DOE/ET-53088-302 IFSR #302

Relation of Wave Energy and Momentum with the Plasma Dispersion Relation in an Inhomogeneous Plasma

H. L. Berk and *D. Pfirsch*

Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

October 1987

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H. L. BERK and D. PFIRSCH*
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

Abstract

The expressions for wave energy and angular momentum commonly used in homogeneous and near homogeneous media, is generalized to inhomogeneous media governed by a nonlocal conductivity tensor. The expression for wave energy applies to linear excitations in an arbitrary three-dimensional equilibrium, while the expression for angular momentum applies to linear excitations of azimuthally symmetric equilibria. For example, the expression for the wave energy, $\mathcal{E}_{\text{wave}}$, for real frequency ω, is given by

$$\begin{aligned}\mathcal{E}_{\text{wave}} &= \frac{c}{16\pi}\frac{\partial}{\partial\omega}\left\{\int d^3r\left[\omega|\mathbf{E}|^2 - \frac{|\nabla\times\mathbf{E}|^2}{\omega}\right]\right. \\ &+ \left.\frac{\text{Re}}{16}\int d^3r\, d^3r'\frac{4\pi i}{c}\mathbf{E}^*(\mathbf{r})\cdot\sigma(\mathbf{r},\mathbf{r}',\omega)\cdot\mathbf{E}(\mathbf{r}')\right\},\end{aligned}$$

where $\sigma(\mathbf{r},\mathbf{r}',\omega)$ is a nonlocal conductivity kernal and $\mathbf{E}(\mathbf{r})$ is the perturbed electric field. The wave energy is interpreted as the energy transferred from linear external sources to the plasma if there is no dissipation. With dissipation, such a simple interpretation is lacking as energy is also thermally absorbed. However, for azimuthally symmetric equilibria the expression for the wave energy in a frame rotating with a frequency ω,

$$\mathcal{E}_{\text{wave}} - \omega L_{\text{wave}}/l$$

where L_{wave} is the wave angular momentum defined in the text and l the azimuthal wavenumber, can be unambiguously separated from thermal energy, and is closely related the real part of a dispersion relation for marginal stability. The imaginary part of the dispersion is closely related to the energy input into a system. Another useful quantity discussed is the impedance form, which can be used for three-dimensional equilibrium without an ignorable coordinate and the expression is closely related to the wave impedance used in antenna theory. Applications to stability theory is also discussed.

*Permanent Address: Max Planck Institute, Garching, FRG

I. Introduction

Variational methods are commonly used to formulate plasma stability problems[1−11] and to obtain estimates of their nonlinear consequences.[12,13] The purpose of this paper is to attempt to relate the variational form that is generally derived, to such physical quantities as energy and momentum in inhomogeneous media. In this work such a relation is obtained for the energy, and, if there is an ignorable coordinate, for the momentum conjugate to this coordinate. These quantities generalize the expressions for the wave energy and wave momentum that are commonly used for homogeneous and near homogeneous medium.[14] A similar expression as ours for wave energy for a somewhat less general system has been derived by Antonsen and Lee.[15]

The energy and momentum expressions give an insight into stability, as the detailed expressions show how each region of phase space contributes to the sign of the energy excitated. At marginal stability a wave can be excited at a real frequency without external energy and momentum input. An expression, that generalizes the result of Ref. 3 to inhomogeneous collisionless media, is derived that relates the real and imaginary part of the global dispersion relation, to the energy and momentum constraints of the problem. When the dissipation of the system is small compared to its reactive response, it is also useful to classify positive and negative energy waves, and positive and negative dissipation as an aide to stability analysis.[11,16]

We shall also discuss a less obvious physical quantity which we call the impedance form, that does not depend on the equilibrium having an ignorable coordinate. This quantity is closely related to the usual impedance of a plasma that is needed to describe how waves couple into a plasma from an external antenna.[17] Additional insight of the relationship of the variational form to entropy production has also been discussed.[18]

Our method of calculation is to assume that all energy and momentum fluxes are driven from external linear sources. At the bounding surfaces the fluxes are given by the Poynting vector and the Maxwell stress tensor. By use of Maxwell equations, fluxes can be related to the currents and fields inside the plasma. This procedure is done partly for general time dependencies, but mainly for perturbations that are proportional to $\exp(-i\omega t)$ where ω is generally a complex frequency. It is especially in the latter formulation, where currents can be expressed in term of a nonlocal conductivity tensor, that relationships between the various physical quantities appear closely related.

In Section II we derive the general quadratic variational form for the linear eigenvalue problem for exponential time dependencies. We define there two linear operators which play a central role for the other physical quantities that are discussed. In Section III the expressions for the energy, impedance form, and angular momentum for real electromagnetic fields and general time dependencies are obtained. In Section IV these physical expressions are evaluated for a given frequency ω and expressed in terms of the conductivity tensor. In Section V we indicate how the impedance form is related to the impedance used in circuit theory.

In Section VI we calculate the nonlocal response, assuming the Vlasov equation, with one ignorable coordinate, governs the determination of the conductivity tensor. We show that without dissipation, the sign of the wave energy we have defined has a useful physical interpretation. With dissipation, the use of wave energy is only useful in a perturbative sense when the dissipation is weak. Nonetheless, with at least one ignorable coordinate, there is a special frame where the wave energy has a precise interpretation. At marginal stability the wave energy in this frame must vanish. This statement is closely related to the condition that the real part of the dispersion relation vanish for real ω. A second independent condition at marginal stability, that the mean power dissipated vanish, is equivalent to the condition that the imaginary part of the dispersion relation vanish.

In Section VII the conclusions of this paper are presented.

II. Quadratic Variational Form

The basic structure of all the quadratic forms we shall derive results from the structure of Maxwell's equations for the perturbed quantities—the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$, the current $\mathbf{j}$, and the charge density ρ of an equilibrium system described by $\mathbf{E}^{(0)}$, $\mathbf{B}^{(0)}$, $\mathbf{j}^{(0)}$, $\rho^{(0)}$:

(1) $$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}$$

(2) $$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\dot{\mathbf{E}}$$

(3) $$\nabla \cdot \mathbf{E} = 4\pi\rho$$

(4) $$\nabla \cdot \mathbf{B} = 0.$$

In eigenvalue problems one seeks time-varying solutions proportional to $\exp(-i\omega t)$ with

(5) $$\omega = \omega_R + i\gamma$$

being generally a complex frequency with a real part ω_R and an imaginary part γ. Maxwell's equations (1) and (2) can then be written in the form

$$\nabla \times \mathbf{E} - \frac{i\omega}{c}\mathbf{B} = 0 \tag{6}$$

$$\nabla \times \mathbf{B} + \frac{i\omega}{c}\mathbf{E} - \frac{4\pi}{c}\mathbf{j} = 0. \tag{7}$$

The current density is in general given by a nonlocal conductivity tensor $\boldsymbol{\sigma}$

$$\mathbf{j}(\mathbf{r}, t) = \int d^3r' \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{E}(\mathbf{r}', t), \tag{8}$$

where in general $i\boldsymbol{\sigma}$ has Hermitian and anti-Hermitian parts. Combining Eqs. (6)–(8) leads to the eigenmode equation for the vector $\mathbf{E}$

$$\frac{\omega}{c}\mathbf{E} - \frac{c}{\omega}\nabla \times (\nabla \times \mathbf{E}) + \frac{4\pi i}{c}\int d^3r' \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{E}(\mathbf{r}', t) \equiv \boldsymbol{\Omega} \cdot \mathbf{E} = 0 \tag{9}$$

which defines the linear operator $\boldsymbol{\Omega}$. Equation (9), coupled with appropriate boundary conditions for $\mathbf{E}$, leads to the eigenvalue problem for ω.

To obtain a variational form we need to multiply Eq. (9) by its adjoint $\mathbf{E}^\dagger$ and integrate over all space. At this point, let us assume for simplicity a perfect conductor surrounding the system, so that surface terms arising from an integration by parts vanish. Then the variational form for Eq. (9) can be written as

$$\int d^3r \left[\frac{\omega}{c}\mathbf{E}^\dagger \cdot \mathbf{E} - \frac{c}{\omega}\,\mathrm{curl}\,\mathbf{E}^\dagger \cdot \mathrm{curl}\,\mathbf{E}\right] + \frac{4\pi i}{c}\int d^3r\, d^3r' \mathbf{E}^\dagger(\mathbf{r}', t) \cdot \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{E}(\mathbf{r}, t)$$
$$\equiv \int d^3r \mathbf{E}^\dagger \cdot \widehat{\boldsymbol{\Omega}} \cdot \mathbf{E} = 0. \tag{10}$$

We also assume that the external conductors are in a vacuum region, so that the space of the integration of $\mathbf{r}$ and $\mathbf{r}'$ are identical. Equation (10) also defines a linear operator $\widehat{\boldsymbol{\Omega}}$ being related to $\boldsymbol{\Omega}$, but being symmetric in its operation on $\mathbf{E}$ and $\mathbf{E}^\dagger$; the variation of Eq. (10) with respect to $\mathbf{E}^\dagger$ gives the eigenmode equation for $\mathbf{E}$, and variation with respect to $\mathbf{E}$ gives the eigenmode equation for the adjoint vector $\mathbf{E}^\dagger$.

We note that if ω is real and $i\boldsymbol{\sigma}$ Hermitian then $\mathbf{E}^\dagger = \mathbf{E}^*$. However, in general $i\boldsymbol{\sigma}$ has both Hermitian and anti-Hermitian components, so that $\mathbf{E}^\dagger \neq \mathbf{E}^*$. Nonetheless, in many physics problems $\boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}', \omega)$ has considerable symmetry, and it is possible to directly relate the adjoint function $\mathbf{E}^\dagger(\mathbf{r}, t)$ to $\mathbf{E}^*(\mathbf{r}, t)$ [1]. When this is true, a second order accurate variational method for eigenvalues is readily constructed. We also note that Eq. (10) resembles a Lagrangian for the electromagnetic field.

