DOE/ET-53088-301 IFSR #301

Theory of Ion Temperature-Gradient-Driven Turbulence and Transport in Low Ion-Collisionality Plasmas

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October 1987

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ABSTRACT

A novel theory for the nonlinear evolution of the trapped-ion temperature-gradient-driven mode, based on the turbulent trapping of resonant ions in the electrostatic potential of the waves, is proposed. A statistical description is adopted whereby the self-consistent evolution of the two-point correlation function of the trapped-particle distribution function is followed in phase space. Threshold-dependent, non-steady state turbulence (nonlinear instability) is shown to develop when the decay of the correlation function is overcome by a source term which derives its free energy from the relaxation of the average distribution function. This nonlinear instability leads to anomalous thermal and particle transport, which in turn reconfigure the equilibrium temperature and density profiles in such a way as to return the system towards its marginal point. Expressions for the nonlinear growth rate and threshold, as well as estimates of the thermal and particle transport level, are derived. The estimated flux levels are sufficiently high as to make any significant departure away from marginality unlikely. The scenario outlined serves to underscore the desirability for pellet injection in experimental devices such

as the Compact Ignition Tokamak [CIT Design Group, Princeton Plasma Physics Laboratory, *PPPL-Report No. 2389* (1986)] which operate in the very low ion collisionality regime where this mode would be expected to become relevant. As with a number of recent theories, the present work further reinforces the notion that unfavorably-drifting trapped particles pose a serious menace to confinement, and suggests inboard, off-axis radio frequency heating as one means of reducing the size of this population, at least for the case of those energetic trapped particles created during auxiliary heating.

I INTRODUCTION

As present and future-generation toroidal fusion experimental devices approach very low collisionality regimes (i.e, $\nu_{*e}, \nu_{*i} \ll 1$, where ν_{*j} is the well-known ratio of the effective collision frequency to the trapped particle bounce frequency of the j-th species), a particularly potent member of the η_i-driven ($\eta_j = d\ln T_j/d\ln n_j$) family of instabilities is induced by the precessional resonance of trapped ions. This instability,[1,2] which is triggered by unfavorable trapped-ion drifts, (i.e., $\bar{\omega}_{di}\omega_{*i} > 0$, where $\bar{\omega}_{di}$ is the bounce-averaged magnetic precessional drift of the trapped ions, and ω_{*i} is the diamagnetic drift frequency), propagates in the ion drift direction, and has a threshold ($\eta_{i,cr} \simeq 2/3$) that is lower than the conventional trapped-ion mode[3] ($\eta_{i,cr} \simeq 3/2$). It is thus fundamentally different from the usual dissipative trapped ion mode[3] in both the direction of its propagation and its destabilization mechanism. The mode is also different from the well-known fluid η_i-mode.[5] Apart from the obvious difference in collisionality regimes, the fluid η_i-mode is sonically-driven (i.e., parallel compression is crucial to the mode), and electron dynamics is basically irrelevant. The mode under consideration here is curvature-driven, with the trapped electron response taken to be dissipative. As will be seen, electron dissipation will be critically important in the nonlinear dynamics of the mode. Moreover, parallel compression is irrelevant, so that it no longer enters as a determining factor in establishing the mode width. Thus, the mode structure has more of a ballooning character than those characteristics generally associated with conventional drift waves. This mode, then, and the turbulence that may be expected to evolve from it, thus garners concern for future Compact Ignition Torus (CIT)[4] experiments, and as such, merits attention.

In an earlier study, Tagger and Pellat[6] explored the nonlinear saturation of the mode due to *coherent* trapping of the resonant ions in the electrostatic potential of the wave. The physical picture there is one where the wave extracts energy from the resonant part of the ion distribution function *á la* nonlinear Landau damping, thus reducing its slope there to zero ("plateau-ing"). Ion collisions act to destabilize this equilibrium by restoring the trough, and a nonlinear steady state is reached when this process is balanced by linear electron dissipation. While the idea of coherent trapping is an interesting one in its own right, it is unlikely to be a faithful description of what actually occurs in practice. To begin with, the width of the modes is comparable to the spacing between them, so that nonlinear particle orbits can more realistically be expected to be stochastic rather than spatially organized. Secondly, given that electrostatic trapping requires a balance between the $\mathbf{E} \times \mathbf{B}$ drift frequency and the precessional (curvature and gradient-driven) drift frequency (i.e., $\omega_{E\times B} \sim \bar{\omega}_{di}$), which in turn implies $e\delta\phi/T_i \sim O(1)$, it is difficult to understand the result of Ref. 6, which finds $e\delta\phi/T_i \ll O(1)$. Finally, by focusing only on the resonant region of the distribution function, the formulation of Tagger and Pellat does not – indeed, cannot – address how the flattening of the distribution function affects the linear free energy source, i.e., η_i. It is in light of these considerations, that we undertake here to investigate the nonlinear dynamics of the trapped-ion mode from the perspective of *turbulent* trapping of resonant ions. In view of the intractability of following exact particle trajectories, we instead adopt a statistical approach and consider the self-consistent evolution of the two-point correlation function of the trapped-particle distribution function in phase space. The situation we envision is one regulated by interactions between an incoherent noise source driven by trapped ion phase-space density granulations, and dissipation through collisions between electrons. We shall presently elaborate on what is intended by this statement, and how the physical picture that ensues differs from the conventional one based on coherent wave-particle interactions.

To begin with, it has become increasingly apparent that the classical paradigm of a plasma as a system of interacting waves and discrete particles is inadequate and too restrictive a description to be able to account for experimental observations. In particular, the broad frequency spectra at fixed wavenumber observed experimentally, are inconsistent with the normal-mode picture which ignores incoherent noise, and concerns itself only with the dielectric properties of the turbulent medium. In such a description, the non-vanishing of the spectrum requires the dielectric, $\epsilon(k,\omega)$, to be zero:

$$\epsilon_{Re}(k,\omega_k) = 0,$$

$$\gamma_k = -\frac{\epsilon_{Im}}{d\epsilon_{Re}/d\omega_k} \ll \omega_k,$$

resulting, under steady-state conditions, in sharply-peaked collective mode resonances at fixed wave-number:

$$I_k \simeq 2\pi\, \delta(\omega - \omega_k)\, |I_k|,$$

where I_k is the correlation function. A more cogent description of plasma turbulence, as pioneered by Dupree,[7] seeks to account for incoherent, i.e., non-mode-like, fluctuations by adding a third class of participants to the turbulent bath. Theses particle-like fluctuations, dubbed "clumps," are analogous to eddies in fluid turbulence, and are to be understood as phase-space density granulations which arise due to the fact that conservation of phase-space along particle orbits hinders the turbulent intermixing of different densities. This theme is depicted in Fig. 1, which illustrates the granulation in the phase-space distribution function by focusing on the orbit divergence of two initially-adjacent particles in phase-space. Fluctuations in the turbulent bath impart random acceleration to these particles, causing their orbits to diverge. If the particles are sufficiently close together in phase-space, they will turbulently scatter as an integral unit, since the fluctuation they experience is nearly the same. In this way, they remain correlated with each other for a time (called the clump lifetime, τ_{cl}) exceeding the typical correlation time of the turbulence (τ_c). Eventually, of course, they become decorrelated from each other, and their correlation function decays to zero. This decay process is compensated for by the regeneration of fine-scale structure due to the free energy associated with the relaxation of the average distribution function. The relaxation is basically a Fokker-Planck process, whereby the diffusion of a "test" macroparticle-like clump in the turbulent bath is counteracted by a drag, induced by the polarization of the dielectric medium. This process is obviously regulated by momentum conservation, and as such, can yield no net relaxation for a single-species plasma with waves damped. In other words, collisions among members of the *same* species, cannot relax and therefore release the free energy associated with the gradient of the average distribution function. The situation changes dramatically in a two-species plasma configuration, since here the dynamical drag of one species is countered by the diffusion of the *other* species. Stated differently, in diffusing due to fluctuations in the other species, particles of a given species react by dragging on the latter through a polarization (shielding) cloud. In this way, unlike-particle collisions uncover an expansion free energy accessibility mechanism, which acts as a source to regenerate the granularity in the phase-space density distribution function. In previous studies of drift wave turbulence,[8,9] a nonlinear steady state was proposed when the clump source balanced clump decay, i.e., when nonlinear (amplitude-dependent) dissipation due to damped collective modes in one species acted to balance linear wave growth *and* nonlinear noise in the other species. The present work offers a somewhat different scenario. Here, we envisage a *non*-steady state situation (nonlinear instability) developing where clump decay is overcome by the rate of regeneration of the granularity.[10] This condition engenders transport, which in turn, acts to modify equilibrium profiles until such time that the source term balances dissipation, and the nonlinear instability is shut off. What is important to note is that in either situation, it is the presence of incoherent noise as an ingredient in the description of turbulence that is responsible for the broadening of the collective resonances discussed earlier, by forcing the dissipation (or inverse dissipation in the case of nonlinear growth) to adjust to its presence.

