \psi_n\right]^* \cdot \psi_m =$$

$$\psi^*_n \cdot \left[\boldsymbol{D}(\mathbf{k}_m) - \boldsymbol{D}(\mathbf{k}^*_n)\right] \cdot \psi_m = (\mathbf{k}^*_n - \mathbf{k}_m) \cdot \mathbf{J}_{nm}. \tag{15}$$

In the limit where $\mathbf{k}^*_n \to \mathbf{k}_m$ one obtains from Eq. (15),

$$\mathbf{J}_m = -(\psi_{m^*})^* \cdot \left[\frac{\partial}{\partial \mathbf{k}} \boldsymbol{D}(\mathbf{k}_m)\right] \cdot \psi_m, \tag{16}$$

where the dots denote the contraction of the fields with the matrix $\boldsymbol{D}$.

Some basic properties of the asymptotic wave energy flow follow immediately from Eqs. (13)-(16). First we note, that the concomitant (14) can actually be identified asymptotically with the wave energy flow. For a single plane wave with a real wave vector the solution (13) consists of only one term and the concomitant assumes the form given by Eq. (16), in agreement with previous expressions for the wave energy flow in homogeneous media.[8,9] More generally, Eq. (16) applies also to complex wave vectors $\mathbf{k}_m$. In this case a contribution to the energy flow requires the presence of both ψ_m and ψ_{m^*} in the asymptotic solution (13). For usual boundary conditions only evanescent waves are allowed, which correspond either to ψ_m or to ψ_{m^*}. Thus there is no contribution from these waves. We note however that solutions with complex conjugate wave vectors can occur in tunneling regions of inhomogeneous media.

For a general superposition of waves the sum in Eq. (14) contains also spatially oscillating interference terms $\mathbf{J}_{nm}$ with $\mathbf{k}_m \neq \mathbf{k}_n^*$. According to Eq. (15) these terms are restricted by an orthogonality constraint, $(\mathbf{k}_n^* - \mathbf{k}_m) \cdot \mathbf{J}_{nm} = 0$, but are otherwise not determined by the Lagrange identity. As previously mentioned, the concomitant expression is in general only uniquely defined up to the curl of an arbitrary bilinear vector field $\mathbf{b}(\psi^*, \psi)$. This quantity has the asymptotic form,

$$\nabla \times \mathbf{b}(\psi^*, \psi) = -\sum_{n,m}(\mathbf{k}_n^* - \mathbf{k}_m) \times \mathbf{b}_{nm} \tag{17}$$

where $\mathbf{b}_{nm}$ is defined as $\mathbf{J}_{nm}$. This expression is zero for $\mathbf{k}_m = \mathbf{k}_n^*$ and otherwise spatially oscillating. In general, the ambiguity of the asymptotic concomitant expression can be removed by taking the spatial average in Eq. (14).

We now consider the case of complex conjugate wave vectors, $\mathbf{k}_m = \mathbf{k}_n^*$ and $\mathbf{k}_m$, Eq. (17) represents the common expression for the wave energy flow. For complex wave vectors $\mathbf{k}_m$,

To be more specific we now discuss the global energy conservation theorem for an inhomogeneity in one dimension. For simplicity of notation we denote by k and J the vector components along the inhomogeneity direction and assume the same transverse wave vector for all waves. The possible values k_n of k are given by the roots of the local dispersion relation. In this case the oscillating terms in Eq. (14) yield $(k_n^* - k_m)J_{nm} = 0$ and therefore the form

$$J(\psi^*, \psi) = \sum_{\substack{n \\ k_n \text{ real}}} |C_n|^2 J_n \tag{18}$$

and is additive with respect to the propagating waves with real wavenumbers. This additivity can be used to define reflection and transmission coefficients for the individual waves. For this purpose we write the energy flow of the wave with wavenumbers k_n in the common form,

$$\begin{aligned} J_n &= v_n W_n \\ v_n &= -\Delta_{,k} / \Delta_{,\omega} \\ W_n &= \psi_n^* \cdot \frac{\partial}{\partial \omega} \boldsymbol{D} \cdot \psi_n \end{aligned} \tag{19}$$

