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Interaction of a Strong Vortex with Decaying Turbulence

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The evolution of a localized, axially symmetric vortex under the action of shear stresses assosciated with decaying two-dimensional turbulent vorticity which is inhomogeneous in the presence of the vortex is studied analytically. For a vortex which is sufficiently strong relative to the coefficient of turbulent eddy viscosity, it is shown that turbulent fluctuations in the vortex interior and diffusion of coherent vorticity by the turbulence localize to the vortex periphery. It is also found that the coefficient of diffusion is small compared to the coefficient of eddy viscosity.

The phenomenon of spatially intermittent behavior in turbulent flows is both intriguing and potentially of importance, not only because it has bearing on what approximations are appropriate for describing turbulence, but because it can affect cascades and transport as well. This type of phenomenon has recently been observed in numerical simulations of decaying 2-D Navier Stokes turbulence, where long lived coherent vortices emerged as the flow evolved from random initial conditions[1]. Cascading turbulence ocurred in nearby but spatially distinct regions of the flow and was viscously dissipated much before the lifetime of coherent vortices. Similar phenonema have been observed in simulations of plasma turbulence, both in fluid-like flows[2] and in flows sensitive to velocity distributions[3]. In the latter case, the existence of the coherent structure can be understood hueristically in terms of a self-binding force associated with a self-electric field which electrostatically traps the particles comprising the structure[4]. Coexistence of the structure with cascading turbulence is possible when the self-binding force exceeds the shear stresses associated with detrapping by the random electric fields of the turbulence[3]. In the case of Navier Stokes turbulence, there are no electric fields. Nevertheless, a type of force balance constraint appears to apply, e.g., in the numerical experiments it was noted that squared vorticity exceeded mean square shear stress for the coherent vortices, while the opposite was true in regions of cascading turbulence[1]. This observation is consistent with the fact that in 2-D anisotropic turbulence, subgrid scale turbulence is supressed if vorticity is strong and shear stresses are weak, and it is generated if the opposite holds[5]. More recently, this idea has been extended to inhomogeneous turbulence, where a general tendency toward growth of small scale excitation in regions where strain exceeds vorticity has been demonstrated[6].

Here, we report the principal results of a detailed study of the interaction of a localized, axially symmetric vortex with decaying, 2-D Navier Stokes turbulence.

Assuming, *a priori* , that the vortex lifetime is long relative to the time scales of the cascade, a two time scale analysis is developed which treats the rapid evolution of inhomogeneous turbulence in the presence of the slowly varying vortex, and slow vortex evolution under the shearing stresses of the inhomogeneous turbulence. The object of the study is to obtain a detailed picture of the spatial and temporal characteristics associated with interaction of regions of strong vorticity and turbulent straining fields. This analysis uncovers the conditions under which vortex decay is significantly slower than that of the typical turbulent eddy (justifying the two-timescale assumption), and elucidates the underlying processes giving rise to this phenomenon.

The dynamics of nonlinear flows in two dimensions is described by the scalar vorticity evolution equation obtained by taking the curl of the Navier Stokes equation. For simplicity, a vortex which has no bulk motion of translation is considered (or equivalently, the co-moving reference frame is selected). The origin of a polar coordinate system is located at the vortex center. Assuming an axially symmetric vortex and isotropic turbulence with no mean (angle averaged) vorticity, the vorticity evolution equation is Fourier transformed in the angle coordinate. The $n = 0$ component then unambiguously represents the vorticity of the vortex while the remaining components represent vorticity of the turbulence (where n is the Fourier transform variable conjugate to the angle). Note that while the vortex is localized radially and remains so on the rapid scale, the turbulence can not be restricted in its radial location because of its tendency to diffuse. Thus, radial overlap of turbulent vorticity with the vortex occurs in general. Assuming two time scales of evolution, the $n = 0$ component of the vorticity evolution equation describes quasi-static, slow time scale evolution of the vortex under the action of turbulent Reynolds stresses which can be time averaged on the rapid scale. The $n > 0$ components of vorticity evolution describe fast scale evolution of turbulence. The evolution equations are:

$$\frac{\partial \overline{q}}{\partial t} + \int_{-i\infty+\gamma_o}^{i\infty+\gamma_o} d\gamma' \sum_{n'} \left(\frac{1}{r} \frac{\partial}{\partial r} (r V_{-n',-\gamma'} \, q_{n',\gamma'}) \right) = \upsilon \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \overline{q}}{\partial r} \right) , \tag{1}$$