In order to help steer the reader through the necessarily detailed analysis, we present here the main physical ideas behind the derivation. We take the electrons to be sufficiently collisional (i.e., $\nu_{eff,e}$ greater than all other frequencies of interest) as to impede the formation of electron clumps. Thus, the electrons are laminar, and the nonlinear clumping mechanism on which we focus our attention is that of trapped ions. It is helpful, at this juncture, to identify some of the important temporal scales in the system. To begin with, an immediate requirement of the low particle-collisionality regimes that we are considering, is that the trapped-particle bounce motion be the fastest time scale in the problem. The next tier in the temporal hierarchy is related to the rate at which two phase-space elements in the turbulent bath (exponentially) decorrelate from each other. This "detuning" process occurs due to *relative* turbulent $\mathbf{E} \times \mathbf{B}$ diffusion and *relative* precessional (banana-center) magnetic drifts of the two particles. The decorrelation rate, which can be thought of as a measure of the granularity or incoherence lifetime, is strongly scale-dependent, peaking at small scales, and decaying to zero at large. We refer to it interchangeably as the clump lifetime or clump decorrelation rate, and denote it by τ_{cl}. There is, additionally, a correlation time associated with the coherent (i.e., mode-like) fluctuations, which acts to limit their nonlinear interaction in the turbulent bath. This time scale, we refer to simply as the correlation time, and denote it by τ_c. Finally, we take note of the fact that the physical process we envisage, namely turbulent trapping, entails a wave-macroparticle resonance ($\omega \sim \bar{\omega}_{di}$), at which rate the trapped ion clumps are ballistically propagated. For the problem at hand, this time scale is the inverse of the energy-dependent trapped-ion precessional drift frequency ($\bar{\omega}_{di}{}^{-1}$), and is longer than the clump decorrelation rate. The evolution of the two-point phase-space density correlation function can then be characterized as proceeding along two disparate time scales: a slow, "centroidal" time scale, determined by the ballistic propagation time of the clump in the wave, and a fast, "relative" time, determined by the clump decorrelation rate. With this intuition in mind, we address the so-called "stochastic acceleration" problem in the first part of the analysis, and proceed to derive an expression for the granularity (clump) decorrelation rate. The result, in the limit of small relative separation, can symbolically be written as [cf. Eq. (22)]

$$\tau_{cl} \simeq \tau_c \ln\Big[(\text{relative separation in phase space})^{-2}\Big],$$

which clearly manifests the strong inhomogeneity of the shearing stresses, as well as the exponentially-divergent behavior of the orbits.

The second part of the analysis concerns itself with the self-consistent interaction between clumps (incoherent noise) and waves (coherent fluctuations). As mentioned earlier, the clump decorrelation process is compensated for by a source term which rekindles granular structure via the relaxation of the average distribution function. More precisely, the rearrangement of phase-space density elements which conspires to produce the more favored average configuration, is channeled into generating granularity, due to the imperfect mixing of regions of different density. The physics of the relaxation can be understood in terms of Fokker-Planck diffusion and drag processes. As a "test" trapped-ion clump is diffusively scattered by coherent fluctuations in the turbulent bath, it attracts a shielding cloud of electrons, which respond to neutralize its presence and maintain quasineutrality (and hence, ambipolarity) in the dielectric medium. Since momentum conservation rules out relaxation via the mechanism of ion-ion collisions (in fact, ion diffusion is exactly cancelled by the dynamical drag imposed on a "test" ion by all the other ions in the bath), it is the very presence of this electron-ion polarization drag that is responsible for releasing the free energy stored in the gradient of the average distribution function. Note the dual role played by electron dissipation in the dynamics: on the one hand, it acts to dampen the collective (normal-mode) response, and on the other, it *enhances* the level of incoherent noise at the expense of the gradient of the average ion distribution function. The logic just outlined serves to indicate that intuition derived from linear theory can be very misleading in predicting finite-amplitude plasma dynamics. The source term can thus be written symbolically as [cf. Eq. (32)]

$$\mathcal{S} \sim \frac{\epsilon_{Im}{}^{i} \epsilon_{Im}{}^{e}}{|\epsilon|^2},$$

where $\epsilon_{Im}{}^{e}$ is the electron dissipation ($\epsilon_{Im}{}^{e} < 0$), and $\epsilon_{Im}{}^{i} \sim (\omega_{*i}/\bar{\omega}_{di})\,[1 + \eta_i(E/T_i - 3/2)]$, and $\epsilon = \epsilon^i + \epsilon^e$. Note that there will be a threshold associated with η_i for the source term to be activated, namely, $\eta_i/\eta_{i,cr} > 1$, where

$$\eta_{i,cr} \equiv \Big(\frac{3}{2} - \frac{E}{T_i}\Big)^{-1}.$$

Finally, the theory is made self-consistent by allowing the incoherent noise thus generated ($\delta\tilde{n}$) to react back on the coherent modes ($\delta\phi$) through quasi-neutrality, and induce even higher collective fluctuation levels [cf. Eq. (27))]:

$$\frac{e\delta\phi}{T_i} = \epsilon^{-1}\,\frac{\delta\tilde{n}}{n_0}.$$

Note that in the absence of incoherent noise, one recovers the usual linear theory result, $\epsilon = 0$.

When the source of incoherent excitations exceeds clump decay processes, a nonlinear instability is instigated with an amplitude-dependent growth rate, which can be written in the symbolic form

$$\gamma_{nl} \sim \tau_c{}^{-1}\Big(\frac{\epsilon_{Im}{}^{e}\,\epsilon_{Im}{}^{i}}{|\epsilon|^2} - 1\Big),$$

where all amplitude-dependence is contained in the (coherent) correlation time. This expression should be compared with the one for the linear growth rate:

$$\gamma_l \sim |\bar{\omega}_{di}|\,\big(\epsilon_{Im}{}^{i} - |\epsilon_{Im}{}^{e}|\big).$$

Once again, note should be made of the very different roles played by electron dissipation in the linear and nonlinear theories.

The instability mechanism just outlined will then lead to anomalous transport. There are then two ways in which a steady state situation can be achieved. The first is for electron dissipation to increase sufficiently enough as to balance incoherent noise emission.[8,9] Since the electrons are laminar, this is not a viable possibility in this case. The only other alternative is for the temperature and density profiles to adjust in such a way as to reduce η_i to within the threshold value for nonlinear instability. This will be the new preferred state about which the plasma will try to maintain itself. The full cycle is schematically illustrated in Fig. 2. The relaxation of the temperature and density profiles will induce other excitation and dissipation processes which lie beyond the scope of the present theory, but which can be expected to maintain the broad spectral profiles. The predictions of the theory strongly underscore the need for pellet injection in CIT and reactor-like configurations. Moreover, given the fact that radio frequency heating in the Ion Cyclotron Range of Frequencies (ICRF) creates a large population of trapped particles whose banana tips lie on the resonance layer, these conclusions reinforce the benefits of moving the resonance layer to the inboard side of the tokamak in order to minimize the number of trapped particles with unfavorable drifts. Once again, it serves to emphasize that the scenario we have just outlined for the nonlinear evolution of the trapped-ion mode is very different from the picture that emerges from a study based on nonlinear, *coherent* wave-particle interactions.[6] It is our opinion that the present theory, in addressing and resolving a number of issues that simply cannot be addressed by anything other than a statisitical theory, provides a more cogent explanation of the situation likely to occur in practice.

We devote the balance of this paper to presenting details of the formal theory which corroborates the heuristic physical discussion just elucidated. In Sec. II, we begin by giving a brief review of the relevant linear theory which, among other things, will serve to set notation for the section that follows. In Sec. III, we present a detailed derivation of the nonlinear evolution equation for the two-point trapped-ion correlation function. The derivation exploits ballooning formalism,[12] and utilizes iterative substitution techniques as in the Direct Interaction Approximation[15] to renormalize the $\mathbf{E}\times\mathbf{B}$ triple-correlation nonlinearity. A transformation to relative and centroidal coordinates serves to highlight the two-scale nature of the problem, and clearly defines the phase-space dimensions which characterize the clump. It is shown that the evolution of clumps is regulated by a source term proportional to the relaxation of the average distribution function, and a decay process due to relative $\mathbf{E}\times\mathbf{B}$ diffusion and magnetic drift. We employ Green's function techniques to solve for the clump decorrelation rate, which we quantitatively define as the time for two very closely-separated particles in phase space to diverge to characteristic clump phase-space dimensions. We then turn to the evaluation of the source term. It is clearly shown how quasineutrality (and by extension, ambipolarity) constrains the trapped-ion source term to scale with dissipation due to electrons. More importantly, an energy-dependent threshold on the magnitude of η_i for arbitrary electron dissipation is derived. If η_i exceeds this threshold, a nonlinear instability sets in which results in the anomalous transport of heat and particle density. An expression for the nonlinear growth rate is then derived. It is found that while the latter is amplitude-dependent, the threshold condition for nonlinear instability is *independent* of amplitudes and depends only on global profiles through η_i. We next turn to transport and provide estimates of the induced heat and particle transport fluxes. It is shown that heat transport is a more efficient mechanism than particle transport, thus serving to saturate the instability by reducing η_i to within its threshold value. Our results are summarized in Sec. IV. The reader should be cautioned that in view of the complexity of the problem, we have not hesitated in making simplifying approximations wherever the need arose in order to make analytical progress. We have pointed out these approximation where they occur, and believe that while rigor has been sacrificed for clarity, the qualitative features of the theory have not been misrepresented.

II Review of Linear Theory

The linear theory of trapped-ion-induced η_i-modes was first discussed by Tagger *et al.*[1] and Tang *et al.*[2] Since the significant majority of trapped particles in tokamak configurations undergo unfavorable drifts, it may be anticipated that they carry potential for destabilization through both resonant and non-resonant interaction with waves set up in the system. These authors recognized the importance of retaining the precessional magnetic drift of the trapped ions in the stability analysis, and found that for very moderate values of η_i, instability can ensue. As an overture to the nonlinear analysis, we devote this section to a review of the linear theory for these modes, focusing our attention on the threshold condition for the onset of instability, and the role that dissipation due to electron collisions plays in influencing this criterion.