where v_n is the corresponding group velocity and W_n the wave energy. At $x \to -\infty$ we assume an incident wave (i) with group velocity $v_i > 0$ and reflected waves (r) with $v_{nr} < 0$. The waves at $x \to +\infty$ are taken as transmitted waves (t) with group velocities $v_{nt} > 0$. Then the global conservation relation (8) can be written in the form

$$J_i + \sum_n J_{nr} = \sum_m J_{mt}. \tag{20}$$

Dividing by the incident flow J_i and defining reflection coefficients $R_n = |J_{nr}/J_i|$ and transmission coefficients $T_m = |J_{mt}/J_i|$ one obtains

$$\sum_n \sigma_n R_n + \sum_m \sigma_m T_m = 1. \tag{21}$$

Here σ_n is the sign of the ratio of the corresponding wave energies. It can therefore be negative if the system supports negative energy waves.

IV. Wave Action, Energy and Momentum Conservation

We now discuss a version of Noether's theorem and relations for wave action, energy and momentum in space and time-dependent media. For Hermitian operators the wave fields can be derived from a variational principle and Noether's theorem provides a general method to obtain field invariants corresponding to the symmetries of the action function. Since this functional depends on derivatives of the field up to an arbitrarily high order, the conserved quantities can be expressed in compact form by the concomitant. The Lagrangian formalism for wave propagation in slowly varying media is thereby generalized to arbitrary inhomogeneities.

In the following we consider wave propagation in a medium with an arbitrary space and time dependence. We assume that the system can be described by a Hermitian operator $\boldsymbol{L}$, which satisfies a Lagrange identity in 4-dimensional space-time x^μ,

$$\mathcal{L}(\psi^*, \psi, x^\mu) - \mathcal{L}(\psi^*, \psi, x^\mu)^* = i\frac{\partial}{\partial x^\mu} J^\mu(\psi^*, \psi), \tag{22}$$

with $\mathcal{L}(\psi^*, \psi, x^\mu) = \psi^* \cdot \boldsymbol{L} \cdot \psi$. We choose the notation A^μ for a four-dimensional vector with time component A^0 and use the repeated index sum notation. The number of components of the fields ψ is unconstrained, but $\mathcal{L}$ is assumed to be a scalar quantity. We define the variational form

$$S = \int d^4x \mathcal{L}(\psi^*, \psi, x^\mu) \tag{23}$$

and note that the wave equation (1) follows from the variational principle $\delta S = 0$ with respect to variations of ψ. Let us now assume an infinitesimal point transformation,

$$\begin{aligned} \tilde{x}^\mu &= x^\mu + \delta x^\mu(x^\mu) \\ \tilde{\psi}(\tilde{x}^\mu) &= \psi(x^\mu) + \delta\psi(x^\mu). \end{aligned} \tag{24}$$

The Jacobian of the transformation of x^μ is

$$1 + \frac{\partial}{\partial x^\mu}\delta x^\mu \tag{25}$$

and the change of ψ at x^μ is defined as

$$\overline{\delta\psi} \equiv \tilde{\psi}(x^\mu) - \psi(x^\mu) = \delta\psi(x^\mu) - \delta x^\mu \frac{\partial}{\partial x^\mu}\psi(x^\mu). \tag{26}$$

With Eqs. (25) and (26) the variation of $S(\psi)$ corresponding to the transformation (24) assumes the form,

$$\delta S = \int d^4\tilde{x} \mathcal{L}(\tilde{\psi}^* \tilde{\psi}, \tilde{x}^\mu) - \int d^4x \mathcal{L}(\psi^*, \psi, x^\mu)$$

$$
\begin{aligned}
&= \int d^4x \left(\delta\mathcal{L} + \mathcal{L}\frac{\partial}{\partial x^\mu}\delta x^\mu\right) \\
&= \int d^4x \left[\overline{\delta\psi^*}\cdot \boldsymbol{L}\cdot\psi + \psi^*\cdot\boldsymbol{L}\cdot\overline{\delta\psi} + \frac{\partial}{\partial x^\mu}(\mathcal{L}\delta x^\mu)\right]. \qquad (27)
\end{aligned}
$$