$$-\gamma \, q_{n,\gamma} - \upsilon \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial q_{n,\gamma}}{\partial r} \right) - \frac{n^2}{r^2} q_{n,\gamma} \right] + \frac{i \overline{U} n}{r} q_{n,\gamma}$$

$$+ \sum_{n' \neq o,n} \left[\frac{1}{r} \frac{\partial}{\partial r} (V_{n',\gamma'} \, q_{n-n',\gamma-\gamma'}) + \frac{in}{r} U_{n',\gamma'} \, q_{n-n',\gamma-\gamma'} \right] = -V_{n,\gamma} \frac{\partial \overline{q}}{\partial r} \tag{2}$$

where $q = \nabla^2 \phi = r^{-1} \partial/\partial r \, (r \, \partial\phi/\partial r) + r^{-2} \, \partial^2\phi/\partial\vartheta^2$ is the vorticity, $U = \partial\phi/\partial r$ is the polar velocity, $V = r^{-1} \partial\phi/\partial\vartheta$ is the radial velocity, ϕ is the stream function, and υ is the viscosity. The vorticity and polar velocity of the vortex are $\overline{q}$ and $\overline{U}$. A Fourier-Laplace transform has been performed, i.e., $q_{n,\gamma}(r) = \int_o^\infty dt \exp(\gamma t) \int_o^{2\pi} d\vartheta \exp(-in\vartheta) \, q(r,\vartheta,t)$ and the inverse Laplace transform with $t = 0$ in the vortex evolution equation accomplishes the average over rapid time scales.

With the time scale separation, the fast scale motions can be treated statistically, i.e., a statistical closure can be applied to the rapid scale fluctuations represented by the $n > 0$ components. Vortex evolution is expressed as an effective eddy diffusion on the slow scale by solving the renormalized fast scale equation for $q_{n',\gamma'}$ (in terms of $V_{n',\gamma'} \partial\overline{q}/\partial r$) and by substituting into the vortex equation. Note, however, that whereas eddy viscosity typically incorporates only smaller spatial scales, here it represents shorter temporal scales and higher n but does not necessarily make any restrictions on radial scales. The objective of the present work is to determine the structure of the turbulence and the vortex diffusion coefficient for a specified model vortex. For simplicity, a rigid rotor is considered with a constant vorticity q_o inside a radius r_o. Outside r_o, the vorticity is taken to fall smoothly to zero in some finite layer. The region of primary focus will be the region of significant vorticity inside of r_o. The corresponding flow velocity is $\overline{U} = U_o r/r_o$ for $r \leq r_o$ and $\overline{U} \approx U_o r_o/r$ outside of the region of nonzero vorticity, where $U_o = 1/2\, r_o q_o$. Consideration is also restricted to large Reynolds numbers so that the viscosity may be neglected.

A renormalized fast scale equation is obtained from Eq. (2) using an EDQNM closure[7] in order to formulate the coherent response of $q_{n,\gamma}$ to the convection $V_{n,\gamma}\partial\overline{q}/\partial r$ of the vortex vorticity by the turbulence. To implement the closure, the driven vorticity $q_{n-n',\gamma-\gamma'}$ and driven flows $U_{n-n',\gamma-\gamma'}$, $V_{n-n',\gamma-\gamma'}$ are iteratively evaluated consistent with the assumption of quasi-Gaussian statistics. The driven vorticity satisfies $P_{n-n',\gamma-\gamma'}\, q_{n-n',\gamma-\gamma'} + V_{n-n',\gamma-\gamma'}\, \partial\overline{q}/\partial r = -r^{-1}\partial/\partial r[\, r(V_{n,\gamma}q_{-n',-\gamma'} + V_{-n',-\gamma'}q_{n,\gamma})] - i(n-n')r^{-1}\, [U_{n,\gamma}q_{-n',-\gamma'} + U_{-n',-\gamma'}q_{n,\gamma}]$ where $P_{n-n',\gamma-\gamma'}$ is the self-similar nonlinear coherent response. The driven flows are expressed in terms of the vorticity by inversion of the Laplacian which relates vorticity to the stream function, e.g., $U_{n-n',\gamma-\gamma'} = \partial/\partial r\, (\phi_{n-n',\gamma-\gamma'})$ where $[r^{-1}\, \partial/\partial r\, (r\partial/\partial r) - (n-n')^2 r^{-2}]\, \phi_{n-n',\gamma-\gamma'} = q_{n-n',\gamma-\gamma'}$. The Laplacian inversion is accomplished with a Green's function: $\phi_{n-n',\gamma-\gamma'} = \int dr'\, G_{n-n'}(r|r')\, q_{n-n',\gamma-\gamma'} \equiv L_{n-n'}^{-1}\, q_{n-n',\gamma-\gamma'}$ where $G_n(r|r') = -(1/2n)(rr')^{1/2}(r/r')^n$ for $r < r'$ and $G_n(r|r') = -(1/2n)(rr')^{1/2}(r'/r)^n$ for $r' < r$. The renormalization procedure yields a diffusion tensor and the turbulent back-reactions or β terms which appear as renormalizations of the velocity. Representing the nonlinear contribution to the response as an eddy viscosity, the turbulent vorticity (averaged over a suitable ensemble) is governed by