III. Second Order Energy, Impedance Form, and Angular Momentum

In this section we consider

$$\begin{aligned} \mathbf{E}_t &\equiv \mathbf{E}^{(0)} + \mathbf{E}, \qquad \mathbf{B}_t \equiv \mathbf{B}^{(0)} + \mathbf{B} \\ \mathbf{j}_t &= \mathbf{j}^{(0)} + \mathbf{j}, \qquad \rho_t = \rho^{(0)} + \rho \end{aligned}$$

as arbitrary time dependent nonlinear quantities. We shall assume that in the equilibrium no energy (or momentum, etc.) is transferred into the system. We take all equilibrium quantities to be time independent. Thus, we have

$$\int_{\mathcal{a}} d\mathcal{a}\mathbf{E}^{(0)} \times \mathbf{B}^{(0)} \cdot \hat{\mathbf{n}} = 0,$$

where $\mathcal{a}$ is a bounding surface of the system where there is no plasma and $\hat{\mathbf{n}}$ its local normal. Now, by multiplying Eq. (2) by $\mathbf{E}$ and Eq. (1) by $\mathbf{B}$, then subtracting the latter from the first, one derives from Maxwell's equations a local conservation law in terms of the nonlinear perturbed quantities,

(11) $$\frac{\partial}{\partial t}\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \mathbf{j} \cdot \mathbf{E} + \frac{c}{4\pi}\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) = 0.$$

In a similar manner one can derive the total energy conservation law,

(12) $$\frac{\partial}{\partial t}\frac{E_t^2 + B_t^2}{8\pi} + \mathbf{j}_t \cdot \mathbf{E}_t + \frac{c}{4\pi}\boldsymbol{\nabla} \cdot (\mathbf{E}_t \times \mathbf{B}_t) = 0.$$

Now, using our equilibrium assumptions and Eq. (11), Eq. (12) becomes

$$\frac{\partial}{\partial t}\frac{\mathbf{E} \cdot \mathbf{E}^{(0)} + \mathbf{B} \cdot \mathbf{B}^{(0)}}{4\pi} + \mathbf{j}^{(0)} \cdot \mathbf{E} + \mathbf{j} \cdot \mathbf{E}^{(0)} + \frac{c}{4\pi}\boldsymbol{\nabla} \cdot \left(\mathbf{E} \times \mathbf{B}^{(0)} + \mathbf{E}^{(0)} \times \mathbf{B}\right) = 0.$$

By integrating these relations over a volume V, to the surface $\mathcal{a}$, global conservation laws result.

(13) $$\dot{\mathcal{E}} \equiv -\int_{\mathcal{a}} \mathbf{S} \cdot \mathbf{n}\, d\mathcal{a} = \int d^3r \left[\frac{\partial}{\partial t}\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \mathbf{j} \cdot \mathbf{E}\right],$$

$$\dot{\mathcal{E}} - \dot{\mathcal{E}}_t \equiv \int_{\mathcal{a}} (\mathbf{S}_t - \mathbf{S}) \cdot \mathbf{n}\, d\mathcal{a}$$

(14) $$= \int d^3r \left[\frac{\partial}{\partial t}\frac{\left(\mathbf{E}^{(0)} \cdot \mathbf{E} + \mathbf{B}^{(0)} \cdot \mathbf{B}\right)}{4\pi} + \mathbf{j} \cdot \mathbf{E}^{(0)} + \mathbf{j}^{(0)} \cdot \mathbf{E}\right],$$

where $\mathbf{S}_t = c\mathbf{E}_t \times \mathbf{B}_t/4\pi$ and $S = c\mathbf{E} \times \mathbf{B}/4\pi$ are the Poynting vectors of the total and perturbed fields.

We now make the assumption that external circuitry can be designed so that the time average

$$\bar{\dot{\mathcal{E}}}_t - \bar{\dot{\mathcal{E}}} \equiv \frac{c}{4\pi} \int d\mathcal{a}\hat{\mathbf{n}} \cdot \left(\overline{\mathbf{E} \times \mathbf{B}^{(0)}} + \overline{\mathbf{E}^{(0)} \times \mathbf{B}} \right) = 0, \tag{15}$$

where the bar denotes time average. This is not an obvious relation, as given that the plasma is a nonlinear medium, linear fields in the plasma can induce nonlinear currents and charges which can give rise to time independent nonlinear field quantities. Thus, in general the time averages in Eq. (15) need not vanish. However, these low-frequency fields must be nonradiative fields, so that we can imagine that the induced fields at the bounding surface are insignificant if they are far enough away from the nearly time independent nonlinear plasma currents and charges. At this (bounding) surface, we assume that the external circuitry has an exact linear response; hence the time average of Eq. (15) vanishes.

With our external circuit assumptions we have that global energy conservation is expressed entirely in terms of Eq. (13). To lowest order in the field perturbations, Eq. (13) depends only on the linear response of the system.

The $\mathbf{j} \cdot \mathbf{E}$ term in Eq. (13) describes both the energy dissipation rate and the rate energy is transformed into ordered motions of the plasma which forms part of the steady-state wave energy. The $\partial/\partial t$ term represents the change of the energy of the electromagnetic fields. For a closed system at the bounding surface, $\mathbf{S} = 0$ and therefore $\dot{\mathcal{E}} = 0$. It is suitable to consider an unstable system as a closed system, as then a finite amplitude can grow from an infinitesimal perturbation. Alternatively, we define a closed system as that in which $\mathbf{n} \times \mathbf{E}$ and $\mathbf{n} \times \mathbf{E}^{(0)}$ at $\mathcal{a}$ vanish. Then $\mathbf{S}_t = 0$ in the surface term in Eq. (14) and it is clear that no external energy is transferred into the system. In a stable system, $\mathbf{S} \neq 0$, and finite amplitudes can arise from an adiabatically increasing energy input of the Poynting flux $\mathbf{S}$.

We now derive a less common quantity for which a global conservation law in a closed system holds. Let us represent $\mathbf{E}$ and $\mathbf{B}$ by a scalar potential ϕ and a vector potential $\mathbf{A}$

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}} \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned}$$

With this and the inhomogeneous Maxwell equations we can deduce

$$\frac{1}{8\pi c}\frac{d}{dt}\int_V \mathbf{E} \cdot \mathbf{A}\, d^3r = \int_V \left[-\frac{1}{2c}\mathbf{j} \cdot \mathbf{A} + \frac{1}{8\pi}\mathbf{A} \cdot \nabla \times \mathbf{B} - \frac{1}{8\pi}\mathbf{E}^2 - \frac{1}{8\pi}\mathbf{E} \cdot \nabla\phi \right] d^3r$$

$$= \int_V \left[-\frac{1}{2c}\mathbf{j}\cdot\mathbf{A} + \frac{1}{8\pi}\left(\mathbf{B}^2 - \mathbf{E}^2\right) + \frac{1}{2}\rho\phi\right] d^3r$$

(16)
$$+ \frac{1}{8\pi}\int_a [\mathbf{A}\cdot\mathbf{n}\times\mathbf{B} - \mathbf{E}\cdot\mathbf{n}\phi]\, da.$$

We can rewrite Eq. (16) as

$$\mathcal{L} \equiv \int_V \left[\frac{1}{2}\left(\frac{1}{c}\mathbf{j}\cdot\mathbf{A} - \rho\phi\right) + \frac{1}{8\pi}\left(\mathbf{E}^2 - \mathbf{B}^2\right)\right] d^3r + \frac{1}{8\pi c}\frac{d}{dt}\int_V \mathbf{E}\cdot\mathbf{A}\, d^3r$$

(17)
$$= \frac{1}{8\pi}\int_a (\mathbf{A}\times\mathbf{n}\cdot\mathbf{B} - \phi\mathbf{E}\cdot\mathbf{n})\, da.$$

At an ideal conducting boundary one has $\mathbf{A}\times\mathbf{n} = 0$ and $\phi = \text{const}$. The surface integral is therefore equal to $-\frac{1}{2}Q\phi_s$ where Q is the perturbation of the total charge of the enclosed system and ϕ_s the potential at the surface. Thus, for the usual case, where $Q = 0$, one has $\mathcal{L} = 0$. As with the energy, the surface integral can be viewed as a source term that can drive stable systems, but as a negligible term for unstable systems. $\mathcal{L} = 0$ is a relation independent of the energy relation. We shall denote $\mathcal{L}$ as the impedance form for reasons that will be clear in Section V.

An equation for the second order angular momentum is obtained the following way: Crossing Maxwell's equations (2) with $\mathbf{B}$ and (1) with $\mathbf{E}$ and adding the resulting equations one obtains the force balance equation

(18)
$$\frac{1}{c}\mathbf{j}\times\mathbf{B} + \rho\mathbf{E} = -\nabla\cdot\left[\frac{\mathbf{B}^2 + \mathbf{E}^2}{8\pi} - \frac{1}{4\pi}(\mathbf{BB} + \mathbf{EE})\right] - \frac{\partial}{\partial t}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}.$$

It contains the divergence of Maxwell's electromagnetic stress tensor

(19)
$$M_{ik} = \frac{\mathbf{B}^2 + \mathbf{E}^2}{8\pi}\delta_{ik} - \frac{1}{4\pi}(B_iB_k + E_iE_k).$$

Since this tensor is symmetric, it follows that with Cartesian coordinates x_i and with the usual summation convention

(20)
$$\frac{\partial}{\partial x_k}(M_{ik}x_l - M_{lk}x_i) = \frac{\partial M_{ik}}{\partial x_k}x_l - \frac{\partial M_{lk}}{\partial x_k}x_i.$$

If we denote by $\mathbf{P}$ the expression

(21)
$$\dot{\mathbf{P}} \equiv \frac{1}{c}\mathbf{j}\times\mathbf{B} + \rho\mathbf{E} + \frac{\partial}{\partial t}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}$$

we find from Eq. (18) with Eq. (20)

$$\frac{1}{2}\epsilon_{jil}\left(\dot{P}_i x_l - \dot{P}_l x_i\right) = -\frac{1}{2}\frac{\partial}{\partial x_k}(M_{ik}x_l - M_{lk}x_i)\,\epsilon_{jil}$$

with ϵ_{jil} the conventional unit antisymmetric tensor (e.g., $\epsilon_{123} = -\epsilon_{132} = 1$, $\epsilon_{113} = 0$, etc.)
from which follows,

$$L_j \equiv \frac{1}{2}\int_V \left(\dot{P}_i x_l - \dot{P}_l x_i\right)\epsilon_{jil} d^3r \;=\; -\frac{1}{2}\int_{\mathcal{a}} n_k \left(M_{ik}x_l - M_{lk}x_i\right)\epsilon_{jil} d\mathcal{a}. \tag{22}$$

The left-hand side is the rate of change of the total angular momentum of the system inside $\mathcal{a}$ in the direction perpendicular to the directions associated with i and l. The $\partial/\partial t$ term in $\dot{\mathbf{P}}$ gives the change of the field contribution to the total angular momentum and the Lorentz and Coulomb forces give the plasma contribution to the total angular momentum. The surface term again represents a source term.