We now use the Lagrange identity (22) to obtain

$$
\begin{aligned}
\delta S &= \int d^4x \left[\overline{\delta\psi^*}(\boldsymbol{L}\cdot\psi) + \overline{\delta\psi}\cdot(\boldsymbol{L}\cdot\psi)^*\right] \\
&+ \int d^4x \frac{\partial}{\partial x^\mu}\left[iJ^\mu\left(\psi^*, \overline{\delta\psi}\right) + \mathcal{L}\delta x^\mu\right]. \qquad (28)
\end{aligned}
$$

If ψ is an extremal of the variational form (23) and S is invariant with respect to the transformation for an arbitrary volume, then Noether's theorem follows in the form,

$$
\frac{\partial}{\partial x^\mu}\left[iJ^\mu\left(\psi^*, \overline{\delta\psi}\right) + \mathcal{L}\delta x^\mu\right] = 0. \qquad (29)
$$

Let us now consider the particular transformation $\psi \to e^{i\theta}\psi$, corresponding to a uniform phase shift of the fields. The infinitesimal changes here are $\delta x^\mu = 0$ and $\delta\psi = \overline{\delta\psi} = i\delta\theta\psi$. Under this transformation the variational form (23) remains invariant, which is called a gauge invariance of the first kind (or sometimes called global gauge invariance). According to Noether's theorem (29) this symmetry leads to the conservation law,

$$
\frac{\partial}{\partial x^\mu}J^\mu(\psi^*, \psi) = 0. \qquad (30)
$$

Integrating Eq. (30) over a volume V that encloses the whole system yields a field invariant,

$$
A = \int_V d^3x\, J^0(\psi^*, \psi). \qquad (31)
$$

The asymptotic expression for $J^0(\psi^*, \psi)$ can be found from Eq. (22) in analogy with the treatment in Sec. III. For a single wave $\propto e^{-i\omega t}$ one obtains

$$
J^0(\psi^*, \psi) = \psi^*\cdot\left(\frac{\partial}{\partial\omega}\boldsymbol{D}\right)\cdot\psi. \qquad (32)
$$

The expression (32) is commonly called the wave action, which is generalized by Eq. (31) to arbitrary inhomogeneities.

For time-independent homogeneous media Eq. (23) is also invariant with respect to infinitesimal translations δx^μ. Using Eq. (26) with $\delta\psi(x^\mu) = 0$ one finds from Eq. (29),

$$\frac{\partial}{\partial x^\mu}\left[J^\mu\left(\psi^*, i\frac{\partial}{\partial x^\nu}\psi\right)\right] = 0. \tag{33}$$

The corresponding field invariants are an energy-like quantity,

$$W = \int_V d^3x\, J^0(\psi^*, i\partial_t\psi) \tag{34}$$

and a momentum-like quantity,

$$P^j = \int_V d^3x\, J^0\left(\psi^*, i\frac{\partial}{\partial x^j}\psi\right), \tag{35}$$

for linear waves. For inhomogeneous, time-dependent media the change of these quantities follows from

$$\frac{\partial}{\partial x^\nu}\mathcal{L}(\psi^*, \psi, x^\mu) = 0 \tag{36}$$

and Eq. (22) in the form,

$$\frac{\partial}{\partial x^\mu}J^\mu\left(\psi^*, i\frac{\partial}{\partial x^\nu}\psi\right) = -\psi^* \cdot \left(\frac{\partial}{\partial x^\nu}\boldsymbol{L}\right)\cdot\psi. \tag{37}$$