$$\left(-\gamma + \frac{in\overline{U}}{r}\right) q_{n,\gamma} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\, D_n^{(r,r)}\, \frac{\partial q_{n,\gamma}}{\partial r}\right) + \frac{n^2}{r^2} D_n^{(\vartheta,\vartheta)}\, q_{n,\gamma} = -V_{n,\gamma}\frac{\partial \overline{q}}{\partial r} \qquad (3)$$

where

$$D_n^{(r,r)} = \sum_{n'\neq o,n}\left(V_{n',\gamma'}P^{-1}_{n-n',\gamma-\gamma'}V_{-n',-\gamma'} + q_{n',\gamma'}\frac{-i(n-n')}{r}L^{-1}_{n-n'}P^{-1}_{n-n',\gamma-\gamma'}V_{-n',-\gamma}\right), \quad (4)$$

$$D_n^{(\vartheta,\vartheta)} = \sum_{n'\neq o,n}\left(U_{n',\gamma'}P^{-1}_{n-n',\gamma-\gamma'}U_{-n',-\gamma'} + r\, q_{n',\gamma'}\frac{\partial}{\partial r}L^{-1}_{n-n'}P^{-1}_{n-n',\gamma-\gamma'}r^{-1}U_{-n',-\gamma'}\right), \quad (5)$$

and

$$P_{n-n',\gamma-\gamma'} = -(\gamma-\gamma') + \frac{i(n-n')\,\overline{U}}{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(rD^{(r,r)}_{n-n'}\frac{\partial}{\partial r}\right) + \frac{(n-n')^2}{r^2}D^{(\vartheta,\vartheta)}_{n-n'}. \quad (6)$$

The slow scale evolution of the vortex is governed by a diffusion coefficient obtained by inverting Eq. (3) and substituting into Eq. (1). Formally, $\partial\overline{q}/\partial t - r^{-1}\,\partial/\partial r(rD_v\partial\overline{q}/\partial r) = 0$, where $D_v = \int d\gamma' \, \Sigma_{n'} V_{-n',-\gamma'} P^{-1}_{n',\gamma'} V_{n',\gamma'}$.

The radial structure of turbulent vorticity in the presence of the vortex is given by Eq. (3). In general, Eq. (3) is complicated since the coefficients of eddy viscosity are functions of radial position. However, the turbulent velocities $V_{n,\gamma}$ and $U_{n,\gamma}$ are integrals of the vorticity $q_{n,\gamma}$; therefore, the radial variation of velocity is smoother than that of vorticity. This is consistent with the results of simulations of 2-D Navier Stokes turbulence, which show that for vorticity with steep gradients, the variation of the stream function remains smooth[1]. Consequently, the variation of the coefficients of eddy viscosity is also smoother than that of vorticity. This follows from the fact that these coefficients are functions of velocity and the response function ($D_n \sim \Sigma_{n'}P^{-1}_{n-n'}V^2_{n'}$). The response function $P_{n-n'}$ is smooth because it represents the response of driven fluctuations which are dominated by turbulence in the exterior of the vortex where turbulent vorticity is homogeneous asymptotically. These features allow the coefficients of eddy viscosity to be replaced by spatially averaged coefficients, to lowest order in the parameter describing the difference of rapid radial variation of the turbulent vorticity in the vortex interior relative to that of velocity.