In a system surrounded by an axisymmetric ideal conducting surface $\mathcal{a}$ with the axis of symmetry being the z axis, the angular momentum L_z in the $j \equiv z$ direction is of interest. Then one finds from Eq. (17)

$$\dot{L}_z = \int_{\mathcal{a}} \left[\frac{B^2+E^2}{8\pi} R n_\theta - \frac{1}{4\pi}\mathbf{B}\cdot\mathbf{n}RB_\theta - \frac{1}{4\pi}\mathbf{E}\cdot\mathbf{n}RE_\theta\right] d\mathcal{a}. \tag{23}$$

The surface integral will vanish if n_θ, $\mathbf{B}\cdot\mathbf{n}$, E_θ, vanish at $\mathcal{a}$. Then L_z is a conserved quantity. Otherwise, the angular momentum is forced to change and the surface term can be viewed as an angular momentum source to the system.

Finally, we note that as our conservation laws depend only on perturbed quantities, the lowest order expressions for $\dot{\mathcal{E}}$, $\mathcal{L}$, and $\dot{L}_z$ can be obtained by substituting in the lowest order linear field amplitudes.

IV. Energy, Impedance Form, and Angular Momentum for Exponential Time Dependence

An alternative and convenient set of expressions for energy, the impedance form and the angular momentum can be obtained by assuming that the time dependence varies as $e^{-i\omega t}$. The physical electric field is of course $\mathrm{Re}\,\mathbf{E}(\mathbf{r},t)$. In this case we can use Maxwell's equations in the form of Eq. (9) which is $\boldsymbol{\Omega}\cdot\mathbf{E} = 0$. We then follow similar procedures to obtain useful forms for the conservation relations.

In order to obtain the energy-like relation we multiply $\boldsymbol{\Omega}\cdot\mathbf{E}$ by $\mathbf{E}^*$ and integrate over the volume V:

$$\int_V \mathbf{E}^*\cdot\boldsymbol{\Omega}\cdot\mathbf{E}\, d^3r \;=\; 0. \tag{24}$$

In this form it is a trivial relation. It becomes a nontrivial one when one performs an integration by parts and then the operator $\widehat{\boldsymbol{\Omega}}$ defined in Eq. (10) is obtained:

$$\int_V \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E}\, d^3r \;=\; \frac{c}{\omega}\int_{a} \mathbf{E}^* \cdot (\mathbf{n} \times (\nabla \times \mathbf{E}))\, d\mathcal{a}. \tag{25}$$

Using Eq. (6), the right-hand side can be expressed by a complex Poynting vector $\hat{\mathbf{S}}$

$$\begin{aligned} \frac{c}{\omega}\int_{a} \mathbf{E}^* \cdot (\mathbf{n} \times (\nabla \times \mathbf{E}))\, d\mathcal{a} &= i\int_{a} \mathbf{E}^* \cdot (\mathbf{n} \times \mathbf{B}) d\mathcal{a} \\ &= -i\frac{8\pi}{c}\int_{a} \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a} \end{aligned} \tag{26}$$

with

$$\hat{\mathbf{S}} \;=\; \frac{c}{8\pi}\mathbf{E}^* \times \mathbf{B}. \tag{27}$$

One can readily show that the real part of $\hat{\mathbf{S}}$ is the actual time-averaged Poynting vector. If we now write Eq. (25) in the form

$$-i\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E}\, d^3r = -\int_{a} \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a} \tag{28}$$

we find for the rate $\dot{\mathcal{E}}$ at which energy is fed to the system inside $\mathcal{a}$

$$\dot{\mathcal{E}} = -\operatorname{Re}\int_S \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a} = \operatorname{Im}\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E}\, d^3r. \tag{29}$$

This relation is equivalent to relation (13) if the frequency ω is fixed.

Besides Eq. (29) there is a second relation contained in Eq. (28) which is its imaginary part. If we multiply Eq. (28) first by $-\omega/|\omega|^2$ and then take its imaginary part we readily obtain our real second order impedance form in a gauge where $\phi = 0$

$$\mathcal{L} = \operatorname{Re}\frac{c}{8\pi}\frac{\omega}{|\omega|^2}\int_V \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E}\, d\mathcal{a} = \operatorname{Im}\frac{\omega}{|\omega|^2}\int_{a} \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a}. \tag{30}$$

In order to further exploit the expressions for $\dot{\mathcal{E}}$ and $\mathcal{L}$ let us split $\widehat{\Omega}$ in the following form

$$\widehat{\Omega} = \widehat{\Omega}_H + i\widehat{\Omega}_A, \tag{31}$$

where $\widehat{\Omega}_H$ is Hermitian and $i\widehat{\Omega}_A$ anti-Hermitian when ω is real. Then we find

$$\dot{\mathcal{E}} \;=\; \operatorname{Im}\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \widehat{\Omega}_H \cdot \mathbf{E}\, d^3r + \operatorname{Re}\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \widehat{\Omega}_A \cdot \mathbf{E}\, d^3r \tag{32}$$

and

$$\mathcal{L} \;=\; \operatorname{Re}\frac{c}{8\pi}\frac{\omega}{|\omega|^2}\int_V \mathbf{E}^* \cdot \widehat{\Omega}_H \cdot \mathbf{E}\, d^3r - \operatorname{Im}\frac{c}{8\pi}\frac{\omega}{|\omega|^2}\int_V \mathbf{E}^* \cdot \widehat{\Omega}_A \cdot \mathbf{E}\, d^3r. \tag{33}$$

For $\omega = \omega_R$ the first term in Eq. (32) for $\dot{\mathcal{E}}$ and the second term in Eq. (33) for $\mathcal{L}$ vanish identically. The second term in $\dot{\mathcal{E}}$ describes the rate at which energy, coupled into our system from outside, is dissipated. We note that in the dissipationless case $\left(\widehat{\Omega}_A = 0\right)$ the impedance is the same as the variational form of Eq. (10), as then $\mathbf{E}^\dagger(r) = \mathbf{E}^*(r)$.

We can also integrate $\dot{\mathcal{E}}$ and L over time from $-\infty$ to t assuming an adiabatic turn on of the perturbations. This means that there is a small positive imaginary part in addition to ω_R. In this case we can expand

$$\widehat{\Omega}_H(\omega) = \widehat{\Omega}_H(\omega_R) + i\gamma\frac{\partial}{\partial\omega_R}\widehat{\Omega}_H(\omega_R) \tag{34}$$

$$\widehat{\Omega}_A(\omega) = \widehat{\Omega}_A(\omega_R) + i\gamma\frac{\partial}{\partial\omega_R}\widehat{\Omega}_A(\omega_R). \tag{35}$$

Note that Eq. (34) yields while Eq. (35) does not yield an additional term, in Eq. (32) for $\dot{\mathcal{E}}$. The contribution from $\widehat{\Omega}_H$ can be interpreted as the buildup of wave energy. After the time integration, which is equivalent to dividing $\dot{\mathcal{E}}$ by 2γ, we obtain

$$\mathcal{E} = \frac{c}{16\pi}\int_V \mathbf{E}^*\cdot\frac{\partial}{\partial\omega_R}\widehat{\Omega}_H(\omega_R)\cdot\mathbf{E}\,d^3r + \frac{1}{2\gamma}\frac{c}{8\pi}\int \mathbf{E}^*\cdot\widehat{\Omega}_A\cdot\mathbf{E}\,d^3r. \tag{36}$$

The second term on the right-hand side is the dissipated energy (which would of course diverge for a true adiabatic turn on where $\gamma \to 0$) whereas the first term is interpreted as the wave energy

$$\mathcal{E}_{\text{wave}} = \frac{c}{16\pi}\int_V \mathbf{E}^*\cdot\frac{\partial}{\partial\omega_R}\widehat{\Omega}_H(\omega_R)\cdot\mathbf{E}\,d^3r. \tag{37}$$

This expression has the same structure as is conventionally used in spatially homogeneous systems but is derived here without any assumptions as to the geometry. The interpretation of wave energy, in the presence of dissipation, will be discussed in Sec. VI.

One can perform the time integral over $\mathcal{L}$ in a similar way to find,

$$\int_{-\infty}^{t}\mathcal{L}\,dt = \frac{1}{2\gamma}\frac{c}{8\pi}\frac{1}{\omega_R}\int_V \mathbf{E}^*\cdot\widehat{\Omega}_H\cdot\mathbf{E}\,d^3r - \frac{c}{8\pi}\frac{1}{\omega_R^2}\int_V \mathbf{E}^*\frac{\partial}{\partial\omega_R}\omega_R\widehat{\Omega}_A\cdot\mathbf{E}\,d^3r. \tag{38}$$

There seems to be no obvious physical interpretation of this relation. However, later we will introduce the concept of impedance where $\mathcal{L}$ is shown to be related to the reactance of our system.