V. Discussion

In the preceding sections we have shown that Lagrange identities for Hermitian operators determine conservation laws that are related to action, energy and momentum conservation. We are mainly concerned with the Vlasov-Maxwell theory of wave propagation in an inhomogeneous time-independent plasma equilibrium. Here Hermitian operators arise for wave propagation perpendicular to the magnetic field and the Lagrange identities can be expected to be valid.[3-4] In general, the plasma response can be written as a differential operator of infinite order. If this operator is applied to plane waves the resulting series often have an infinite radius of convergence and accurate approximations can often be obtained for truncations of finite orders. It is assumed that a corresponding truncation of the differential operator can be justified, leading to differential equations of an arbitrary high order. An example can be found in Refs. 3,4 and 6 if wave propagation in a thermal plasma perpendicular to a uniform magnetic field is considered. (We know that it is only for perpendicular propagation that Hermiticity is satisfied, otherwise Landau damping introduces an anti-Hermitian component to the wave operators.) For waves with a given frequency the action (31) and the energy (34) as well as the corresponding flows are related by a constant factor. The demonstration of global energy conservation for these Hermitian systems is the main result of this work.

For time-dependent Hermitian operators one can readily generalize the 3-dimensional spatial conservation law to a continuity equation for the wave action. The energy then becomes an independent quantity, which in general is no longer conserved. One should however, notice that the Hermitian form of the time dependent Vlasov-Maxwell operator is less obvious and this structure may apply to special cases with an adiabatically slow time variation only.

To illustrate the general procedure we now discuss a specific example describing wave propagation in a cold plasma with no external fields. The wave equation for the electric field $\mathbf{E}$ is written in the form

$$\Delta\mathbf{E} - \frac{1}{c^2}\partial_t^2\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) - \frac{\omega_p^2}{c^2}\mathbf{E} = 0 \tag{38}$$

with the plasma frequency ω_p and speed of light c. The operator in Eq. (38) is Hermitian and we can therefore obtain conserved quantities from the concomitant expression. For any second order derivative the Lagrange identity assumes the form

$$\mathbf{E}^*\frac{\partial^2}{\partial x^2}\cdot\mathbf{E} - \mathbf{E}\cdot\frac{\partial^2}{\partial x^2}\mathbf{E}^* = \frac{\partial}{\partial x}\left(\mathbf{E}^*\cdot\frac{\partial}{\partial x}\mathbf{E} - \mathbf{E}\cdot\frac{\partial}{\partial x}\mathbf{E}^*\right), \tag{39}$$

which defines the concomitants of the Laplacian and of the time derivative in Eq. (38). For the operator $\nabla\nabla\cdot$ one can derive two alternative identities. If the divergence is first transferred to the adjoint field one obtains

$$\mathbf{E}^*\cdot(\nabla\nabla\cdot\mathbf{E}) - \mathbf{E}\cdot(\nabla\nabla\cdot\mathbf{E}^*) = \nabla\cdot[\mathbf{E}^*\cdot\nabla\mathbf{E} - \mathbf{E}\cdot\nabla\mathbf{E}^*]. \tag{40}$$

Conversely, if the gradient is first transferred, the identity assumes the form

$$\mathbf{E}^*\cdot(\nabla\nabla\cdot\mathbf{E}) - \mathbf{E}\cdot(\nabla\nabla\cdot\mathbf{E}^*) = \nabla\cdot[\mathbf{E}^*(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{E}^*)]. \tag{41}$$

In both identities the adjoint operators are the same, but the concomitants differ by the expression,

$$(\mathbf{E}^*\cdot\nabla)\mathbf{E} \ + \ \mathbf{E}(\nabla\cdot\mathbf{E}^*) - [(\mathbf{E}\cdot\nabla)\mathbf{E}^* + \mathbf{E}^*(\nabla\cdot\mathbf{E})] =$$

$$\nabla \cdot (\mathbf{E}^*\mathbf{E}) \quad - \quad \nabla \cdot (\mathbf{E}\mathbf{E}^*) = \nabla \times (\mathbf{E} \times \mathbf{E}^*), \tag{42}$$

which is the curl of a bilinear vector field. This is an example of the ambiguity in the concomitant as discussed in Sec. III. We now combine Eqs. (39) and (40) to obtain the conservation law for the operator (38),

$$\partial_t J^0(\mathbf{E}^*, \mathbf{E}) + \nabla \cdot \mathbf{J}(\mathbf{E}^*, \mathbf{E}) = 0 \tag{43}$$

with

$$J^0(\mathbf{E}^*, \mathbf{E}) = -\frac{1}{c^2}\left[\mathbf{E}^*\partial_t\mathbf{E} - \mathbf{E}\partial_t\mathbf{E}^*\right]$$

$$\mathbf{J}(\mathbf{E}^*, \mathbf{E}) = \mathbf{E}^* \times (\nabla \times \mathbf{E}) - \mathbf{E} \times \nabla \times \mathbf{E}^*.$$