We focus on the low particle-collisionality regime, i.e., ν_{*e}, $\nu_{*i} \ll 1$, so that trapping becomes important for both species. The mode frequency is ordered to be comparable to the ion precessional magnetic drift frequency, and we take the electron and ion temperatures to be comparable, so that the effective electron collision frequency is large, while that of the ions is negligible. In summary, the frequency ordering is given by

$$\omega_{bi}, \omega_{ti} \gg \omega_{*i} > |\omega| \sim \omega_{di} \gg \nu_{eff,i},$$

$$\omega_{be}, \omega_{te} \gg \nu_{eff,e} > \omega_{*e} > |\omega|,$$

where ω_b $(\omega_t) = \oint dl/v_\parallel$ is the bounce (transit) frequency, $\omega_* = v_t^2 \hat{\mathbf{e}}_\parallel \cdot \nabla n \times \mathbf{k}_\perp / 2\Omega_c = -k_\theta \rho v_t / 2L_n$ is the diamagnetic drift frequency, $\hat{\mathbf{e}}_\parallel = \mathbf{B}/B$ is the unit vector in the direction of the ambient magnetic field, $\rho = v_t/\Omega_c$ is the Larmor radius, $v_t = (2T/m)^{1/2}$ is the thermal speed, $\Omega_c = eB/mc$ is the cyclotron frequency, $L_n = -(d \ln n/dr)^{-1}$ is the density gradient scale length, $k_\theta = m/r$ is the poloidal wave vector, ω is the mode frequency, $\omega_d = \mathbf{k}_\perp \cdot \mathbf{v}_d \simeq E\, \hat{\mathbf{e}}_\parallel \cdot \boldsymbol{\kappa} \times \mathbf{k}_\perp / \Omega_c = -k_\theta \rho\, E G(\hat{s}, \theta)/R$ is the magnetic drift frequency, $E = v^2/2$ is the energy per unit mass, $\boldsymbol{\kappa} = \hat{\mathbf{e}}_\parallel \cdot \nabla \hat{\mathbf{e}}_\parallel$ is the magnetic curvature, $G(\hat{s}, \theta)$ contains the pitch angle dependence of ω_d and can be expressed in terms of elliptic integrals, $\hat{s} = rq'/q$ is the magnetic shear, $\nu_{eff} = \nu\, (E/T)^{-3/2}/\epsilon_0$ is the energy-dependent, Krook-like effective collision frequency, ν is the usual collision frequency, and $\epsilon_0 = r/R$ is the inverse aspect ratio [the notation is chosen so as to prevent confusion with the dielectric function, $\epsilon(k, \omega)$]. The analysis proceeds from the linearized drift kinetic equation:

$$\delta f_j = -\frac{e}{T_j} F_{Mj} \delta\phi + \delta h_j,$$

$$\Big(\frac{v_\parallel}{qR} \frac{\partial}{\partial \theta} - i\omega + i\omega_{dj} + \nu_{eff,j}\Big)\, \delta h_j = -i\, \frac{e_j}{T_j} \left(\omega - \omega_{*j}^T\right) F_{Mj} \delta\phi, \qquad (1)$$

where we have linearized the particle distribution function about a Maxwellian, i.e., $f_j = F_{Mj} + \delta f_j$, where $F_{Mj} = (n_0/(2\pi T_j)^{3/2} \exp(-E/T_j)$, and we have invoked the electrostatic approximation. The mode is assumed to be localized in a region small relative to the plasma radius, but large compared to the banana width. Thus, we conduct the analysis in the radially local limit and ignore effects associated with finite Larmor radius ($k_\perp \rho_i \ll 1$)

and finite banana width. In Eq. (1), $\omega_{*j}^T = \omega_{*j}[1+\eta_j(E/T_j - 3/2)]$, where $\eta_j = d\ln T_j/d\ln n_j$. To proceed, we note that the non-adiabatic circulating particle response is negligible, i.e., $\delta h_{j,c} \sim O(\omega_{dj}/\omega_{tj}) \ll 1$. This removes a potential dissipation mechanism, namely, Landau-damping on the non-adiabatically-responding circulating ions. The trapped-particle response is obtained by bounce-averaging the drift-kinetic equation:

$$\Big(\omega - \bar{\omega}_{dj}\bar{E} + i\,\frac{\nu_j}{\epsilon_0}\,\bar{E}^{-3/2}\Big)\,\delta h_{j,t} = \frac{e_j}{T_j}\,(\omega - \omega_{*j}^T)\,F_{Mj}\overline{\delta\phi},$$

where an overhead bar denotes bounce-averaging, $\overline{(\cdots)} = \oint d\theta\,(\cdots)/v_\parallel$, $\bar{\omega}_{dj} = k_\theta\rho_j v_j\,G(\hat{s},\theta_0)/2R$, where now $G(\hat{s},\theta_0)$ depends only on the azimuthal coordinate of the turning point, and $\bar{E} = E/T_j$. Note how parallel compression is thus removed from the analysis. The electron and ion densities are then obtained by taking velocity moments of δh_j:

$$\delta n_e = \frac{e\overline{\delta\phi}}{T_e}\,\Big(n_0 - 2\sqrt{2\epsilon_0}\,\{\frac{\omega-\omega_{*e}^T}{i\nu_e/\epsilon_0}\,\bar{E}^{3/2}\,\exp(-\bar{E})\}_v\Big),$$

$$\delta n_i = -\frac{e\overline{\delta\phi}}{T_i}\,\Big(n_0 - 2\sqrt{2\epsilon_0}\,\{\frac{\omega-\omega_{*i}^T}{\omega-\bar{\omega}_{di}\bar{E}}\,\exp(-\bar{E})\}_v\Big)$$

where $2\sqrt{2\epsilon_0}$ is the fraction of trapped particles and the notation $\{\cdots\}_v = \pi^{-1/2}\int d\bar{E}\,\bar{E}^{1/2}$ signifies an integration over velocity. The dispersion relation, $\epsilon(k,\Omega)$ (where $\Omega = \omega/\bar{\omega}_{di}$), is given by the quasineutrality relation:

$$\epsilon(k,\Omega)\,\frac{e\overline{\delta\phi}}{T_i} = \frac{\delta n_e - \delta n_i}{2\sqrt{2\epsilon_0}\,n_0} = 0. \qquad (2)$$

When the velocity integrations are carried out, the resulting expression for $\epsilon(k,\Omega)$ becomes

$$\begin{aligned}\epsilon(k,\Omega) = {}& \frac{1+\tau^{-1}}{2\sqrt{2\epsilon_0}} + \big[\Omega - \epsilon_0^{-1}(1-\tfrac{3}{2}\eta_i)\big]\big[1+\Omega^{1/2}\,Z(\Omega^{1/2})\big] \\ & -\frac{\eta_i}{2\epsilon_0}\,\big[1+2\Omega+2\Omega^{3/2}\,Z(\Omega^{1/2})\big] + i\,\frac{2\epsilon_0\omega_{*i}}{\sqrt{\pi}\,\nu_e}\,(\frac{\epsilon_0}{\tau}\,\Omega + 1 + \frac{3}{2}\eta_e),\end{aligned} \qquad (3)$$

where $\tau = T_e/T_i$, and

$$Z(\zeta) = \pi^{-1/2}\int_{-\infty}^{+\infty} dt\,\frac{\exp(-t^2)}{t-\zeta}$$

is the plasma dispersion function.[12]

This dispersion relation, with $\nu_e \to \infty$, has been solved numerically by Tang *et al.*[2] who find that there is a threshold value associated with η_i given roughly by $\eta_{i,cr} \sim 1$. Since the mode is propagating in the ion direction, electron dissipation can be expected to exert a stabilizing influence on the mode, thus increasing the threshold criterion.

III Formal Nonlinear Theory

For reasons discussed at length in Sec. I, a formal nonlinear theory of plasma dynamics is best couched within a statistical framework which tracks the correlation of two points in phase-space at close separation. Our aim, in this section, is to formulate the equation describing the temporal evolution of the ensemble-averaged two-point correlation function in phase-space. Noting that strong electron collisionality will effectively inhibit the formation of electron clumps, i.e., electrons are laminar, we focus attention only on trapped-ion clump dynamics. We begin with the bounce-averaged nonlinear drift kinetic equation,[12] and employ the ballooning representation for the perturbed quantities:

$$\delta A = \sum_n \exp(in\zeta) \sum_m \exp(-im\theta) \int_{-\infty}^{+\infty} d\vartheta \ \exp[i(m-nq)\,\vartheta + inS]\,\delta A_n(\vartheta)$$

to write it as

$$\Big(\frac{\partial}{\partial t} + \bar{v}_{di}\bar{E}\,\frac{\partial}{\partial y}\Big)\,\delta h = \frac{e}{T_i}\Big(\frac{\partial}{\partial t} + \tilde{\omega}_{*i}^T\,\frac{\partial}{\partial y}\Big)\,F_0\overline{\delta\phi} - \frac{\pi}{cB}\sum_{m'}\sum_{k,k',\omega,\omega'} kk'\hat{s}$$

$$\times(2\pi m'+\vartheta'_k-\vartheta''_k)\exp[i(2\pi m'n'q+ky)]\left(\overline{[\delta\phi^*_{k',\omega'}]}\,\delta h_{k'',\omega''}-\overline{\delta\phi}_{k'',\omega''}\,[\delta h^*_{k',\omega'}]\right), \quad (4)$$