We proceed now with the derivation of a complex angular momentum relation. In analogy to the derivation of the force balance equation (18) we cross $\boldsymbol{\Omega}\cdot\mathbf{E}$ with $i\mathbf{B}^*/8\pi$; then we take the azimuthal component of the resulting vector, multiply it by the radius R and integrate over the volume V. This leads to

$$\frac{i}{8\pi}\int_V (\mathbf{B}^*\times\boldsymbol{\Omega}\cdot\mathbf{E})_\theta\,R\,d^3r = -\frac{c}{8\pi\omega^*}\int_V \left(R\hat{\theta}\times(\nabla\times\mathbf{E}^*)\right)\cdot\boldsymbol{\Omega}\cdot\mathbf{E}\,d^3r = 0. \tag{39}$$

Integrating the curl curl term in $\boldsymbol{\Omega}$ by parts we find for the time derivative of the complex total angular momentum (note that the real part of $\dot{L}_z$ is the time rate of change of the actual angular momentum)

$$\begin{aligned} \dot{L}_z &= -\frac{c}{8\pi\omega^*}\int_V \left(R\hat{\theta}\times(\nabla\times\mathbf{E}^*)\right)\cdot\widehat{\boldsymbol{\Omega}}\cdot\mathbf{E}\,d^3r \\ &= -\frac{c^2}{8\pi|\omega|^2}\int_{\mathcal{a}} \left(R\hat{\theta}\times(\nabla\times\mathbf{E}^*)\right)\cdot(\mathbf{n}\times(\nabla\times\mathbf{E}))\,d\mathcal{a}. \end{aligned} \tag{40}$$

$\hat{\theta}$ is a unit vector in θ direction. Let $\mathcal{a}$ be an axisymmetric ideal conducting surface $\nabla\times\mathbf{E}^*\sim\mathbf{B}^*$ and $\hat{\theta}$ both lie in the surface. Its cross product is then parallel to $\mathbf{n}$, which makes the surface integral vanish. Then, $\dot{L}_z=0$ is a nontrivial expression. In order to get a more convenient expression we express $R\hat{\theta}\times(\nabla\times\mathbf{E})$ in cylindrical coordinates R, θ, z and its corresponding components are

$$\begin{aligned} R\hat{\theta}\times(\nabla\times\mathbf{E}) &= R((\nabla\times\mathbf{E})_z,\quad 0,\quad -(\nabla\times\mathbf{E})_R) \\ &= R\left(\frac{1}{R}\frac{\partial}{\partial R}RE_\theta-\frac{1}{R}\frac{\partial}{\partial\theta}E_R,\quad 0,\quad -\frac{1}{R}\frac{\partial}{\partial\theta}E_z+\frac{\partial}{\partial z}E_\theta\right) \\ &= \nabla(RE_\theta)-\frac{\partial}{\partial\theta}(E_R,E_\theta,E_z). \end{aligned} \tag{41}$$

When inserting this in the volume integral of (40), the $\nabla(RE_\varphi)$ term yields

$$\begin{aligned} &-\frac{c}{8\pi\omega^*}\int_V \nabla(RE_\theta^*)\cdot\left(\frac{\omega}{c}\mathbf{E}+\frac{4\pi i}{c}\mathbf{j}\right)d^3r \\ &= \frac{c}{8\pi\omega^*}\int_V RE_\theta^*\left(\frac{\omega}{c}\nabla\cdot\mathbf{E}+\frac{4\pi i}{c}\nabla\cdot\mathbf{j}\right)d^3r-\int_{\mathcal{a}} RE_\theta\mathbf{n}\cdot\left(\frac{\omega}{c}\mathbf{E}+\frac{4\pi i}{c}\mathbf{j}\right)d\mathcal{a}. \end{aligned} \tag{42}$$

With $\nabla\cdot\mathbf{j}=i\omega\rho$, $\nabla\cdot\mathbf{E}=4\pi\rho$ the volume integral vanishes. So does the surface integral if $\mathcal{a}$ possesses the above mentioned properties. For the axisymmetric case we can therefore write Eq. (40) as

$$\dot{L}_z = \frac{c}{8\pi\omega^*}\int_V \frac{\partial}{\partial\theta}(E_R^*,E_\theta^*,E_z^*)\cdot\widehat{\boldsymbol{\Omega}}\cdot\mathbf{E}\,d^3r=0. \tag{43}$$

In axisymmetry E_R, E_θ, E_z can be chosen to be proportional to $e^{il\theta}$, thus

$$\dot{L}_z = -\frac{icl}{8\pi\omega^*}\int_V \mathbf{E}^*\cdot\widehat{\boldsymbol{\Omega}}\cdot\mathbf{E}\,d^3r=0. \tag{44}$$

$\omega^*\dot{L}_z/l$ is identical with the complex energy relation given in Eq. (28) and the imaginary part of $-\dot{L}_z/l$ is identical with the impedance form given by Eq. (30). Therefore, the angular momentum does not provide an independent relation. However, for axisymmetric systems it is a more physical quantity than the impedance form.

By comparison with the derivation in Sec. III one finds that the time derivative of the actual angular momentum is the real part of $\dot{L}_z$. We then have

$$\begin{aligned} \operatorname{Re} \dot{L}_z &= \frac{cl\gamma}{8\pi|\omega|^2} \operatorname{Re} \int \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E} \, d^3r + \frac{cl}{8\pi} \frac{\omega_R}{|\omega|^2} \operatorname{Im} \int \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E} \, d^3r \\ &= \frac{cl\gamma}{8\pi|\omega|^2} \operatorname{Re} \int \mathbf{E}^* \cdot \widehat{\Omega} \cdot \mathbf{E} \, d^3r + \frac{l\omega_R}{|\omega|^2} \operatorname{Re} \dot{\mathcal{E}}. \end{aligned} \tag{45}$$

With the representation (31) of $\widehat{\Omega}$ we can also write

$$\begin{aligned} \operatorname{Re} \dot{L}_z &= \frac{cl\gamma}{8\pi|\omega|^2} \left[\operatorname{Re} \int \mathbf{E}^* \cdot \widehat{\Omega}_H \cdot \mathbf{E} \, d^3r - \operatorname{Im} \int \mathbf{E}^* \cdot \widehat{\Omega}_A \cdot \mathbf{E} \, d^3r \right] \\ &+ \frac{cl\omega_R}{8\pi|\omega|^2} \left[\operatorname{Re} \int \mathbf{E}^* \cdot \widehat{\Omega}_A \cdot \mathbf{E} \, d^3r + \operatorname{Im} \int \mathbf{E}^* \cdot \widehat{\Omega}_H \cdot \mathbf{E} \, d^3r \right]. \end{aligned} \tag{46}$$

Now using Eqs. (34) and (35) in the limit $\gamma \to 0$

$$\begin{aligned} \operatorname{Re} \dot{L}_z &= \frac{cl\gamma}{8\pi\omega_R^2} \int \mathbf{E}^* \cdot \frac{\partial}{\partial \omega} \left(\omega_R \widehat{\Omega}_H (\omega_R) \right) \cdot \mathbf{E} \\ &+ \frac{cl}{8\pi\omega_R} \int \mathbf{E}^* \cdot \widehat{\Omega}_A (\omega_R) \cdot \mathbf{E} \, d^3r + \mathcal{O}\left(\gamma^2\right). \end{aligned} \tag{47}$$

If $\widehat{\Omega}_A = 0$, the integration over time for adiabatic turn on at $t = -\infty$ yields

$$\operatorname{Re} L_z = \frac{cl}{16\pi\omega_R^2} \int \mathbf{E}^* \frac{\partial}{\partial \omega_R} \left[\omega_R \widehat{\Omega}_H (\omega_R) \right] \cdot \mathbf{E} \, d^3r \tag{48}$$

which is the wave angular momentum.

V. Introduction of the Concept of Impedance

To obtain an insight into the significance of the imaginary part of the complex energy relation (28) we introduce the concept of impedance.

Assume that through a small part, $\delta\mathcal{A}$ of the surface $\mathcal{A}$, power is fed into the system, e.g., with a coaxial cable. If we express this input power in terms of an input current I_i and an input voltage V_i, we can identify[17]

$$\frac{1}{2} I_i V_i^* = -\int_{\delta\mathcal{A}} \hat{\mathbf{S}} \cdot \mathbf{n} \, d\mathcal{A}. \tag{49}$$

This allows us to write Eq. (28) as

$$\frac{1}{2} I_i V_i^* = -i\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \hat{\boldsymbol{\Omega}} \cdot \mathbf{E}\, d^3r + \int_{\mathcal{a}-\delta\mathcal{a}} \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a}. \tag{50}$$

We can now define an input impedance

$$Z = \mathcal{R} - iX \tag{51}$$

by

$$V_i = ZI_i$$

and find for the resistance $\mathcal{R}$ and the reactance X the expressions

$$\mathcal{R} = \frac{2}{|I_i|^2}\left[\mathrm{Im}\,\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \hat{\boldsymbol{\Omega}} \cdot \mathbf{E}\, d^3r + \mathrm{Re}\int_{\mathcal{a}-\delta\mathcal{a}} \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a}\right] \tag{52}$$

$$X = \frac{2}{|I_i|^2}\left[\mathrm{Re}\,\frac{c}{8\pi}\int_V \mathbf{E}^* \cdot \hat{\boldsymbol{\Omega}} \cdot \mathbf{E}\, d^3r - \mathrm{Im}\int_{\mathcal{a}-\delta\mathcal{a}} \mathbf{n} \cdot \hat{\mathbf{S}}\, d\mathcal{a}\right]. \tag{53}$$

If one can neglect the surface terms and ω is real, $\mathcal{R}$ is given by the anti-Hermitian part of $\hat{\Omega}$ and X by the Hermitian part of $\hat{\boldsymbol{\Omega}}$ and is proportional to the impedance form as given by Eq. (30). More generally $\mathcal{R}$ is always proportional to $\dot{\mathcal{E}}$, Eq. (29), and characterizes the real power fed into the system, whereas X, which is closely related to $\mathcal{L}$, describes the reactive power, i.e., the component of the power input in which current and voltage are 90° out of phase.

VI. Wave Energy and Momentum from Vlasov Equation

As an example we will present the structure of the wave energy and momentum that follows from a system that satisfies the Vlasov equation. The conductivity kernal in inhomogeneous media has been calculated in several works.[5,6,8,9,10] For completeness we present a somewhat novel derivation in the Appendix.

We consider an equilibrium distribution function that is a function of energy $H = m_j v^2/2 + q_j \Phi_0(r_1 z)$ and angular momentum $P_\theta = m_j R v_\theta + q_j A_\theta(R, z)/c$. The equilibrium plasma is axisymmetric but inhomogeneous in the radial and axial directions as in a tokamak, spheromak, field reversed pinch, etc.