If in this model the plasma frequency is taken both space and time dependent, Eq. (43) describes the conservation of the wave action. If the medium is time independent, action conservation becomes simply related to energy conservation. In this case $\partial_t\mathbf{E}$ can be taken as a solution of Eq. (38) and the energy current $J^\mu(\mathbf{E}^*, \partial_t\mathbf{E})$ as defined by Eq. (33) will then also satisfy Eq. (43). To compare this expression with the physical wave energy we use Maxwell's equations for a cold plasma without external fields in the form

$$\begin{aligned} \nabla \times \partial_t\mathbf{B} &= \frac{1}{c}\partial_t^2\mathbf{E} + \frac{1}{c}\omega_p^2\mathbf{E} \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\partial_t\mathbf{B}. \end{aligned} \tag{44}$$

The energy $J^0(\mathbf{E}^*, \partial_t\mathbf{E})$ can then be rewritten as

$$\begin{aligned} J^0(\mathbf{E}^*, \partial_t\mathbf{E}) &= -\frac{1}{c^2}\left[c\mathbf{E}^* \cdot \nabla \times \partial_t\mathbf{B} - \omega_p^2|\mathbf{E}|^2 - |\partial_t\mathbf{E}|^2\right] \\ &= \frac{1}{c^2}\left[|\partial_t\mathbf{E}|^2 + \omega_p^2|\mathbf{E}|^2 + |\partial_t\mathbf{B}|^2 - c\nabla \cdot (\partial_t\mathbf{B} \times \mathbf{E}^*)\right]. \end{aligned} \tag{45}$$

Similarly, the energy current assumes the form,

$$\begin{aligned} \mathbf{J}(\mathbf{E}^*, \partial_t\mathbf{E}) &= \partial_t\left[\mathbf{E}^* \times (\nabla \times \mathbf{E})\right] - \partial_t\mathbf{E}^* \times (\nabla \times \mathbf{E}^*) - \partial_t\mathbf{E} \times (\nabla \times \mathbf{E}^*) \\ &= -\frac{1}{c}\partial_t(\mathbf{E}^* \times \partial_t\mathbf{B}) + \frac{1}{c}\left(\partial_t\mathbf{E}^* \times \partial_t\mathbf{B} + \partial_t\mathbf{E} \times \partial_t\mathbf{B}^*\right). \end{aligned} \tag{46}$$

Inserting Eqs. (45) and (46) into Eq. (43) one obtains the energy conservation law,

$$\partial_t(|\partial_t \mathbf{E}|^2 + \omega_p^2|\mathbf{E}|^2 + |\partial_t \mathbf{B}|^2) + \nabla \cdot \left[c\partial_t \mathbf{E}^* \times \partial_t \mathbf{B} + c\partial_t \mathbf{E} \times \partial_t \mathbf{B}^*\right] = 0. \tag{47}$$

This is in accordance with Poynting's theorem for the cold plasma model (44). We note, however, that the energy and the corresponding flow cannot be uniquely defined by the conservation relation (43) for arbitrary inhomogeneities. Actually any transformation

$$\begin{aligned} W' &= W + \nabla \cdot \mathbf{a} \\ \mathbf{J}' &= \mathbf{J} - \partial_t \mathbf{a} + \nabla \times \mathbf{b} \end{aligned} \tag{48}$$

with bilinear vector fields $\mathbf{a}$ and $\mathbf{b}$ leaves Eq. (43) form invariant. This ambiguity arises in the derivation of Eq. (47) with $\mathbf{a} = \frac{1}{c}\mathbf{E}^* \times \partial_t \mathbf{B}$ and $\mathbf{b} = \mathbf{E}^* \times \mathbf{E}$. Asymptotically, however, for plane waves with given frequencies and wave vectors the ambiguous terms vanish and then unique expressions are obtained.