where $k'' = k+k'$, and it is understood that the ϑ variation of the distribution function has been removed by the bounce-averaging. In the above, ζ is the toroidal angle, ϑ is the ballooning coordinate which forms a Fourier-conjugate pair with the radial distance away from the mode rational surface in real (configuration) space, S is the piece of the eikonal that includes the slow envelope radial variation, $\vartheta_k = \partial S/\partial q$, $y = r\zeta/q$, $k = nq/r$, $\bar{E} = E/T_i$, $\bar{\omega}_d = \bar{\omega}_{di}\bar{E}$, $\bar{v}_{di} = (r/q)\,\bar{\omega}_{di}$ is the bounce-averaged magnetic drift speed, $\tilde{\omega}_{*i}^T = (r/q)\,\omega_{*i}^T$, and $[A] = A(\vartheta + 2\pi m')$. The principal advantage of the ballooning representation is that by utilizing the two-scale nature of modes with disparate parallel and perpendicular length scales, it acts to extract the dominant radial variation from the eikonal as $k_r = -k\hat{s}\,(\vartheta - \vartheta_k)$, so that the analysis can be carried out, to leading order, locally on a single flux surface. The slow radial variation enters this leading-order analysis only as a parameter. An appropriate linear superposition of these aperiodic "quasi-modes" can then be constructed to simulate the physically periodic behavior of the true eigenfunction. We will choose the rational surface localized on the outer region of adverse curvature as the reference point for the ballooning coordinate, i.e., $\vartheta_k = 0$. The nonlinear term in Eq. (4) originates from the $\mathbf{E} \times \mathbf{B}$ advection of the phase-space distribution function. This nonlinearity somewhat undermines the utility of the ballooning representation, since it acts to couple different flux surfaces together. This coupling is manifested mathematically by the square-bracketed term in Eq. (4), which signifies that the field inside the brackets is to be evaluated at the rational surface Bloch-shifted away from the one of interest by $2\pi m'$. The net effect of this shift, after renormalization, is to introduce non-local (global) behavior, and thereby significantly complicate the analysis. The flute-like nature of the trapped-ion mode, on the other hand, leads us to believe that the physical picture will not be qualitatively affected by this non-locality. Its neglect amounts to a pessimistic approximation, since the coupling of eigenmodes centered on different rational surfaces to the one at $\vartheta_k = 0$, can be expected to "detune," and hence exert a stabilizing role on the instability. Noting that the renormalization ultimately affects the *decay* but not the *drive* of the two-point correlation function, the physical consequence of keeping track of non-locality can be expected to be a reduction in the clump lifetime. On the other hand, there is another physical process, namely the self-binding of phase-space density structures,[13] that inherently cannot be accounted for in this or any other two-point closure theory, but which acts to *enhance* the clump lifetime. We conjecture, based on experience with numerical simulations,[14] that this is a more important effect than non-locality, and argue that the net result of neglecting these two effects is to arrive at an optimistic estimate on the decay of the correlation function. Following this line of thinking, we are led to ignore non-locality. Finally, we will also ignore any pitch-angle variation, and assume that all trapped particles have unfavorable precessional drift. This is a very good approximation, especially for shear values of order unity.[3] Only *marginally* trapped ions have favorable drifts, but these have high mean collision rates. The approximation is thus well justified.

With these approximations, the time evolution of the two-point equation is obtained upon multiplying the one-point equation for $\delta h(1)$ [where the argument '1' indicates the phase-space point at (r_1, y_1, E_1)] by $\delta h(2)$, ensemble-averaging, and symmetrizing the result. One then arrives at

$$\Big[\frac{\partial}{\partial t} + \bar{v}_{di}\,(\bar{E}_1\,\frac{\partial}{\partial y_1} + \bar{E}_2\,\frac{\partial}{\partial y_2}) + \boldsymbol{T}\Big]\,\langle\delta h(1)\delta h(2)\rangle = \boldsymbol{\mathcal{S}}, \tag{5}$$

where $\langle\cdots\rangle$ denotes an ensemble-average,

$$\begin{aligned}\boldsymbol{T}\langle\delta h(1)\delta h(2)\rangle = \frac{\pi c}{B}\sum_{m'}\sum_{k,k',k'',\omega,\omega',\omega''} k'k''\,(2\pi m')\\ \times\Big[\Big\langle\exp[i(k''-k')y_1 - iky_2]\delta h^*_{k,\omega}(2)\,\overline{[\delta\phi^*_{k',\omega'}(1)]}\delta h_{k'',\omega''}(1)\Big\rangle\exp[2\pi i m' n' q(1)]\hat{s}_1\\ +\Big\langle\exp[i(k'-k'')y_2 + iky_1]\,\delta h_{k,\omega}(1)\,\overline{[\delta\phi_{k',\omega'}(2)]}\delta h^*_{k'',\omega''}(2)\Big\rangle\exp[-2\pi i m' n' q(2)]\hat{s}_2\Big]\end{aligned} \tag{6}$$

is the nonlinear $\mathbf{E}\times\mathbf{B}$ triple correlation, and

$$\begin{aligned}\mathcal{S} = \frac{e}{T_i}\,F_0\Big[\langle\delta h(2)\,\frac{\partial\overline{\delta\phi(1)}}{\partial t}\rangle + \langle\delta h(1)\,\frac{\partial\overline{\delta\phi(2)}}{\partial t}\rangle\Big]\\ +\frac{e}{T_i}\,\Big[\langle\delta h(2)\tilde{\omega}^T_{*i}F_0(1)\,\frac{\partial\overline{\delta\phi(1)}}{\partial y_1}\rangle + \langle\delta h(1)\tilde{\omega}^T_{*i}F_0(2)\,\frac{\partial\overline{\delta\phi(2)}}{\partial y_2}\rangle\Big]\end{aligned} \tag{7}$$

is a source term which drives the evolution of the two-point correlation function. Our objective for the remainder of this section is to evaluate the triplet and source terms, thereby expressing them in forms that make their physical content transparent.

III.A Evaluation of the Triplet Nonlinearity

We first focus on the problem of closing Eq. (5) via an appropriate renormalization procedure. Our approach follows standard methods of renormalized strong turbulence theory, where one iteratively substitutes the field that is driven nonlinearly by the direct beat interaction of a test mode (k,ω) with a mode (k',ω') in the bath of background fluctuations, for the mode (k'',ω'') in

Eq. (6). The small-scale nature of the turbulent correlations is best brought out by making a coordinate transformation to relative and centroidal coordinates:

$$\begin{pmatrix} y_\pm \\ E_\pm \\ r_\pm \end{pmatrix} = \begin{pmatrix} y_1 \\ E_1 \\ r_1 \end{pmatrix} \pm \begin{pmatrix} y_2 \\ E_2 \\ r_2 \end{pmatrix}. \tag{8}$$

The spectral average over the toroidal angle, i.e., $\oint dy_+$, can then be carried out, yielding $\delta(k + k' - k'')$, where $\delta(\cdots)$ is the Dirac delta function. Equation (6) then becomes

$$\begin{aligned} \boldsymbol{T}\langle\delta h(1)\delta h(2)\rangle = \frac{\pi c}{B}\sum_{m'}\sum_{k,k',k'',\omega,\omega'} & kk'\hat{s}_1\,(2\pi m')\,\exp[2\pi i m' n' q(1)] \\ \times\Big(\exp(iky_-)\,\langle\delta h^*_{k,\omega}(2)\,\overline{[\delta\phi^*_{k',\omega'}(1)]}\delta h^{(2)}_{k+k',\omega+\omega'}(1)\rangle & - \exp[-i(k+k')y_-] \\ \times\,\langle\delta h^{(2)}_{k+k',\omega+\omega'}(2)\,\overline{[\delta\phi^*_{k',\omega'}(1)]}\delta h^*_{k',\omega'}(1)\rangle\Big) & + (1 \leftrightarrow 2), \end{aligned} \tag{9}$$

where $\delta h^{(2)}_{k'',\omega''}$ is the leading-order (i.e., beat interaction), phase-coherent part of the nonlinearity, and the notation $(1 \leftrightarrow 2)$ stands for a second set of terms identical to the first, but with arguments interchanged and complex-conjugated, as appropriate. The beat fluctuation is given by

$$\begin{aligned} \delta h^{(2)}_{k+k',\omega+\omega'} = g_{k+k',\omega+\omega'}\left(\frac{\pi c}{B}\right)\sum_m kk'\hat{s}\,(2\pi m)\,\exp(-2\pi i n' m q) \\ \times\left(\overline{[\delta\phi_{k',\omega'}]}\delta h_{k,\omega} - \overline{\delta\phi_{k,\omega}}[\delta h_{k',\omega'}]\right), \end{aligned} \tag{10}$$

where

$$g_{k+k',\omega+\omega'} = \left[-i\,(\omega+\omega') + i\,(\bar{\omega}_{di} + \bar{\omega}'_{di})\,\bar{E} + \tau_c^{-1}\right]^{-1}$$

is the propagator. Substitution of Eq. (10) into the expression for the triplet nonlinearity will yield terms which can broadly be categorized as smoothing (integral-like) and diffusive (differential) in nature. From the discussion in Sec. I, and also Fig. 1, it is clear that the turbulent scattering of trapped-ion clumps, which is the physical process described by Eq. (6), is far more sensitive to singular, diffusive operations than the smoothing, integral-like terms. We are led to retain from the triplet, then, only those contributions leading to diffusion. As discussed by Boutros-Ghali and Dupree,[21] the neglected terms act to reduce diffusion because of momentum constraints, which provides further guarantee that our calculation of the clump lifetime will be an optimistic one. We thus get

$$\mathcal{T}\langle\delta h(1)\delta h(2)\rangle = \left(\frac{\pi c}{B}\right)^2 \sum_{m,m'} \sum_{k,k',\omega,\omega'} (kk'\hat{s}_1)^2 \,(2\pi m)\,(2\pi m')$$

$$\Big[\exp[2\pi i(m'-m)n'q(1) + iky_-]\, g_{k+k',\omega+\omega'}(1)\,\langle\overline{[\delta\phi(1)]}^2\rangle_{k',\omega'}\,\langle\delta h(1)\delta h^*(2)\rangle_{k,\omega}$$

$$+ \exp\Big(2\pi i n'[m'q(1) - mq(2)] - i(k+k')y_-\Big)\, g_{k+k',\omega+\omega'}(2)\,\langle\overline{[\delta\phi^*(1)]}\,\overline{[\delta\phi(2)]}\rangle_{k',\omega'}$$

$$\times\,\langle\delta h^*(1)\delta h(2)\rangle_{k,\omega}\Big] + (1\leftrightarrow 2). \qquad (11)$$