We take an electric field (and all other perturbed quantities) to be of the form

$$\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{E}}(R, z)\exp(il\theta - i\omega t)$$
$$\hat{\mathbf{E}} = \left(\hat{E}_R, \hat{E}_\theta, \hat{E}_z\right).$$

We then find in the appendix,

$$
\begin{aligned}
i\int d^3r\,\jmath(R,z)\cdot\widehat{\mathbf{E}}^*(R,z) &= i\int d^3r\int d^3r'\widehat{\mathbf{E}}^*(R,z)\cdot\boldsymbol{\sigma}(\mathbf{r},\mathbf{r}',\omega)\cdot\widehat{\mathbf{E}}(R',z') \\
&= \sum_j q_j^2\int d^3r\,d^3v\left\{\frac{R}{\omega}\frac{\partial F}{\partial P_\theta}v_\theta\left|\widehat{E}_\theta\right|^2\right. \\
&\quad\left. + \sum_n\frac{\left[\mathbf{v}\cdot\widehat{\mathbf{E}}(R,z)e^{il\delta\theta}\right]_n\left[\mathbf{v}\cdot\widehat{\mathbf{E}}(R,z)e^{il\delta\theta}\right]_n^*}{\omega-n\omega_b-l\bar{\dot{\theta}}}\left(\frac{\partial F}{\partial H}+\frac{l}{\omega}\frac{\partial F}{\partial P_\theta}\right)\right\},
\end{aligned}
\tag{54}
$$

where the following quantities need to be defined: $T\equiv 2\pi/\omega_b$ is the period of an orbit in the (R,z) plane; with the definition that $\mathbf{r}(\tau)\equiv[r(\tau),z(\tau),\theta(\tau)]$ and $\mathbf{v}(\tau)$ are trajectories over perturbed orbits that arrive at the point $\mathbf{r}$, $\mathbf{v}$ at time $\tau=0$,

$$
\begin{aligned}
\bar{\dot{\theta}} &= \frac{1}{T}\int_{-T/2}^{T/2}d\tau\dot{\theta}(\tau) \\
\theta(\tau) &= \delta\theta(\tau)+\bar{\dot{\theta}}\tau+\theta \\
\left[\mathbf{v}\cdot\widehat{\mathbf{E}}(R,z)e^{il\delta\theta}\right]_n &= \frac{1}{T}\int_{-T/2}^{T/2}d\tau\exp\left(-in\omega_b\tau\right)\mathbf{v}(\tau)\cdot\widehat{\mathbf{E}}(\tau)\exp\left[il\delta\theta(\tau)\right].
\end{aligned}
$$

We note that orbits can be ergodic in the (R,z) plane, in which case the Fourier series converts to a Fourier integral, with the conversion of the summation to the following integral

$$
\sum_n\frac{\left(\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right)_n\left(\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right)_n^*}{\left(\omega-l\bar{\dot{\theta}}-n\omega_b\right)}\rightarrow\int d\omega'\frac{\left(\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right)_{\omega'}\left(\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right)_{\omega'}^*}{\left(\omega-l\bar{\dot{\theta}}-\omega'\right)}
$$

$$
\left(\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right)_{\omega'} = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau\widehat{\mathbf{E}}(\tau)\cdot\mathbf{v}(\tau)e^{il\delta\theta(\tau)-i\omega'\tau}.
$$

With adiabatic switching on of the input at a frequency ω_R, the rate in which energy is put into the system is found from Eqs. (10) and (29) to be

$$
\begin{aligned}
\dot{\mathcal{E}} &= \lim_{\gamma\to 0^+}\int d^3r\left[\gamma\frac{\left|\widehat{\mathbf{E}}\right|^2}{8\pi}+\frac{\gamma c^2|\nabla\times\tilde{\mathbf{E}}|^2}{(\omega_R^2+\gamma^2)\,8\pi}\right]-\int d^3r\,d^3v\sum_j q_j^2\left\{\frac{R\gamma}{\omega_R^2+\gamma^2}\frac{\partial F}{\partial P_\theta}v_\theta\left|\widehat{E}_\theta\right|^2\right. \\
&\quad\left.+\sum_n\left[\frac{\left|\left[\mathbf{v}\cdot\widehat{\mathbf{E}}e^{il\delta\theta}\right]_n\right|^2\gamma}{\left(\omega_R-n\omega_b-l\bar{\dot{\theta}}\right)^2+\gamma^2}\left(\frac{\partial F}{\partial H}+\frac{l\left(2\omega_R-n\omega_b-l\bar{\dot{\theta}}\right)}{\omega_R^2+\gamma^2}\frac{\partial F}{\partial P_\theta}\right)\right]\right\}
\end{aligned}
\tag{55}
$$

with $\tilde{\mathbf{E}}=\widehat{\mathbf{E}}\exp(il\theta)$. The total energy, put into the system is, $\mathcal{E}=\dot{\mathcal{E}}/2\gamma$. If there are no resonances at $\omega=\omega_R$ or $\omega=\omega-n\omega_b-l\bar{\dot{\theta}}$, for all phase space values, then $\mathcal{E}$ is the total energy put into the system and it is identical to the wave energy, $\mathcal{E}_{\text{wave}}$, given by the expression in Eq. (37). Specifically, for the Vlasov equation,

$$
\begin{aligned}
\mathcal{E}_{\text{wave}} &= \int d^3r \left[\frac{\left|\widehat{\mathbf{E}}\right|^2}{16\pi} + \frac{c^2 \left|\nabla \times \tilde{\mathbf{E}}\right|^2}{16\pi\omega_R^2} \right] - \frac{1}{2}\int d^3r\, d^3\mathbf{v} \sum_j q_j^2 \left\{ \frac{R}{\omega}\frac{\partial F}{\partial P_\theta} v_\theta \left|\widehat{E}_\theta\right|^2 \right. \\
&\left. + \sum_n \left[\frac{\left|\left[\mathbf{v}\cdot\widehat{\mathbf{E}}e^{il\delta\theta}\right]_n\right|^2 \left[\frac{\partial F}{\partial H} + \frac{l}{\omega_R^2}\frac{\partial F}{\partial P_\theta}\left(2\omega_R - n\omega_b - l\bar{\dot{\theta}}\right)\right]}{\left[\omega_R - n\omega_b - l\bar{\dot{\theta}}\right]^2} \right] \right\}. \qquad (56)
\end{aligned}
$$

However, the simple interpretation of wave energy is blurred when orbit resonances are present or when $\omega_R = 0$. When $\omega_R = 0$, we note from Eq. (55) that $\dot{\mathcal{E}} \propto 1/\gamma$, so that the power input into the system is arbitrarily large, unless divergent terms happen to balance. If the denominator, $\omega_R - n\omega_b - l\bar{\dot{\theta}}$, goes through zero over continuous band of phase space, then we observe that

$$
\lim_{\gamma\to 0^+} \frac{\gamma}{\left(\omega_R - n\omega_b - l\bar{\dot{\theta}}\right)^2 + \gamma^2} \to \pi\delta\left(\omega_R - n\omega_b - l\bar{\dot{\theta}}\right),
$$

and the rate of energy input is a constant and is given by

$$
\lim_{\gamma\to 0^+} \dot{\mathcal{E}} = -\pi \int d^3r\, d^3v \sum_j q_j^2 \left(\frac{\partial F}{\partial H} + \frac{l}{\omega_R}\frac{\partial F}{\partial P_\theta}\right) \sum_n \left|\left[\mathbf{v}\cdot\mathbf{E}e^{il\delta\theta}\right]_n\right|^2 \delta\left(\omega - l\bar{\dot{\theta}} - n\omega_b\right). \qquad (57)
$$

We note that this expression is the Landau damping expression and represents the dissipation due to resonant particles. Its sign depends on the sign of the numerator,

$$
\text{Num} \equiv -\left(\frac{\partial F}{\partial H} + \frac{l}{\omega_R}\frac{\partial F}{\partial P_\theta}\right),
$$

and if Num > 0 we have positive dissipation with the system absorbing energy and if Num < 0, we have negative dissipation with the system releasing energy. If there is a fixed amount of dissipation in the limit $\gamma \to 0$, it means that, within the limits of the linearized theory, an "infinite" amount energy must be supplied to establish a configuration, where the input energy is absorbed by the resonant particles. In addition to the "infinite" amount of energy expended in establishing the wave configuration there is a finite amount of wave energy present which is given by Eq. (56), where it is necessary to allow for the modification

$$
\int d^3r\, d^3v \frac{\text{Num}\,1}{\left(\omega - n\omega_b - l\bar{\dot{\theta}}\right)^2} \to P\int d^3r\, d^3v \frac{\text{Num}\,1}{\left(\omega - n\omega_b - l\bar{\dot{\theta}}\right)^2} \equiv M,
$$

where P stands for principal part and Num 1 can be extracted straightforwardly from a comparison with Eq. (56). Let us suppose Num 1 > 0, as it would be if F depends only on H and $\partial F/\partial H < 0$. It would then appear that $M > 0$, which certainly is the case if there were no resonances. However, with resonances M can have either sign. For example, note that

$$P \int_{-\infty}^{\infty} \frac{dx}{(x-y)^2 (x^2 + \Delta x^2)} = \frac{\pi (y^2 - \Delta x^2)}{(y^2 + \Delta x^2)^2 \Delta x}, \tag{58}$$

which can have either sign, when $y \simeq \Delta x$, and significantly is positive if $|y| \gg \Delta x$. Physically, the inability in being able to establish a sign to the wave energy when there are resonances, can be interpreted as follows. Since we have assumed adiabatic switching, which takes an arbitrary long time to establish a configuration, and since, with resonances, a finite amount of energy is dissipatively absorbed in each cycle, an infinite amount of energy has to be transferred into the system. The nondissipative part of the transferred energy, which we have called the wave energy, is only a remnant of the total input energy. With dissipation present, this remnant cannot in general be signed since it is only a finite portion of the total energy that has been transferred from external to internal energy.

In order to have a reasonable interpretation of wave energy, when there are resonances, the resonant interaction should be small, and the lowest order wave energy should be calculated by deleting the resonant particle region from the calculation of Eq. (56). To next order, the lowest order rate of dissipation can be calculated from the resonant interaction, while the modification to the real wave energy, which can be of either sign, can justifiably be deleted since it is a correction to the lowest order wave energy. We shall apply this point below when we discuss stability.

We note that Eq. (55) can give an exact statement of stability if F is only a function of H, and $\partial F/\partial H < 0$. Then we note that for any $\gamma > 0$, Eq. (55) shows that $\mathcal{E}$ is greater than zero for any electric field $\mathbf{E}$. However, in order to have instability, one must conserve energy and have $\dot{\mathcal{E}} = 0$. Thus, in this case it is impossible to have $\gamma > 0$. Thus we have demonstrated another way of proving a well-known stability result. The expression for the wave energy can be used to ascertain stability in more complicated systems. A particularly interesting application, using Nyquist diagrams can be found in Ref. 15.