Under this approximation, the turbulent vorticity equation can be solved exactly. Constructing a Green's function from the homogeneous solutions to Eq. (3), the turbulent vorticity in the vortex interior is given by

$$q_{n,\gamma}(r) = \int_0^{r_0} dr'\; g_n(r\,|r')\; D_n^{-1}\; V_{n,\gamma}(r')\; \partial\bar{q}/\partial r', \tag{7}$$

where

$$g_n(r\,|r') = (-i/2)\pi r' \begin{Bmatrix} J_n(r/\Delta r)\; H_n(r'/\Delta r), & r \le r' \\ J_n(r'/\Delta r)\; H_n(r/\Delta r), & r \ge r' \end{Bmatrix}. \tag{8}$$

The function J_n is a Bessel function of the first kind; H_n is a Hankel function, $H_n = J_n + iY_n$, chosen to facilitate the matching onto exterior solutions. The radial scale Δr is the interior turbulence scale given by $\Delta r = \{\gamma/D_n - iU_o n/D_n r_o\}^{-1/2}$ where D_n is an isotropic eddy viscosity ($V \approx U \Rightarrow D_n^{(r,r)} \approx D_n^{(\vartheta,\vartheta)} \rightarrow D_n$) adopted for simplicity. Note that the Laplace transform variable represents the damping rate of vorticity $q_{n,\gamma}$ and is given approximately by D_n/r_n^2 where r_n is a turbulence scale in the exterior region.

A natural strong vortex limit, $U_o n r_o/D_n >> 1$, is suggested by these equations. Physically, this limit means that the vortex turnover rate exceeds the eddy decay rate of an eddy of size r_o. In this limit, $|r_o/\Delta r| >> 1$ and the Green's function $g_n(r\,|r')$ can be expanded asymptotically from the large argument expansions of the Bessel functions. If $|\mathrm{Im}(r_o/\Delta r)| >> 1$ as well, then the radial integral over the Green's function in Eq. (7) can be performed using saddle point methods to obtain the dominant asymptotic contribution to the integral. The turbulent vorticity is then given by

$$q_{n,\gamma}(r) = \frac{-i}{\pi}\left(\frac{\gamma}{D_n} - \frac{iU_o n}{D_n r_o}\right)^{-1/2}\left(1 - i\,\frac{b_r}{b_i}\right)^{3/2}\left(\frac{r_o}{r}\right)^{1/2}\frac{b_i}{|b|^2} \times$$
$$\exp\left[b_i(r + r_o) + ib_r(r + r_o) - i\pi(1/2 + n)\right] D_n^{-1}\, V_{n,\gamma}(r_o)\left(\frac{\partial \bar{q}}{\partial r}\right)_{r_o} \qquad (9)$$

where

$$b_r - ib_i = \mathrm{Re}(\Delta r^{-1}) + \mathrm{Im}(\Delta r^{-1}) = \left(\frac{\gamma^2}{D_n^2} + \frac{U_o^2 n^2}{D_n^2 r_o^2}\right)^{1/4}\exp\left[(-i/2)\,\mathrm{Tan}^{-1}\left(\frac{U_o n}{r_o \gamma}\right)\right]$$

The internal turbulent vorticity $q_{n,\gamma}(r)$ matches at $r = r_o$ onto an external vorticity solution whose amplitude is consistent with the level of vorticity fluctuations in the external region. From Eq. (9), it is evident that the interior vorticity is localized to an exponentially narrow layer inside $r = r_o$ of width b_i^{-1}. The existence of this layer is significant. It means that the coefficient of vortex decay will be exponentially small in the vortex interior. Moreover, because the layer becomes narrower with increasing vortex amplitude, the coefficient of vortex decay will decrease with increasing vortex amplitude. The precise conditions for this localization of turbulent vorticity to the vortex periphery depend on the radial scale of turbulent eddies relative to the vortex size. For eddies which are larger than the vortex, $r_n > r_o$, the strong vortex condition introduced previously, $U_o n r_o/D_n >> 1$, is a necessary and sufficient condition. In terms of the eddy decay rate γ, this condition states that the ratio of vortex turnover rate to eddy decay rate must exceed the ratio of the squares of eddy size to the vortex size, $U_o n/r_o\gamma >> (r_n/r_o)^2$. For eddies which are smaller than the vortex, $r_n < r_o$, it is sufficient for the vortex turnover rate to exceed the eddy decay rate, i.e., $U_o n/r_o\gamma >> 1$. The constraint on vortex vorticity for localization of turbulent vorticity is thus less stringent for eddies which are smaller than the vortex than it is for large eddies.