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Appendix — Operator Identities

In this appendix we consider vector systems of differential equations of arbitrary order n and derive explicit expressions for the adjoint operator and the concomitant. The operator has the general form

$$L_{ij} = \sum_{m=0}^{n} \sum_{\alpha_1 \cdots \alpha_m = 1}^{3} a_{ij}^{\alpha_1 \cdots \alpha_m} \partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_m}} \tag{A1}$$

and for simplicity of notation is written in the symbolic form

$$\boldsymbol{L} = \sum_{m=0}^{n} \boldsymbol{L}^m, \tag{A2}$$

where

$$\begin{aligned} \boldsymbol{L}^m &= \mathbf{a}^m \nabla^m \\ \boldsymbol{a}^m &\equiv a_{ij}^{\alpha_1 \cdots \alpha_m} \quad ; \quad \nabla^m = \underbrace{\nabla \otimes \nabla \cdots \otimes \nabla}_{m \text{ times}}. \end{aligned}$$

Here summation over the indices of the gradients is always implicitly assumed. We start with the derivative of m-th order, $\boldsymbol{L}^m$. Using repeatedly the product rule one obtains for arbitrary fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ the identity,

$$\begin{aligned} \boldsymbol{\phi} \cdot \boldsymbol{L}^m \cdot \boldsymbol{\psi} &= \nabla \left[\boldsymbol{\phi} \cdot \boldsymbol{a}^m \nabla^{m-1} \cdot \boldsymbol{\psi}\right] + \left[(-\nabla)\boldsymbol{\phi} \cdot \boldsymbol{a}^m\right] \nabla^{m-1} \cdot \boldsymbol{\psi} \\ &= \nabla \left\{ \sum_{p=1}^{m} \left[(-\nabla)^{p-1} \boldsymbol{\phi} \cdot \boldsymbol{a}^m\right] \nabla^{m-p} \cdot \boldsymbol{\psi} \right\} \\ &+ \left[(-\nabla)^m \boldsymbol{\phi} \cdot \boldsymbol{a}^m\right] \cdot \boldsymbol{\psi}. \end{aligned} \tag{A3}$$

Comparison with Eq. (2) shows that

$$\begin{aligned} \boldsymbol{L}^{m\dagger} &= (-\nabla)^m \boldsymbol{a}^{m\dagger} \\ i\mathbf{J}(\boldsymbol{\phi}, \boldsymbol{\psi}) &= \sum_{p=1}^{m} \left[(-\nabla)^{p-1} \boldsymbol{\phi} \cdot \boldsymbol{a}^m\right] \nabla^{m-p} \cdot \boldsymbol{\psi}, \end{aligned} \tag{A4}$$

with $\left(a_{ij}^{\alpha_1 \cdots \alpha_m}\right)^\dagger = \left(a_{ji}^{\alpha_1 \cdots \alpha_m}\right)^*$. If the matrix $a_{ij}^{\alpha_1 \cdots \alpha_m}$ is not completely symmetric with respect to the indices $\alpha_1 \cdots \alpha_m$, the expression (A4) is not uniquely defined and depends on the ordering in which the product rule in (A3) is applied to different gradients. On the other hand $\boldsymbol{L}^m$ and $\boldsymbol{L}^{m\dagger}$ are unique since $\boldsymbol{a}^m$ and $\boldsymbol{a}^{m\dagger}$ are here contracted with the completely symmetric tensor ∇^m. As a consequence two different representations for the concomitant can only differ by the curl of a bilinear vector field, $\nabla \times \mathbf{b}(\phi, \psi)$. An example of this ambiguity is discussed in Sec. V.