If F depends on H and P_θ, then the terms in Eq. (55) are not of one sign, and it may be possible to arrange fields so that $\dot{\mathcal{E}} = 0$. We note that to achieve instability, we must also have that the rate of change of angular momentum, $\dot{L}_z$, be zero. Using Eq. (45) we obtain,

$$
\begin{aligned}
\dot{L}_z &= \frac{\gamma l}{\omega_R^2+\gamma^2}\int d^3r\left[\frac{\omega_R\left|\widehat{\mathbf{E}}\right|^2}{8\pi}-\frac{\left|\nabla\times\tilde{\mathbf{E}}\right|^2\omega_R}{8\pi\left[\omega_R^2+\gamma^2\right]}\right] \\
&+ \frac{\gamma}{|\omega|^2}\sum_j q_j^2\int d^3r\,d^3v\left\{Rv_\theta\frac{\partial F}{\partial P_\theta}\frac{\left|\widehat{\mathbf{E}}_\theta\right|^2\omega_R}{\left(\omega_R^2+\gamma^2\right)}\right. \\
&+ \left.\sum_n\left[\frac{\frac{\partial F}{\partial H}\left(\omega-l\bar{\dot\theta}-n\omega_b\right)+\frac{l}{\left(\omega_R^2+\gamma^2\right)}\frac{\partial F}{\partial P_\theta}\left[\omega_R\left(\omega_R-l\bar{\dot\theta}-n\omega_b\right)-\gamma^2\right]}{\left(\omega-l\dot\theta-n\omega_b\right)^2+\gamma^2}\right]\left|\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right|^2\right\} \\
&+\frac{l\omega_R}{|\omega|^2}\dot{\mathcal{E}} \qquad (59)
\end{aligned}
$$

with $\dot{\mathcal{E}}$ given by Eq. (55).

We now note that the quantity,

$$G\left(\omega_R,\gamma\right)=\mathcal{E}-\frac{|\omega|^2}{l\omega_R}L_z$$

has no divergences as $\gamma\rightarrow 0$ and can be interpreted as the wave energy in a special rotating frame of reference. In fact $G\left(\omega_R,0\right)$ is related to the real part of the dispersion relation for real ω, as

$$
\begin{aligned}
\int d^3r\mathbf{E}^*\cdot\widehat{\boldsymbol{\Omega}}_H\left(\omega_R\right)\cdot\mathbf{E} &= \omega_R G\left(\omega_R,0\right)=\int d^3r\left[\frac{\left|\widehat{\mathbf{E}}\right|^2\omega_R}{16\pi}-\frac{\left|\nabla\times\tilde{\mathbf{E}}\right|^2}{16\pi\omega_R}\right] \\
&+ \sum_j\frac{q_j^2}{2}\int d^3r\,d^3v\left\{Rv_\theta\frac{\partial F}{\partial P_\theta}\frac{\left|\widehat{E}_\theta\right|^2}{\omega_R}\right. \\
&+ \left.\sum_n\left[\frac{\partial F}{\partial H}+\frac{l}{\omega_R}\frac{\partial F}{\partial P_\theta}\right]\left|\left(\widehat{\mathbf{E}}\cdot\mathbf{v}e^{il\delta\theta}\right)_n\right|^2\frac{P}{\left(\omega_R-l\bar{\dot\theta}-n\omega_b\right)}\right\}. \qquad (60)
\end{aligned}
$$

(We should however point out that as in general $\mathbf{E}^\dagger\neq\widehat{\mathbf{E}}^*$ that

$$\mathrm{Re}\int d^3r\mathbf{E}^\dagger\cdot\widehat{\boldsymbol{\Omega}}\left(\omega_R\right)\cdot\mathbf{E}\neq\int d^3r\mathbf{E}^*\cdot\widehat{\boldsymbol{\Omega}}_H\left(\omega_R\right)\cdot\mathbf{E},$$

where the left-hand side is the real part of the dispersion relation for real ω, that follows from Eq. (10), and the right-hand side is the expression in Eq. (60). However, at marginal stability both terms will be zero.) The condition for marginal stability, can thus be interpreted as energy in a preferred rotating frame of reference, is zero, and the rate of energy input, with $\gamma = 0$, is zero. The latter statement, demands from Eq. (57),

$$\begin{aligned} \dot{\mathcal{E}}\left(\omega_R, \gamma = 0\right) &= -\pi \sum_j q_j^2 \int d^3r\, d^3v \sum_n \left| \left[\mathbf{v} \cdot \widehat{\mathbf{E}} e^{il\delta\theta} \right]_n \right|^2 \left(\frac{\partial F}{\partial H} + \frac{l}{\omega_R} \frac{\partial F}{\partial P_\theta} \right) \\ &\times \ \delta\left(\omega_R - n\omega_b - l\bar{\dot{\theta}}\right). \end{aligned} \tag{61}$$

This result is a generalization to electromagnetic perturbations in inhomogeneous media, of the result obtained by Dawson and Oberman for the interpretation of the marginal stability condition of the electrostatic spatially homogeneous problem.

If there are no resonances, Eq. (61) is not appropriate for marginal stability, and instead, in addition to Eq. (60), marginal stability would require that

$$\lim_{\gamma \to 0} \frac{\dot{\mathcal{E}}}{\gamma} = 0, \qquad \text{or} \qquad \lim_{\gamma \to 0} \frac{\dot{L}_z}{\gamma} = 0,$$

with either condition yielding that the wave energy, given by Eq. (56), be zero.

In many problems, where the dissipation is small (i.e., where the resonant terms in Eq. (55) are multiplied by small coefficients), one can treat the dissipation term as a perturbation. To lowest order with $\omega = \omega_R$ and no dissipation, Eq. (9) is an Hermitian operator, and then $\mathbf{E}^\dagger = \mathbf{E}^*$. The lowest order dispersion relation is given by Eq. (10), and also satisfies Eq. (60) when it is set to zero. If Eq. (10) is solved to next order, we find,

$$2\left(\omega - \omega_R\right) \mathcal{E}_{\text{wave}}^{(0)}\left(\omega_R\right) + i\dot{\mathcal{E}}\left(\omega_R, \gamma = 0\right) = 0$$

with $\mathcal{E}_{\text{wave}}^{(0)}$ the wave energy, given by Eq. (56), neglecting contributions from the resonance region, and $\dot{\mathcal{E}}\left(\omega_R, \gamma = 0\right)$, the rate of dissipation, is given by Eq. (61). Thus

$$\omega - \omega_R = \frac{-i\dot{\mathcal{E}}\left(\omega_R, \gamma = 0\right)}{2\mathcal{E}_{\text{wave}}^{(0)}\left(\omega_R\right)}.$$

Instability then arises if $\dot{\mathcal{E}}/\mathcal{E}_{\text{wave}}^{(0)} < 0$, which can be caused by combining positive energy waves, $\mathcal{E}_{\text{wave}}^{(0)} > 0$, with negative dissipation, $\dot{\mathcal{E}} < 0$, or negative energy waves, $\mathcal{E}_{\text{wave}}^{(0)} < 0$, with positive dissipation $\dot{\mathcal{E}} > 0$. Stability arises with positive energy coupled with positive dissipation, or negative energy waves coupled with negative dissipation.

VII. Conclusion

In this work we have related the formal quadratic form used in linear stability theory of inhomogeneous media which have perturbed currents determined by a nonlocal conductivity tensor, with expressions of physical interest. These expressions include the time rate of change of energy, the wave energy of general inhomogeneous media, the time rate of change of angular momentum, and the wave angular momentum of an azimuthal symmetric equilibrium.

In presenting the formalism we needed to separate the source of energy (or momentum which can be substituted for energy in the immediate discussion below) from the cavity which encloses the plasma where the energy is stored. To do this we postulated the presence of a remote linear excitation source, where the remoteness is necessary to eliminate the interaction of the source with nonlinear second order fields. A criticism with this type of definition is that the stored wave energy now includes the energy stored in the transmission lines or wave guides, which should be very weakly coupled to the mode associated with the cavity. We also noted in the text that the concept of wave energy is only precise for a dissipationless case, and only with a small rate of dissipation does our definition of wave energy have the meaning of the reactive stored excited energy of the cavity. If there is no dissipation, our dilemma would normally resolve itself as the cavity mode, for real ω, would lead to evanescent linear solutions in a connecting wave guide, so that the fields in the external wave guides tends to zero exponentially in space as one moves away from the plasma. At a remote source, both the linear and nonlinear fields are negligibly small. By making the wave guide dimensions arbitrarily small, the energy stored in the wave guide could be made arbitrarily small compared to the energy in the cavity. The purpose of the source is to supply to stored energy, but in the steady state zero energy is being extracted from the source. From another point of view, with an adiabatic turn-on with a rate γ, the linear field at the source are proportional to γ which to leading order is zero. If the waves in the external guide are not cut-off, then we have radiative boundary conditions which is another form of dissipation. In this case it is now not surprising that the definition of wave energy, which includes the energy stored in the guide, is not associated totally with the energy in the cavity.

For unstable systems, we do not need external sources and we can consider such a system as closed. Then assuming that fields do not radiate to infinity, the perturbed energy, and (for axisymmetric systems) the perturbed angular momentum, are conserved. Hence, in unstable systems distortions evolve with these quantities remaining the same as their equilibrium values. For the collisionless case, but with dissipation present through Landau damping, we have shown that the dispersion relation of an axisymmetric equilibrium has the following physical interpretation. The real part of the dispersion relation is closely related to the perturbed energy in a frame rotating with the wave at a rotation frequency ω, while the imaginary part of the dispersion relation is closely related to the rate in which energy is fed into the system. In this special frame the wave energy has a precise meaning, as we can then show that the wave energy separates from the energy going into heat. It is possible that this definition of wave energy also has meaning in more general systems with collisions. Thus we have a physical interpretation of the dispersion relation at marginal stability. We note that Oberman and Dawson[3] have made a similar observation for an electrostatic instability (in a homogeneous medium). In our analysis we have generalized their result so as to obtain an interpretation of the overall dispersion relation in more complex systems. We also note that our expression for the wave energy (and angular momentum) generalizes the concept of wave energy used for spatially homogeneous medium, and of course reproduces the result in the spatially homogeneous case.