The m'-summation may be eliminated by exploiting the observation that the potential fluctuation spectrum is typically broad in wavenumber. Thus, if we expand the exponent in the second term as follows,

$$n'\,[m'q(1) - mq(2)] = k'r_1\,(m-m') - m'\frac{r_-}{\Delta'},$$

where $\Delta' = (k'\hat{s}_1)^{-1}$, and noting a corresponding factor in the first term, we can anticipate that the k'-summation would force $m = m'$. Finally, cognizant of the fact that the physical process of interest is one of a wave-particle resonant interaction, we may reasonably assume that

$$\langle\delta h^2\rangle_{k,\omega} \simeq \langle\delta h^2\rangle_k\, 2\pi\,\delta(\omega - \bar{\omega}_{di}).$$

The key physical point here is that the "test" clump correlation function decays on a much faster time scale ($\sim \tau_{cl}$) than the time that it takes for it to propagate ballistically from one point to the next (described by the propagator $g_{k',\omega'}$). Since we are interested in constructing an equation for the time evolution of the correlation function, we may approximate the time dependence of $\langle\delta h^2\rangle$ by delta functions *inside the time history integrals*, i.e., inside $\sum_\omega$. This is the so-called Markovian assumption, which allows us to remove summation over ω, and set $g_{k+k',\omega+\omega'} \rightarrow g_{k',\omega'}$. Incorporating these approximations into Eq. (11), we obtain

$$\mathcal{T}\langle\delta h(1)\delta h(2)\rangle = \sum_k k^2\Big[\left(\frac{\pi c}{B}\right)^2\sum_m (2\pi m)^2 \sum_{k',\omega'} k'^2\hat{s}_1^2 g_{k',\omega'}(1)\langle\overline{[\delta\phi(1)]}^2\rangle_{k',\omega'}\Big]$$

$$\times\,\langle\delta h(1)\delta h^*(2)\rangle_{k,\omega} + \sum_k k^2\,\exp(-iky_-)\Big[\left(\frac{\pi c}{B}\right)^2\sum_m (2\pi m)^2\sum_{k',\omega'} k'^2\hat{s}_1^2 g_{k',\omega'}(2)$$

$$\times \exp(-ik'y_- - 2\pi i m r_-/\Delta')\,\langle\overline{[\delta\phi^*(1)]}\,\overline{[\delta\phi(2)]}\rangle_{k',\omega'}\Big]\,\langle\delta h^*(1)\delta h(2)\rangle_{k,\omega} + (1\leftrightarrow 2).$$

Converting back from Fourier to position space,

$$\mathcal{T} = -\Big(\mathcal{D}_{11}\,\frac{\partial^2}{\partial y_1^2} + \mathcal{D}_{12}\,\frac{\partial^2}{\partial y_1 \partial y_2}\Big) + (1 \leftrightarrow 2), \tag{12}$$

where

$$\mathcal{D}_{11} = \Big(\frac{\pi c}{B}\Big)^2 \sum_m (2\pi m)^2 \sum_{k',\omega'} k'^2 \hat{s}_1^2\, g_{k',\omega'}(1)\, \langle \overline{[\delta\phi(1)]}^2 \rangle_{k',\omega'} \tag{13}$$

is the familiar (renormalized) quasilinear diffusion operator which emanates from a one-point renormalized theory, while

$$\mathcal{D}_{12} = \Big(\frac{\pi c}{B}\Big)^2 \sum_m (2\pi m)^2 \sum_{k',\omega'} k'^2 \hat{s}_1^2\, g_{k',\omega'}(2) \times \exp(-ik'y_- - 2\pi i m r_-/\Delta')\, \langle \overline{[\delta\phi^*(1)]}\,\overline{[\delta\phi(2)]} \rangle_{k',\omega'} \tag{14}$$

is a bivariate diffusion operator which is unique to the two-point calculation. Physically, $\mathcal{D}_{11}$ represents the *independent*, i.e., uncorrelated, diffusion of two points in phase-space. Clearly, when these two points are only closely separated from each other, they "feel" very nearly the same level of fluctuation, and hence are turbulently scatterred in unison. This feature, by definition, cannot be recovered from a one-point analysis, and is absent from $\mathcal{D}_{11}$. In the two-point renormalization we have just presented, it manifests itself mathematically in the form of a correlated diffusion operator, $\mathcal{D}_{12}$. The physical structure of Eq. (12) becomes more transparent upon transforming to relative and centroidal coordinates, using Eq. (8):

$$\Big(\mathcal{D}_{11}\,\frac{\partial^2}{\partial y_1^2} + \mathcal{D}_{12}\,\frac{\partial^2}{\partial y_1 \partial y_2}\Big) + (1 \leftrightarrow 2) \simeq (2\mathcal{D} - \mathcal{D}_{12} - \mathcal{D}_{21})\,\frac{\partial^2}{\partial y_-^2},$$

where an implicit average over y_+ has been assumed, as before. The triplet nonlinearity then finally reduces to the following simple form

$$\mathcal{T} \simeq -\mathcal{D}_-\,\frac{\partial^2}{\partial y_-^2}, \tag{15}$$

where

$$\mathcal{D}_- = 2\,(\mathcal{D} - \mathcal{D}_{12}) \simeq 2\mathcal{D}\Big[1 - \cos(k_0 y_- + \frac{r_-}{\Delta_0})\Big] \tag{16}$$

is the relative spatial diffusion operator, $\mathcal{D}$ is given by Eq. (13), and $\Delta_0 \simeq 1/k_0\hat{s}$ and $k_0^{-1} \simeq r_+/(q\langle n^2\rangle^{1/2})$ characterize the mean square spread of the clump in the radial and poloidal directions, respectively. The reader may note that both the radial and poloidal extent of the clump is determined by a single scale size, namely, k_0. This follows from the structure of the ballooning representation discussed earlier, where r_- enters only as a parameter. The reader may note that while our calculation has uncovered spatial diffusion, it has yielded no diffusion in velocity space. Physically, this is due to the fact that trapped particles, in their rapid bounce excursions, average out any velocity-space diffusion. The mathematical manifestation of this fact in the theory occurs when the term $\delta E_{\|}\, \partial/\partial v_{\|} = -\partial \delta\phi/dl\, \partial/\partial v_{\|}$ is annihilated upon bounce-averaging the drift kinetic equation. The behavior of $\mathcal{D}_-$ as a function of relative separation is indicated schematically in Fig. 3. As the relative separation goes to zero, there is no relative diffusion between two points in phase-space. At the other extreme, as the distance between the points increases, they become less and less correlated with each other and asymptotically, diffuse independently of each other. Then, $\mathcal{D}_-$ asymptotes to the constant value of $2\mathcal{D}$. For small enough separation, i.e., $y_- < k_0^{-1}$, $r_- < \Delta_0$, we may make a parabolic expansion for the variation of $\mathcal{D}_-$:

$$\mathcal{D}_- \simeq \mathcal{D}\left(k_0 y_- + \frac{r_-}{\Delta_0}\right)^2. \tag{17}$$

III.B Evaluation of Clump Lifetime

The bivariate diffusion coefficients, obtained in the previous section, give rise to an enhanced correlation between two closely-separated points in phase-space over and above the correlation that exists in the background turbulence. It is the object of the present section to evaluate this so-called clump lifetime or decorrelation time, and show that it exceeds the correlation time of the turbulent fluctuation bath. Having carried out the renormalization of the triplet $\mathbf{E} \times \mathbf{B}$ nonlinearity, we substitute it back into Eq. (5) to obtain

$$\left(\frac{\partial}{\partial t} + \bar{v}_{di}\bar{E}_- \frac{\partial}{\partial y_-} - \mathcal{D}_- \frac{\partial^2}{\partial y_-^2}\right) \langle \delta h(1) \delta h(2) \rangle = \mathcal{S}. \tag{18}$$

Equation (18) states that the evolution of the two-point correlation function in phase-space is regulated by a source term which, as we shall show in the next subsection, is proportional to the relaxation of the average ion distribution function, and a decay process due to relative $\mathbf{E} \times \mathbf{B}$ diffusion and relative magnetic drift. This inhomogeneous equation may be solved by Green's function techniques in the usual way:

$$\left(\frac{\partial}{\partial t} + \bar{v}_{di}\bar{E}_- \frac{\partial}{\partial y_-} - \mathcal{D}_- \frac{\partial^2}{\partial y_-^2}\right) \mathcal{G}(r_-, y_-, \bar{E}_-, t | r'_-, y'_-, \bar{E}'_-, t') = 0, \tag{19}$$

where $\mathcal{G}$ is the Green's function, and the equation is to be solved subject to the initial condition

$$\mathcal{G}(r_-, y_-, \bar{E}_-, t | r'_-, y'_-, \bar{E}'_-, t) = \delta(r_- - r'_-)\delta(y_- - y'_-)\delta(\bar{E}_- - \bar{E}'_-).$$

The two-point correlation function is then given by

$$\langle \delta h(1)\delta h(2)\rangle = \int dr'_- dy'_- d\bar{E}'_- dt' \\ \times \mathcal{G}(r_-, y_-, \bar{E}_-, t | r'_-, y'_-, \bar{E}'_-, t')\, \mathcal{S}(r'_-, y'_-, \bar{E}'_-, t').$$