For the general operator (A2) the identity (A3) holds for each term from $m = 1$ up to $m = n$, while the term $m = 0$ produces no boundary term. Summation over m then yields

$$\begin{aligned} \boldsymbol{L}^\dagger &= \sum_{m=0}^{n} (-\nabla)^m \boldsymbol{a}^{m\dagger} \\ i\mathbf{J}(\phi, \psi) &= \sum_{m=1}^{n} \sum_{p=1}^{m} \left[(-\nabla)^{p-1} \phi \cdot \boldsymbol{a}^m \right] \nabla^{m-p} \cdot \psi. \end{aligned} \tag{A5}$$

For Hermitian operators Eq. (A5) does not express explicitly the Hermitian form.

A general form for Hermitian operators is

$$\boldsymbol{L}_H = \sum_{m,n=0}^{N} (-\nabla)^m \boldsymbol{G}^{mn} \nabla^n \tag{A6}$$

where the matrix $\boldsymbol{G}^{mn}$ satisfies the condition,

$$G_{ij}^{mn} = (G_{ji}^{nm})^*. \tag{A7}$$

Defining

$$\begin{aligned} \boldsymbol{T}(\psi) &= \sum_{n=0}^{N} (-1)^m \boldsymbol{G}^{mn} \nabla^n \psi \\ Q(\phi, \psi) &= \sum_{m=0}^{N} \left[(-\nabla)^m \phi^* \right] \cdot \boldsymbol{T}(\psi) \\ \mathbf{I}(\phi^*, \psi) &= \sum_{m=1}^{N} \sum_{p=1}^{m} \left[(-\nabla)^{p-1} \phi^* \right] \nabla^{m-p} \cdot \boldsymbol{T}(\psi), \end{aligned} \tag{A8}$$

we obtain by the same procedure as in Eq. (A3) the identity,

$$\phi^* \cdot \boldsymbol{L}_H \cdot \psi = Q(\phi, \psi) + \nabla \cdot \mathbf{I}(\phi^*, \psi). \tag{A9}$$

The expression $Q(\phi, \psi)$ is a Hermitian form satisfying

$$\begin{aligned}
Q^*(\psi, \phi) &= \sum_{m,n} \left[(\nabla^m \psi^*) \cdot \boldsymbol{G}^{mn} \cdot (\nabla^n \phi)\right]^* \\
&= \sum_{m,n} (\nabla^m \psi) \cdot \boldsymbol{G}^{mn*} \cdot (\nabla^n \phi^*) \\
&= \sum_{mn} (\nabla^n \phi^*) \cdot \boldsymbol{G}^{mn} \cdot (\nabla^m \psi) = Q(\phi, \psi).
\end{aligned} \tag{A10}$$

Interchanging ϕ and ψ in Eq. (A9) and taking the complex conjugate yields,

$$\psi \cdot (\boldsymbol{L}_H \cdot \phi)^* = Q^*(\psi, \phi) + \nabla \cdot \mathbf{I}^*(\psi^*, \phi). \tag{A11}$$

Subtracting Eq. (A11) from Eq. (A9) and observing Eq. (A10) one obtains the Lagrange identity for $\boldsymbol{L}_H$ with the concomitant

$$i\mathbf{J}(\phi^*, \psi) = \mathbf{I}(\phi^*, \psi) - \mathbf{I}^*(\psi^*, \phi). \tag{A12}$$

We finally give the form of the algebraic operators, that correspond to $\boldsymbol{L}$ and $\boldsymbol{L}^\dagger$ in the asymptotic limit. These are obtained by substituting $\nabla \to i\mathbf{k}$ in Eqs. (A2) and (A5) yielding,

$$\begin{aligned}
\boldsymbol{L} &\to \sum_{m=0}^{n} \boldsymbol{a}^m (i\mathbf{k})^m = \boldsymbol{D}(\mathbf{x}, \mathbf{k}) \\
\boldsymbol{L}^\dagger &\to \sum_{m=0}^{n} \boldsymbol{a}^{m\dagger} (i\mathbf{k}^*)^{m*} = \boldsymbol{D}^\dagger(\mathbf{x}, \mathbf{k}^*).
\end{aligned} \tag{A13}$$

Let $\boldsymbol{L}$ be Hermitian then $\boldsymbol{D}$ is Hermitian for real $\mathbf{k}$. This symmetry, however, no longer holds for complex $\mathbf{k}$.