For media whose equilibrium does not contain a spatial symmetry, the use of a global momentum (or angular momentum) is no longer a natural concept. However, we have found a quantity, which we call the impedance form that is conserved in closed unstable systems, and is related to surface terms in driven systems. The physical basis for it is less intuitive than with angular momentum, but nonetheless it is a useful quantity. For example, the surface contribution of this term for real ω gives rise to the wave impedance of the system. The imaginary part of the impedance form is closely related to the power input (or rate of dissipation) into the system. For axisymmetric systems, the impedance form and the expressions for the rate of input of angular momentum contain the same information.

The expressions for wave energy, and the rate of dissipation can be very useful in stability analysis. For example, when dissipation is small, it is useful to categorize excitations as being positive energy (energy added to the equilibrium) or negative energy (energy removed from the equilibrium). The sign of the expression in Eq. (56), immediately shows if energy has been removed or added to the system for the adiabatic turn-on of an arbitrary set of electric fields. In Sec. VI, where we calculated the conductivity tensor for the Vlasov equation, one can ascertain which region of phase space contributes positive or negative energy contributions. If there is a small amount of dissipation present, one can immediately infer stability or instability depending on the relative sign of the dissipation with respect to the wave energy, positive (negative) dissipation coupled to positive (negative) energy waves implies wave stability, while positive (negative) dissipation coupled to negative (positive) energy waves leads to instability. This concept is frequently invoked in stability analyses.[15,11] Here we note that this concept, which follows straightforwardly from the algebraic structure of a dispersion relation, is physically justified.

Acknowledgements

This research was supported by the U. S. Department of Energy Contract No. DE-FG05-80ET-53088.

Appendix A

Here we derive the conductivity tensor for a Vlasov plasma with unperturbed distribution functions for particles of species j

(A1)
$$f_j^{(0)} = F_j\left(H^{(0)}, P_\theta^{(0)}\right)$$

with

(A2)
$$H^{(0)} = \frac{m_j}{2}v^2 + q_j\phi^{(0)}(R,z)$$

and

(A3)
$$P_\theta^{(0)} = R\left(m_j v_\theta + \frac{q_j}{c}A_\theta^{(0)}(R,z)\right).$$

The notation refers to a cylindrical coordinate system R, θ, z. The functions $f_j^{(0)}$ represent a class of axisymmetric equilibria where the energy and the canonical momentum $P_\theta^{(0)}$ are constants of the motion.

We now assume that perturbations are turned on adiabatically at $t = -\infty$. If the initial values H and P_θ at $t = -\infty$ are

(A4)
$$H_0 = H^{(0)}(t = -\infty)$$

(A5)
$$P_{\theta 0} = P_\theta^{(0)}(t = -\infty)$$

one can represent the perturbed distribution functions in the form

(A6)
$$f_j = F_j\left(H_0(\mathbf{r}, \mathbf{v}, t), P_{\theta 0}(\mathbf{r}, \mathbf{v}, t)\right).$$

$H_0(\mathbf{r}, \mathbf{v}, t)$, $P_{\theta 0}(\mathbf{r}, \mathbf{v}, t)$, which have the numerical values H_0 and $P_{\theta 0}$, respectively, are expressed as functions of the particles' positions $\mathbf{r}$ and velocities $\mathbf{v}$ at time t according to the perturbed orbits.

With the distribution function (A6) we can calculate the current density:

(A7)
$$\mathbf{j}(\mathbf{r}, t) = \sum_j q_j \int d^3v\, \mathbf{v} F_j\left(H_0, P_{\theta 0}\right).$$

Because of the linearity of the time response, and because the equilibrium is independent of θ, we can consider perturbed fields that have an exponential time dependence and an exponential dependence on θ,

(A8)
$$\mathbf{E}^{(1)}(\mathbf{r}, t) = \widehat{\mathbf{E}}^{(1)}(R, z)\exp[-i\omega t + il\theta]; \quad \widehat{\mathbf{E}}^{(1)} = \left(\widehat{E}_R^{(1)}, \widehat{E}_\theta^{(1)}, \widehat{E}_Z^{(1)}\right).$$

Using a first order vector field

$$\mathbf{E}^{\dagger}(\mathbf{r},t) = \widehat{\mathbf{E}}^{\dagger}(R,z)\exp[i\omega t - il\theta] \tag{A9}$$

we find,

$$\int \mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}^{\dagger}(\mathbf{r},t)d^3r = \sum_j q_j \int d^3r\, d^3v F_j\left(H_0, P_{\theta 0}\right)\mathbf{v}\cdot\widehat{\mathbf{E}}^{\dagger}(R,z)\exp[i\omega t - il\theta]. \tag{A10}$$

In order to be able to evaluate the right-hand side we need to find H_0, $P_{\theta 0}$ as functions of $\mathbf{r}$, $\mathbf{v}$, t to first order in the perturbations.

The perturbed system can be described by the Hamiltonian

$$H(\mathbf{r},\mathbf{p},t) = \frac{1}{2m_j}\left(\mathbf{p} - \frac{q_j}{c}\mathbf{A}(\mathbf{r},t)\right)^2 + q_j\phi^{(0)}(\mathbf{r}). \tag{A11}$$

In writing $\phi^{(0)}(\mathbf{r})$ instead of $\phi(\mathbf{r},t)$ we have assumed a radiation gauge for the perturbations:

$$\phi^{(1)}(\mathbf{r},t) = 0. \tag{A12}$$

The unperturbed Hamiltonian is in agreement with (A2):

$$H^{(0)} = \frac{1}{2m_j}\left(\mathbf{p} - \frac{q_j}{c}\mathbf{A}^{(0)}(R,z)\right)^2 + q_j\phi^{(0)}(R,z). \tag{A13}$$

The perturbation to first order is

$$H_j^{(1)} = -\frac{q_j}{m_j c}\left(\mathbf{p} - \frac{q_j}{c}\mathbf{A}^{(0)}(R,z)\right)\cdot\mathbf{A}^{(1)} = -\frac{q_j}{c}\mathbf{v}\cdot\mathbf{A}^{(1)}. \tag{A14}$$

In cylindrical coordinates we assume all scalar quantities of first order to be proportional to $\exp(il\theta)$. The equations for H and P_θ are then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{q_j}{c}\mathbf{v}\cdot\frac{\partial\mathbf{A}^{(1)}}{\partial t} = q_j\mathbf{v}\cdot\mathbf{E}^{(1)} \tag{A15}$$

$$\frac{dP_\theta}{dt} = -\frac{\partial H}{\partial\theta} = -il\frac{q_j}{c}\mathbf{v}\cdot\mathbf{A}^{(1)} = \frac{l}{\omega}q_j\mathbf{v}\cdot\mathbf{E}^{(1)}. \tag{A16}$$

From these equations one finds

$$H = H_0 + q_j\int_{-\infty}^{t}\mathbf{v}(\tau)\cdot\widehat{\mathbf{E}}^{(1)}\left(\mathbf{r}(\tau)\right)e^{-i\omega\tau}d\tau \equiv H_0 + H_1 \tag{A17}$$

$$P_\theta = P_{\theta 0} + \frac{l}{\omega}H_1, \tag{A18}$$

where the τ integral is over unperturbed orbits. We can then expand

$$F_j(H_0, P_{\theta 0}) = F_j(H, P_\theta) - \left(\frac{\partial F_j}{\partial H} + \frac{l}{\omega}\frac{\partial F_j}{\partial P_\theta}\right) H_1 + \cdots \tag{A19}$$

The unperturbed orbits along which the integrals in Eq. (A17) are to be taken, have to be characterized by the position $\mathbf{r}$ an orbit passes through at time t and the velocity at this time:

$$\begin{aligned} \mathbf{v}(\tau) &= \mathbf{v}(\tau - t; \mathbf{r}, \mathbf{v}); \qquad \mathbf{v}(t) = \mathbf{v}(0; \mathbf{r}, \mathbf{v}) = \mathbf{v} && \text{(A20)} \\ R(\tau) &= R(\tau - t; \mathbf{r}, \mathbf{v}); \qquad R(t) = R(0; \mathbf{r}, \mathbf{v}) = R && \\ z(\tau) &= z(\tau - t; \mathbf{r}, \mathbf{v}); \qquad z(t) = z(0; \mathbf{r}, \mathbf{v}) = z && \text{(A21)} \\ \theta(\tau) &= \theta + (\tau - t)\bar{\dot{\theta}} + \delta\theta(\tau) - \delta\theta(t). && \text{(A22)} \end{aligned}$$

In expressing $\theta(\tau)$ in the form (A22) we assume the unperturbed orbits are periodic with a time $T = 2\pi/\omega_b$, whereupon they return to the initial coordinates R, z, and $\mathbf{v}$. We have also defined

$$\begin{aligned} \dot{\theta}(\tau) &= \bar{\dot{\theta}} + \delta\dot{\theta}(\tau), \qquad \bar{\dot{\theta}} = \frac{1}{T}\int_{-T/2}^{T/2} \dot{\theta}(\tau) d\tau \\ \int_t^\tau \dot{\theta}(\tau') d\tau' &= \theta(\tau) - \theta(t) = (\tau - t)\bar{\dot{\theta}} + \delta\theta(\tau) - \delta\theta(t) && \text{(A23)} \end{aligned}$$

which agrees with Eq. (A22). Thus, $\delta\theta(\tau)$ also is a periodic function of τ. The integrand in H_1, Eq. (A17), can then be written as

$$\mathbf{v}(\tau) \cdot \hat{\mathbf{E}}(R(\tau), z(\tau))\, e^{il\left(\delta\theta(\tau) + i(\tau - t)\bar{\dot{\theta}} + i\theta - \delta\theta(t)\right) - i\omega\tau}.$$

That part of it that is periodic on τ can be Fourier transformed to the following form

$$\mathbf{v}(\tau) \cdot \hat{\mathbf{E}}(R(\tau), z(\tau))\, e^{il(\delta\theta(\tau) - \delta\theta(t))} = \sum_n g_{nl} e^{in\omega_b(\tau - t)} \tag{A24}$$

$$g_{nl}\left(R, \mathbf{v}, z; \hat{E}\right) = \frac{1}{T}\int_{-T/2}^{T/2} d(\tau - t) e^{-in\omega_b(\tau - t)} \mathbf{v}(\tau) \cdot \hat{\mathbf{E}}(R(\tau), z(\tau))\, e^{il(\delta\theta(\tau) - \delta\theta(t))}. \tag{A25}$$

This allows the time integral in (A17) to be performed:

$$H_1 \equiv \hat{H}_1 \exp[il\theta - i\omega t] = \sum_n g_{nl} \frac{e^{il\theta - i\omega t}}{i\left(n\omega_b + l\bar{\dot{\theta}} - \omega\right)}. \tag{A26}$$

According to the definitions (A20) (A21) the quantities g_{nl} are functions of R, z, and $\mathbf{v}$.