The exact solution to Eq. (19), given that the decay process depends on scale-dependent relative diffusion, is very difficult. Following Dupree,[7] we instead settle on a different course of analysis which capitalizes on the fact that the clump lifetime is intimately related to the time rate of change of the stochastic divergence of particle orbits. Thus, we can extract the needed information more directly by focusing on the second-moment Langevin equations of stochastic motion. Defining the moments by

$$\langle\langle \cdots \rangle\rangle = \int dr'_- dy'_- dE'_- dt'\, \mathcal{G}(r_-, y_-, \bar{E}_-, t | r'_-, y'_-, \bar{E}'_-, t')\,(\cdots),$$

we can straightforwardly derive

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\langle y_- \rangle\rangle &= \bar{v}_{di}\,\langle\langle \bar{E}_- \rangle\rangle, \\
\frac{\partial}{\partial t}\begin{pmatrix} \langle\langle r_-^n \rangle\rangle \\ \langle\langle \bar{E}_-^n \rangle\rangle \end{pmatrix} &= 0, \qquad\qquad n = 1, 2\,, \\
\frac{\partial}{\partial t}\begin{pmatrix} \langle\langle r_- y_- \rangle\rangle \\ \langle\langle \bar{E}_- y_- \rangle\rangle \end{pmatrix} &= \bar{v}_{di}\begin{pmatrix} \langle\langle r_- \bar{E}_- \rangle\rangle \\ \langle\langle \bar{E}_-^2 \rangle\rangle \end{pmatrix}, \\
\frac{\partial}{\partial t}\langle\langle r_- \bar{E}_- \rangle\rangle &= 0, \\
\frac{\partial}{\partial t}\langle\langle y_-^2 \rangle\rangle &= \langle\langle \mathcal{D}_- \rangle\rangle + 2\bar{v}_{di}\,\langle\langle y_- \bar{E}_- \rangle\rangle.
\end{aligned} \tag{20}$$

The last equation states that time evolution of the mean square toroidal separation is determined by relative $\mathbf{E} \times \mathbf{B}$ diffusion and relative magnetic precession. Equations (20) can be combined to give a single second-order differential equation for $\langle\langle y_-^2 \rangle\rangle$:

$$\frac{\partial^2}{\partial t^2}\langle\langle y_-^2 \rangle\rangle - \tau_c^{-1} \frac{\partial}{\partial t}\langle\langle y_-^2 \rangle\rangle = \frac{2\bar{v}_{di}}{k_0 \Delta_0 \tau_c} \langle\langle r_- \bar{E}_- \rangle\rangle + 2\bar{v}_{di}^2 \langle\langle \bar{E}_-^2 \rangle\rangle, \qquad (21)$$

which is to be solved subject to the initial conditions

$$\langle\langle y_-^2 \rangle\rangle (t=0) = y_-^2,$$

$$\Big[\frac{\partial}{\partial t}\langle\langle y_-^2 \rangle\rangle\Big](t=0) = \tau_c^{-1} \left(y_- + \frac{r_-}{k_0 \Delta_0}\right)^2 + 2\bar{v}_{di} y_- \bar{E}_-.$$

In the above, $\tau_c^{-1} = k_0^2 \mathcal{D}$ is the nonlinear scrambling time of the fluctuation spectrum. Equation (21) has the time-asymptotic ($t > \tau_c$) solution:

$$\langle\langle y_-^2 \rangle\rangle(t) \simeq \Big[2\left(\bar{v}_{di}\tau_c \bar{E}_- + \frac{1}{2}(y_- + \frac{r_-}{k_0\Delta_0})\right)^2 + \frac{1}{2}(y_- + \frac{r_-}{k_0\Delta_0})^2\Big] \exp(t/\tau_c).$$

At this juncture, we define the clump lifetime as the time necessary for two very closely separated points in phase-space to diverge to characteristic clump dimensions, i.e.,

$$k_0^2 \langle\langle y_-^2 \rangle\rangle (t = \tau_{cl}) \equiv 1.$$

With this definition, τ_{cl} can be easily seen to be

$$\tau_{cl} = \tau_c \ln\Big[2\left(k_0 \bar{v}_{di} \tau_c \bar{E}_- + \frac{1}{2}(k_0 y_- + \frac{r_-}{\Delta_0})\right)^2 + \frac{1}{2}(k_0 y_- + \frac{r_-}{\Delta_0})^2\Big]^{-1}, \qquad (22)$$

where the expression is strictly valid only for the argument of the logarithm larger than unity. Two features of Eq. (22) are worthy of the reader's attention. First, the vanishing of the logarithm defines an ellipsoid in phase-space demarcating the physical extent of the clump. Second, the logarithmic dependence of τ_{cl} is indicative of the exponentially-divergent nature of neighboring trajectories. The singularity in Eq. (22) as the relative separation goes to zero, once again reflects the fact that orbits infinitesimally separated from each other will be correlated for an indefinitely long period of time. Finally note that through its dependence on τ_c, τ_{cl} is proportional to the mean-square fluctuation amplitude. This has the eminently reasonable interpretation that the larger the fluctuation amplitudes in the spectrum, the longer the turbulent trapping of clumps in the potential trough of the waves.

III.C Evaluation of the Source

The triplet nonlinearity which has occupied our attention in the last two subsections, physically serves to characterize the degeneration of the microscale statistical correlation between two points in phase-space. In the present subsection, we shift our focus to the process by which the fine-scale granularity is regenerated. The formulation of the problem derives inspiration from the physics of discreteness as described by the Lenard-Balescu equation. Thus, we decompose the distribution function into a coherent and an incoherent response:

$$\delta h_{k,\omega} = \delta h^{(c)}_{k,\omega} + \delta\tilde{h}_{k,\omega}. \tag{23}$$

When substituted into the expression for the source term, the coherent response will give rise to a term which physically accounts for the relaxation of the average distribution function by diffusive mixing due to the turbulent spectrum of ion fluctuations. The substitution of the incoherent response, accounts for the frictional drag exerted on the test trapped-ion clump by the bath of turbulent fluctuations. The similarity of this picture with the Fokker-Planck process described by the Balescu-Lenard turbulent collision operator should not be lost on the reader. The major difference is that here, clumps of correlated ions have replaced the discreteness associated with individual ions. As in the Balescu-Lenard case, momentum conservation requires that same-species diffusion and drag balance. An expansion-free energy source is made accessible by the left-over piece in the polarization drag, which physically represents the Čerenkov emission induced by the dressed ion clump as it moves through the plasma. As we shall see, this free energy source provides the driving energy for the incoherent fluctuations.

To corroborate these statements, we begin by substituting Eq. (23) into Eq. (7), and Fourier analyzing as before:

$$\mathcal{S} = \mathrm{Re}\Big[-i\frac{e}{T_i}\langle f_i(1)\rangle \sum_{k',\omega'} \exp(ik'y_-)\,(\omega' - \omega^{T\,\prime}_{*i}) \\ \times \Big\langle \overline{\delta\phi(1)}\left(\delta h^{(c)}(2) + \delta\tilde{h}(2)\right)^*\Big\rangle_{k',\omega'} + (1 \leftrightarrow 2)\Big]. \tag{24}$$

The first term in $\mathcal{S}$, i.e., the term proportional to $\langle\overline{\delta\phi}\delta h^{(c)*}\rangle$, is the radial diffusion term, which is also obtained in quasilinear theory. We denote it by 'D'. The second piece, i.e., the term proportional to $\langle\overline{\delta\phi}\delta\tilde{h}^*\rangle$, is missing from

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$$\delta n_e - \delta n_i = 0.$$

The ion density fluctuations are given by

$$\begin{aligned}\delta n_i &= -n_0 \frac{e\overline{\delta\phi}_{k,\omega}}{T_i} + 2\sqrt{2\epsilon_0}\,\{\delta h^{(c)}_{k,\omega} + \delta\tilde{h}_{k,\omega}\}^i_v \\ &= -n_0 \frac{e\overline{\delta\phi}_{k,\omega}}{T_i}\,(1 - 2\sqrt{2\epsilon_0}\,\chi^i_{k,\omega}) + 2\sqrt{2\epsilon_0}\,\{\delta\tilde{h}_{k,\omega}\}^i_v,\end{aligned} \tag{25}$$

where

$$\begin{aligned}\chi^i_{k,\omega} &= \pi^{-1/2}\int_0^\infty d\bar{E}\,\bar{E}^{1/2}\,\frac{\omega - \omega^T_{*i}}{\omega - \bar{\omega}_{di}\bar{E}}\,\exp(-\bar{E}) \\ &\simeq \pi^{-1/2}\int_0^\infty d\bar{E}\,\bar{E}^{1/2}\,i\pi\delta(\omega - \bar{\omega}_{di}\bar{E})\,(\omega - \omega^T_{*i})\,\exp(-\bar{E}) \\ &= i\pi^{1/2}\left|\frac{\omega}{\bar{\omega}^3_{di}}\right|^{1/2}(\omega - \omega^T_{*i})\,\exp\Big(-\Big|\frac{\omega}{\bar{\omega}_{di}}\Big|\Big).\end{aligned}$$

The electron density fluctuations are given as in Sec. II:

$$\delta n_e = n_0 \frac{e\overline{\delta\phi}_{k,\omega}}{T_e}\,(1 - 2\sqrt{2\epsilon_0}\,\chi^e_{k,\omega}), \tag{26}$$

where

$$\chi^e_{k,\omega} = -i2\pi^{-1/2}\,\frac{\omega_{*i}\tau}{\nu_e/\epsilon_0}\Big(1 + \frac{3}{2}\eta_e + \frac{\omega}{\omega_{*i}\tau}\Big).$$