Substitution of Eq. (9) into Eq. (1) yields the vortex diffusion equation, $\partial\bar{q}/\partial t - r^{-1}\,\partial/\partial r\,(rD_v(r)\partial\bar{q}/\partial r_o) = 0$, where

$$D_v = \int_{-i\infty+\gamma_o}^{i\infty+\gamma_o} d\gamma \sum_n |V_{n,\gamma}|^2\, i\,\frac{\Delta r}{\pi}\left(1-i\,\frac{b_r}{b_i}\right)^{3/2}\left(\frac{r_o}{r}\right)^{1/2}\frac{b_i}{|b|^2} D_n^{-1}\exp\left[i(r+r_o)(b_r-ib_r)-i\pi(1/2+n)\right].$$

The inverse Laplace transform is a contour integration which must take account of the analytic properties of the integrand. Because the turbulence is decaying, the turbulent velocity spectrum has a simple pole at $\gamma=\gamma_n$: $|V_{n,\gamma}|^2 = |V_n|^2\,(\gamma+\gamma_n)^{-1}$ where $\gamma_n > 0$. The remaining part of the integrand has branch points at $\gamma = \pm\, iU_o n/r_o$. For purposes of the integration, branch cuts are taken form these points to $-\infty$ along lines parallel to the real γ axis. The integration follows a Landau contour[8] which is deformed from $\pm i\infty+\gamma_o$ to proceed along a vertical line displaced to the left of the Imγ axis by an amount $\nu > \gamma_n$. Indentation along horizontal lines intersecting singularities keeps the singularities to the left of the contour and prevents crossing branch cuts. The dominant contribution comes from the pole at $\gamma=\gamma_n$, giving

$$D_v = \sum_n^{U_o r_o/D_n} \frac{\exp[(r_o+r)b_i]\,|V_n|^2}{\left(1+\dfrac{U_o{}^2 n^2}{r_o{}^2\gamma_n{}^2}\right)^{1/2}\gamma_n} S(rb_r) \tag{10}$$

where $S(rb_r) = [(1+i)\cos(r+r_o)b_r - (1-i)\sin(r+r_o)b_r]\cos[\pi(n-1/8)]$ is an oscillatory structure function of unit amplitude and the upper limit of the summation has been chosen to exclude modes whose excitation is confined to regions outside the vortex.

Like the turbulent vorticity, vortex diffusion is localized to the vortex periphery in a layer of width b_i^{-1}. The matching to exterior solutions effectively guarantees that $D_v \approx D_n[1+U_o{}^2n^2/r_o{}^2\gamma_n{}^2]^{-1/2}\exp[(r-r_o)b_i]$. When $U_o n/r_o\gamma_n >>1$ for the relevant values of n, it follows that $D_v \ll D_n$. In this case the vortex has a lifetime which is greater than the decay time of the turbulence. It is also apparent that the turnover rate and decay rate for the vortex are not comparable as would be the case in homogeneuos turbulence. Here, the turnover rate increases with the vorticity of the vortex, while the decay rate decreases. From random initial conditions, such as in the simulations[1], a region of large vorticity satisfying $U_0 n/r_0\gamma_n \gg 1$ could be expected to emerge as an isolated, coherent vortex as the turbulence relaxed. This condition is consistent with the observation that squared vorticity exceeded mean square shear stress in the coherent vortices of the simulations. Here shear stress is represented statistically by an eddy decay rate. It should be noted that the two time scale assumption which permited the statistical description is shown to be justified a posteriori when $U_0 n/r_0\gamma_n \gg 1$.

In summary, the interaction of a symmetric, localized vortex with decaying turbulence has been investigated. It is found that when the vorticity of the vortex is sufficiently large, turbulence is restricted to regions outside the vortex and to a narrow layer at the vortex periphery. The diffusion of the vortex under the action of turbulent shear stresses is also localized to the edge of the vortex and is found to be small relative to eddy viscosities. The vortex strength required for this behavior depends on the relative sizes of the vortex and turbulent eddies. In general, the vortex turnover rate must exceed the larger of the eddy decay rate or the eddy decay rate times the ratio of eddy size squared to vortex size squared.

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References

1. J. McWilliams, J. Fluid Mech. **146**, 21 (1984).
2. G.G. Craddock, Ph.D. Dissertation, University of Texas, 1987.
3. R.H. Berman, D.J. Tetreault, and T.H. Dupree, Phys. Fluids **28**, 155 (1985).
4. T.H. Dupree, Phys. Fluids **25**, 277 (1982).
5. J.R. Herring, J. Atmos. Sciences **32**, 2254 (1975).
6. J. Weiss, La Jolla Institute Report No. LJI-TN-81-121, 1981 (unpublished).
7. S.A. Orszag, in *Fluid Dynamics: Les Houches*, Edited by R. Balin and J.-L. Peube, (Gordon and Breach, New York, 1973), p. 235.
8. L. Landau, J. Phys. USSR, **10**:25 (1946). For a discussion see N.A. Krall and A.W. Trivelpiece, *Principles of Plasma Physics,* (McGraw Hill, New York, 1973), p. 379.