With the expression H_1 in Eq. (A26), we can evaluate Eq. (A10) making use of Eq. (A19):

$$\int \mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}^\dagger(\mathbf{r},t)d^3r = \sum_j q_j \int d^3r\, d^3v F_j(H,P_\theta)\,\mathbf{v}\cdot\mathbf{E}^\dagger(r,t)$$

$$(A27) \qquad - \sum_j q_j \int d^3r\, d^3v \left(\frac{\partial F_j}{\partial H} + \frac{l}{\omega}\frac{\partial F_j}{\partial P_\theta}\right)\hat{H}_1 \mathbf{v}\cdot\hat{\mathbf{E}}^\dagger(R,z).$$

In the first expression of the right-hand side we have

$$(A28) \qquad \sum_j q_j \int d^3v F_j(H,P_\theta)\,\mathbf{v} = \mathbf{j}_0\left(\mathbf{A},\phi^{(0)}\right)$$

with

$$\mathbf{j}_0\left(\mathbf{A}^{(0)},\phi^{(0)}\right) : \text{equilibrium current density.}$$

This zero order term gives zero when inserted in Eq. (A27) as one then integrates a spatially periodic expression in θ. However, the first order term of $\mathbf{j}_0\left(\mathbf{A},\phi^{(0)}\right)$ contributes to the second order terms of Eq. (A27) when we expand $\mathbf{j}_0\left(\mathbf{A},\phi^{(0)}\right)$ to first order,

$$(A29) \qquad \mathbf{j}_0\left(\mathbf{A},\phi^{(0)}\right) = \mathbf{j}_0\left(\mathbf{A}^{(0)},\phi^{(0)}\right) + \mathbf{A}^{(1)}\cdot\frac{\partial}{\partial\mathbf{A}^{(0)}}\mathbf{j}_0\left(\mathbf{A}^{(0)},\phi^{(0)}\right) + \cdots .$$

We notice that $\mathbf{j}_0$ depends only on A_θ which gives

$$(A30) \qquad \mathbf{A}^{(1)}\cdot\frac{\partial}{\partial\mathbf{A}^{(0)}}\mathbf{j}_0 = A_\theta^{(1)}\cdot\frac{\partial}{\partial A_\theta^{(0)}}\mathbf{j}_0 = \frac{c}{i\omega}E_\theta^{(1)}\frac{\partial}{\partial A_\theta^{(0)}}\mathbf{j}_0\left(A_\theta^{(0)},\phi^{(0)}\right)$$

and that $\mathbf{j}_0$ has only a θ component.

In the second term in Eq. (A27) the phase space integral is taken as an integral over the following well-known canonically conjugate variables[19]: the energy E and its canonically conjugate time variable, $\hat{t}$, (the $\hat{t}$ integration is then over unperturbed orbits at constant energy) and two other pairs of canonical variables;

$$(A31) \qquad d^3r\, d^3v = d\varGamma_6 = d\varGamma_4 d\mathcal{E}\, d\hat{t},$$

where $d\varGamma_6$ is the 6-dimensional phase space element and $d\varGamma_4$ a 4-dimensional one. $d\varGamma_4$ is constant along the orbits as $d\mathcal{E}$ is. Having performed the $\hat{t}$-integral one has arrived at an integral which has no more dependence on $\hat{t}$ along the unperturbed orbits. We can therefore once again integrate over t and divide by the interval of integration which is our large period T. This allows the replacements,

$$\int d^3r\, d^3p \cdots \to \int d\Gamma_4 d\mathcal{E} \frac{dt}{T} \int_{-T/2}^{T/2} d\hat{t} \cdots = \int d\Gamma_6 \frac{1}{T} \int_{-T/2}^{T/2} d\hat{t} \cdots$$

$$= \int d^3r\, d^3v \frac{1}{T} \int_{-T/2}^{T/2} d\hat{t} \cdots . \tag{A32}$$

We now evaluate the time integral along the unperturbed orbits. Since $\partial F_j/\partial H$ and $\partial F_j/\partial P_\theta$ are constant along the unperturbed orbits this time integral effects only $\hat{H}_1 \mathbf{v} \cdot \hat{\mathbf{E}}^\dagger(R, z)$. In $\hat{H}_1$, as given by Eq. (A26), the quantities g_{nl} are functions of $\mathbf{v} = \mathbf{v}(t)$ and $\mathbf{r} = \mathbf{r}(t)$. When integrating over $\hat{t}$, the time t becomes $\hat{t}$ and the $\mathbf{v}$ and $\mathbf{r}$ dependencies become $\mathbf{v}(\hat{t})$, $\mathbf{r}(\hat{t})$ with the initial conditions $\mathbf{v}(t) = \mathbf{v}$ and $\mathbf{r}(t) = \mathbf{v}$: Equation (A24) written for the time $\hat{t}$ instead of t is

$$\mathbf{v}(\tau) \cdot \hat{\mathbf{E}}\left(R(\tau), z(\tau)\right) e^{il\left(\delta\theta(\tau) - \delta\theta(\hat{t})\right)} = \sum_n \tilde{g}_{nl}\left(R(\hat{t}), z(\hat{t})\right) e^{in\omega_b(\tau - \hat{t})}. \tag{A33}$$

Comparison with Eq. (A25) yields then

$$g_{nl}\left(R(t), z(t), \mathbf{v}(t)\right) e^{in\omega_b(\tau - t) + il\delta\theta(t)} = \tilde{g}_{nl}\left(R(\hat{t}), z(\hat{t})\mathbf{v}(\hat{t})\right) e^{in\omega_b(\tau - \hat{t}) + il\delta\theta(\hat{t})}$$

or

$$\tilde{g}_{nl}\left(R(\hat{t}), z(\hat{t}), \mathbf{v}(\hat{t})\right) = g_{nl}\left(R(t), z(t), \mathbf{v}(t)\right) e^{in\omega_b(\hat{t} - t) - il\left(\delta\theta(\hat{t}) - \delta\theta(t)\right)}, \tag{A34}$$

where $R(t) = R$ and $z(t) = z$ and $\mathbf{v}(t) = \mathbf{v}$. With this relation we find

$$\frac{1}{T} \int_{-T/2}^{T/2} d\hat{t} H_1 \mathbf{v} \cdot \hat{\mathbf{E}}^\dagger e^{(i\omega\hat{t} - il\theta)} = q_j \frac{1}{T} \int_{-T/2}^{T/2} d(\hat{t} - t) \sum_n g_{nl}(R, z) \frac{e^{in\omega_b(\hat{t} - t) - il\left(\delta\theta(\hat{t}) - \delta\theta(t)\right)}}{i\left(n\omega_b + l\dot{\theta} - \omega\right)} \mathbf{v}(\hat{t})$$

$$\cdot\ \hat{\mathbf{E}}^t\left(R(\hat{t}), z(\hat{t})\right)$$

$$= -iq_j \sum_n \frac{g_{nl}(R, z, \mathbf{v}; \hat{\mathbf{E}}) g_{-n-l}\left(R, z, \mathbf{v}; \hat{\mathbf{E}}^\dagger\right)}{\left(n\omega_b + l\bar{\dot{\theta}} - \omega\right)}. \tag{A35}$$

One can readily show that

$$g_{nl}\left(R, z, \mathbf{v}, t; \hat{\mathbf{E}}\right) g_{-n,-l}\left(R, z, \mathbf{v}, t; \hat{\mathbf{E}}^\dagger\right) = \tilde{g}_{n,l}\left(R, z, \mathbf{v}, t; \hat{\mathbf{E}}\right) \tilde{g}_{-n,-l}\left(R, z, \mathbf{v}, t; \hat{\mathbf{E}}^\dagger\right)$$

with

$$\tilde{g}_{n,l} = \frac{1}{T} \int_{-T/2}^{T/2} d\tau \exp\left[-in\omega_b t\right] \mathbf{v}(\tau) \cdot \hat{\mathbf{E}}(\tau) \exp\left[il\delta\theta(\tau)\right] \equiv \left[\mathbf{v} \cdot \hat{\mathbf{E}} e^{il\delta\theta}\right]_n .$$

We can then represent Eq. (A28) in the form

$$\begin{aligned}\int \mathbf{E}^{\dagger}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r},t)d^3r &= \int \mathbf{E}^{\dagger}(\mathbf{r},t)\cdot\sigma(\mathbf{r},\mathbf{r}',\omega)\cdot\mathbf{E}(\mathbf{r}',t)d^3r\,d^3r' \\ &= \frac{c}{i\omega}\int d^3r\,\widehat{E}_\theta^{\dagger}\widehat{E}_\theta\frac{\partial j_{0\theta}}{\partial A_\theta^{(0)}}\left(A_\theta^{(0)},\phi^0\right) \\ &- \sum_j q_j^2\int d^3r\,d^3v\left(\frac{\partial F_j}{\partial H}+\frac{l}{\omega}\frac{\partial F_j}{\partial P_\theta}\right)\sum_n\frac{\tilde{g}_{nl}(;\widehat{\mathbf{E}})g_{-n-l}\left(;\widehat{\mathbf{E}}^{\dagger}\right)}{i\left(n\omega_b+l\bar{\dot{\theta}}-\omega\right)}. \end{aligned} \tag{A36}$$

We further note that if $\widehat{\mathbf{E}}^{\dagger}(R,z)=\widehat{\mathbf{E}}^{*}(R,z)$, that $\tilde{g}_{-n,-l}\left(;\widehat{\mathbf{E}}^{*}\right)=\tilde{g}_{n,l}^{*}(;\widehat{\mathbf{E}})$. Hence, by expressing the first order current as $\mathbf{j}(\mathbf{r},t)=\hat{\mathbf{j}}(R,z)\exp(il\theta-i\omega t)$, we have Eq. (54) of the text.

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