Substituting these expressions into quasineutrality, we arrive at

$$\epsilon_{k,\omega}\,\frac{e\overline{\delta\phi}_{k,\omega}}{T_i} = \frac{\delta\tilde{n}_{k,\omega}}{n_0}, \tag{27}$$

where

$$\epsilon_{k,\omega} = \frac{1+\tau^{-1}}{2\sqrt{2\epsilon_0}} - \tau^{-1}\chi^e_{k,\omega} - \chi^i_{k,\omega}$$

is the dielectric, and $\delta\tilde{n}_{k,\omega} = \{\delta\tilde{h}_{k,\omega}\}_v$ is the incoherent part of the ion density fluctuations. Equation (27) is the self-consistent link in the nonlinear cycle shown schematically in Fig. 2, and is the critical point of departure from conventional methods of nonlinear analysis based on coherent mode-coupling [cf. Eq. (2)]. It represents the shielding of the incoherent fluctuations by the eigenmodes, and shows how clump fluctuations can drive collective resonances. We shall use Eq. (27) to re-express the bounce-averaged potential fluctuations in terms of shielded clumps. First consider the coherent term:

$$\langle \overline{\delta\phi(1)}\delta h_i^{(c)*}(2)\rangle_{k',\omega'} = n_0 \frac{e}{T_i^{5/2}} \mathcal{R}^{i*}_{k',\omega'}(2) \langle \overline{\delta\phi(1)}\,\overline{\delta\phi^*(2)}\rangle_{k',\omega'},$$

where $\mathcal{R}$ is the coherent response function

$$\delta h_i^{(c)} = \frac{n_0}{(2\pi T_i)^{3/2}} \mathcal{R}^i_{k,\omega} \frac{e\overline{\delta\phi}}{T_i},$$

and $\chi^i = \{\mathcal{R}^i\}_v$. Using Eq. (27), we have

$$\langle \overline{\delta\phi(1)}\delta h_i^{(c)*}(2)\rangle_{k',\omega'} = \frac{n_0}{eT_i^{1/2}} \mathcal{R}^{i*}_{k',\omega'}(2)\, \epsilon^{-1}_{k',\omega'}(1)\epsilon^{*-1}_{k',\omega'}(2) \langle \frac{\delta\tilde{n}(1)}{n_0} \frac{\delta\tilde{n}^*(2)}{n_0}\rangle_{k',\omega'}.$$

We make use of the fact that $\langle \delta\tilde{n}^2\rangle$ is related to $\langle \delta\tilde{n}\delta\tilde{h}\rangle$ by a velocity moment:

$$\begin{aligned}\langle \delta\tilde{n}(1)\delta\tilde{n}^*(2)\rangle_{k',\omega'} &= \pi^{-1/2}T_i^{3/2}\int_0^\infty d\bar{E}_2\, \bar{E}_2^{1/2}\, \langle \delta\tilde{n}(1)\delta\tilde{h}^*(2)\rangle_{k',\omega'} \\ &= 2\pi^{1/2}T_i^{3/2}\left|\frac{\omega'}{\bar{\omega}'^3_{di}}\right|^{1/2} \langle \delta\tilde{n}(1)\delta\tilde{h}^*(2)\rangle_{k'},\end{aligned}$$

where we have used

$$\begin{aligned}\langle \delta\tilde{n}(1)\delta h^*(2)\rangle_{k',\omega'} &= \left[g_{k',\omega'}(1) + g^*_{k',\omega'}(2)\right] \langle \delta\tilde{n}(1)\delta h^*(2)\rangle_{k'} \\ &\simeq 2\,\mathrm{Re}\, g_{k',\omega'}(+)\, \langle \delta\tilde{n}(1)\delta h^*(2)\rangle_{k'}\end{aligned}$$

to relate the two-time and equal-time correlation functions. Using this result together with

$$\begin{aligned}\mathcal{R}^{i*}_{k',\omega'}(2) &= -i\pi^{1/2}\,\delta(\omega' - \bar{\omega}_{di}{}'\bar{E}_+)\,(\omega' - \omega^{T'}_{*i})\,\exp(-|\omega'/\bar{\omega}'_{di}|) \\ &= \pi^{1/2}\,\delta(\omega' - \bar{\omega}_{di}{}'\bar{E}_+)\left|\frac{\bar{\omega}'^3_{di}}{\omega'}\right|^{1/2} \chi^{i*}_{k',\omega'},\end{aligned}$$

we finally arrive at

$$\langle\overline{\delta\phi(1)}\delta h^{(c)*}(2)\rangle_{k',\omega'} = \frac{n_0 T_i}{e}\, 2\pi\, \delta(\omega' - \bar{\omega}_{di}{}'\bar{E}_+)\, \chi^{i*}_{k',\omega'}(2)$$
$$\times\, \epsilon^{-1}_{k',\omega'}(1)\, \epsilon^{*\,-1}_{k',\omega'}(2)\, \langle \frac{\delta\tilde{n}(1)}{n_0} \frac{\delta\tilde{h}^*(2)}{n_0} \rangle_{k'}. \qquad (28)$$

Next, we turn to the evaluation of the polarization drag term induced by the incoherent response. In the spirit of the calculation we have just made, we have

$$\langle\overline{\delta\phi(1)}\delta\tilde{h}^*(2)\rangle_{k',\omega'} = \frac{n_0 T_i}{e}\, 2\pi\, \delta(\omega' - \bar{\omega}_{di}{}'\bar{E}_+)\, \epsilon^{-1}_{k',\omega'}(1)\, \langle \frac{\delta\tilde{n}(1)}{n_0} \frac{\delta\tilde{h}^*(2)}{n_0} \rangle_{k'},$$

Inserting the unity operator $\epsilon^{*\,-1}_{k',\omega'}(2)\, \epsilon^{*}_{k',\omega'}(2)$ into the above expression, we obtain

$$\langle\overline{\delta\phi(1)}\delta\tilde{h}^*(2)\rangle_{k',\omega'} = \frac{n_0 T_i}{e}\, 2\pi\, \delta(\omega' - \bar{\omega}_{di}{}'\bar{E}_+)\, \Big[\frac{1+\tau^{-1}}{2\sqrt{2\epsilon_0}}$$
$$-\tau^{-1}\chi^{e*}_{k',\omega'}(2) - \chi^{i*}_{k',\omega'}(2)\Big]\, \epsilon^{-1}_{k',\omega'}(1)\, \epsilon^{*\,-1}_{k',\omega'}(2)\, \langle \frac{\delta\tilde{n}(1)}{n_0} \frac{\delta\tilde{h}^*(2)}{n_0} \rangle_{k'}. \qquad (29)$$

Substituting expressions (28) and (29) into Eq. (24), we note that the ion piece of the polarization drag exactly cancels the radial diffusion, so that our final expression for the source term becomes

$$\mathcal{S} \simeq n_0 \sum_{k',\omega'} \Big(\omega' - \omega_{*i}{}' [1 + \eta_i\, (\bar{E}_+ - \frac{3}{2})]\Big)\, \frac{\operatorname{Im} \epsilon^{e}_{k',\omega'}}{|\epsilon_{k',\omega'}|^2} \Big\langle \frac{\delta\tilde{n}}{n_0} \frac{\delta\tilde{h}^*}{n_0} \Big\rangle_{k',\omega'} F_{Mi}(\bar{E}_+), \quad (30)$$

where $\operatorname{Im} \epsilon^{e}_{k',\omega'} = -\tau^{-1} \operatorname{Im} \chi^{e}_{k',\omega'} < 0$. Equation (30) has incorporated into it all the features advertised in Sec. I. Two points are particularly noteworthy. First, is the fact that the source term is proportional to electron dissipation. The basic physical reason for this proportionality can be understood as follows: As the ion distribution function relaxes by the scattering of trapped-ion clumps down the average gradient, electrons are impelled to respond in order to redress the charge imbalance, and thereby maintain quasineutrality. Thus, quasineutrality (and by extension, ambipolarity) constrains ion relaxation to scale with electron dissipation. *Prima facie*, it may come as a surprise that electron dissipation works towards *enhancing* rather than reducing the source term. A cogent physical explanation for this behavior can be sought in the induced emission process, as explained in Sec. I. It serves to point out the pitfalls of extrapolating intuition gleaned from linear theory to predicting finite-amplitude plasma dynamics. Secondly, the fact that $|\omega|/\omega_{*i}^T \sim \bar{\omega}_{di}/\omega_{*i} \sim \epsilon < 1$, implies that the source term is proportional to the free energy stored in the radial gradient of the trapped-ion average distribution function. Thus, a (positive) source will be available to extract energy from the equilibrium gradient through the ion clump channel when

$$\frac{\eta_i}{\eta_{i,cr}(\bar{E}_+)} > 1,$$

where

$$\eta_{i,cr}(\bar{E}_+) \equiv (\frac{3}{2} - \bar{E}_+)^{-1}. \tag{31}$$

This energy-dependent threshold is depicted in Fig. 4. It shows that for any value of η_i, there will be some energy range within which ion clumps will contribute to the source term. This again is in marked contrast to the predictions of the linear theory, and suggests the possibility of a subcritical (i.e., nonlinear) instability. We may thus write our final expression for the source term in the suggestive form

$$\mathcal{S} \simeq 2\pi\, n_0 \sum_{k'} \omega_{*i}{}' \left(\frac{\eta_i}{\eta_{i,cr}(\bar{E}_+)} - 1\right) \frac{\mathrm{Im}\, \epsilon^e_{k',\bar{\omega}'_{di}\bar{E}_+}}{|\epsilon_{k',\bar{\omega}'_{di}\bar{E}_+}|^2} \langle \frac{\delta\tilde{n}}{n_0} \frac{\delta\tilde{h}^*}{n_0} \rangle_{k'} F_{Mi}(\bar{E}_+), \tag{32}$$

A more convenient expression for the source, which would help expedite the analytic manipulations in the next section, can be obtained by looking at the limit of a continuum of modes, i.e., $\sum_{k'} \to \int dk'$. We expand the dielectric using the pole approximation

$$\epsilon_{k',\bar{\omega}'_{di}\bar{E}_+} = \left[k' - k(\bar{\omega}_{di}\bar{E}_+)\right] \frac{\partial}{\partial k'} \mathrm{Re}\, \epsilon_{k',\omega'}\Big|_{k'=k(\bar{\omega}_{di}\bar{E}_+)} + i\, \mathrm{Im}\, \epsilon_{k,\bar{\omega}_{di}\bar{E}_+},$$

where we have used $\mathrm{Re}\, \epsilon_{k(\bar{\omega}_{di}\bar{E}_+),\bar{\omega}_{di}\bar{E}_+} = 0$. This leads to

$$\mathcal{S} \simeq 2\pi\, n_0 v_* \left(\frac{\eta_i}{\eta_{i,cr}(\bar{E}_+)} - 1\right)$$
$$\times \int dk'\, k' \frac{|\epsilon_{Im}{}^e|}{(k'-k)^2\, |\partial\epsilon_{Re}/\partial k|^2 + |\epsilon_{Im}|^2} \langle \frac{\delta\tilde{n}}{n_0} \frac{\delta\tilde{h}^*}{n_0} \rangle_{k'} F_{Mi}(\bar{E}_+).$$

Further simplification is effected by treating the denominator as a resonance function, so that the numerator is sampled at k. Our reduced expression is then given by

$$\mathcal{S} \simeq 2\pi^2 n_0 \omega_{*i} \Big(\frac{\eta_i}{\eta_{i,cr}(\bar{E}_+)} - 1\Big) \frac{\bar{\epsilon}_{Im}{}^e}{|\partial \bar{\epsilon}_{Re}/\partial k|\, |\bar{\epsilon}_{Im}|} \langle \frac{\delta \tilde{n}}{n_0} \frac{\delta \tilde{h}^*}{n_0} \rangle_k F_{Mi}(\bar{E}_+), \qquad (33)$$

where the superbars over the dielectric functions indicate that they are to be evaluated at $\Omega = \bar{E}_+$.

III.D Nonlinear Instability

The two-point correlation function, $\langle \delta h(1) \delta h(2) \rangle$, is composed of three parts: a (coherent) piece that contains information about collective plasma interactions, an (incoherent) piece that contains information pertinent to singular behavior at short separation, and finally, cross-correlations between the two. The quantity that deserves special attention is the second one, and we may extract it from $\langle \delta h(1) \delta h(2) \rangle$ as follows:

$$\langle \delta \tilde{h}(1) \delta \tilde{h}(2) \rangle = \langle \delta h(1) \delta h(2) \rangle - \langle \delta h^{(c)}(1) \delta \tilde{h}(2) \rangle - \langle \delta \tilde{h}(1) \delta h^{(c)}(2) \rangle - \langle \delta h^{(c)}(1) \delta h^{(c)}(2) \rangle. \qquad (34)$$

From the developments in the previous sections, we may write down the equation describing the evolution of the full correlation function:

$$\Big(\frac{\partial}{\partial t} + \tau_{cl}{}^{-1}\Big) \langle \delta h(1) \delta h(2) \rangle = \mathcal{S}, \qquad (35)$$

where τ_{cl} and $\mathcal{S}$ are given by Eqs. (22) and (33), respectively. An equation for the coherent piece of the two-point correlation function is obtained by following through manipulations similar to those carried out in Sec. III. The result is

$$\Big(\frac{\partial}{\partial t} + \bar{v}_d \bar{E}_- \frac{\partial}{\partial y_-} - \mathcal{D}_+ \frac{\partial^2}{\partial y_-^2}\Big) \langle \delta h^{(c)}(1) \delta h^{(c)}(2) \rangle = 2D,$$

where $\mathcal{D}_+ \simeq 2\mathcal{D}$, as given by Eq. (13), and D is the radial diffusion piece of the source term [cf. the comments below Eq. (24)]. Note that $\mathcal{D}_+$ is simply the uncorrelated (i.e., large relative separation in phase-space) limit of $\mathcal{D}_-$, as one would expect on physical grounds. Inverting the left-hand-side of the above equation in the same manner as was done in Sec. III.B, we obtain

$$\Big(\frac{\partial}{\partial t} + \tau_c^{-1}\Big) \langle \delta h^{(c)}(1) \delta h^{(c)}(2) \rangle = 2D, \tag{36}$$

where, as before, $\tau_c^{-1} = k_0^2 \mathcal{D}$ is the correlation time of the fluctuations (cf. Sec. III.C). The same analysis may be carried through for the cross-correlation, with the result[7]

$$\Big(\frac{\partial}{\partial t} + \tau_c^{-1}\Big) \langle \delta \tilde{h}(1) \delta h^{(c)}(2) \rangle = F, \tag{37}$$

where F is the polarization drag piece of the source term [cf. comments below Eq. (24)]. Using Eqs. (35), (36), and (37) in Eq. (34), we arrive at an equation describing the nonlinear dynamics of the singular part of the two-point correlation function:

$$\langle \delta \tilde{h}(1) \delta \tilde{h}(2) \rangle \simeq \Big(\frac{\tau_{cl}}{\gamma_{nl}\tau_{cl} + 1} - \frac{\tau_c}{\gamma_{nl}\tau_c + 1} \Big) S, \tag{38}$$

where γ_{nl} is the nonlinear growth rate. In order to solve for the latter, we need to relate $\langle \delta \tilde{h} \delta \tilde{h} \rangle$ to $\langle \delta \tilde{n} \delta \tilde{h} \rangle_k$, which appears in the expression for the source term. We use the following two results:

$$\begin{aligned}
\langle \delta \tilde{n} \delta \tilde{n}^* \rangle_{k,\omega} &\simeq \frac{T_i^3}{\pi} \int d\bar{E}_- d\bar{E}_+ \, \bar{E}_+ \, \langle \delta \tilde{h} \delta \tilde{h}^* \rangle_{k,\omega} \\
&= \frac{T_i^3}{\pi} \int d\bar{E}_- d\bar{E}_+ \, \bar{E}_+ \, 2\pi \delta(\omega - \bar{\omega}_{di} \bar{E}_+) \, \langle \delta \tilde{h} \delta \tilde{h}^* \rangle_k \\
&= 2T_i^3 \, \frac{|\omega|}{\bar{\omega}_{di}{}^2} \int d\bar{E}_- \int dy_- \, \exp(iky_-) \Big\langle \delta \tilde{h} \delta \tilde{h}^* \Big| \bar{E}_+ = |\omega/\bar{\omega}_{di}| \Big\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle \delta \tilde{n} \delta \tilde{n}^* \rangle_{k,\omega} &= \frac{T_i^{3/2}}{\sqrt{\pi}} \int d\bar{E}_+ \, \bar{E}_+^{1/2} \, \langle \delta \tilde{n} \delta \tilde{h}^* \rangle_{k,\omega} \\
&= \frac{T_i^{3/2}}{\sqrt{\pi}} \int d\bar{E}_+ \, \bar{E}_+^{1/2} \, 2\pi \delta(\omega - \bar{\omega}_{di} \bar{E}_+) \, \langle \delta \tilde{n} \delta \tilde{h}^* \rangle_k \\
&= 2\sqrt{\pi} \Big| \frac{\omega}{\bar{\omega}_{di}{}^3} \Big|^{1/2} T_i^{3/2} \Big\langle \delta \tilde{n} \delta \tilde{h}^* (\bar{E}_+ = |\omega/\bar{\omega}_{di}|) \Big\rangle_k .
\end{aligned}$$

Comparing these two relations, we can write

$$\left\langle \delta\tilde{n}\delta\tilde{h}^*(\bar{E}_+ = |\omega/\bar{\omega}_{di}|)\right\rangle_k = \frac{T_i^{3/2}}{\pi}\left|\frac{\omega}{\bar{\omega}_{di}}\right|^{1/2} \times \int d\bar{E}_- \int dy_- \ \exp(-iky_-)\left\langle \delta\tilde{h}\delta\tilde{h}^* \middle| \bar{E}_+ = |\omega/\bar{\omega}_{di}|\right\rangle. \qquad (39)$$

Thus, upon integrating the evolution equation for the incoherent fluctuation correlation function over E_-, Fourier-transforming the result with respect to y_-, using Eq. (39), and substituting Eq. (33) for the source, we get the following nonlinear dispersion relation

$$\left|2\frac{\omega}{\bar{\omega}_{di}}\right|^{1/2}\omega_{*i}\left(\frac{\eta_i}{\eta_{i,cr}(|\omega/\bar{\omega}_{di}|)} - 1\right)\frac{{\epsilon_{Im}}^e}{|\partial\epsilon_{Re}/\partial k|\,|\epsilon_{Im}|}\ \exp(-|\omega/\bar{\omega}_{di}|) \times \int dy_- \ \exp(-iky_-)\int d\bar{E}_- \left(\frac{\tau_{cl}}{\gamma_{nl}\tau_{cl}+1} - \frac{\tau_c}{\gamma_{nl}\tau_c+1}\right) = 1, \qquad (40)$$

where ω is the solution to the linear dispersion relation as discussed in Sec. II. Due to the appearance of the transcendental function, τ_{cl}, in Equation (40), it is intractable to obtain a closed-form analytic expression for the nonlinear growth rate. However, we may obtain an approximate solution by making the simplification $\gamma_{nl}\tau_{cl}+1 \simeq \gamma_{nl}\tau_c+1$ in the first denominator in the integrand. One then gets

$$\gamma_{nl} \simeq {\tau_c}^{-1}\left[\left|2\frac{\omega}{\bar{\omega}_{di}}\right|^{1/2}\aleph(k)\left(\frac{\eta_i}{\eta_{i,cr}(|\omega/\bar{\omega}_{di}|)}-1\right)\frac{{\bar{\epsilon}_{Im}}^e}{|\partial\bar{\epsilon}_{Re}/\partial k|\,|\bar{\epsilon}_{Im}|}\ \exp(-|\omega/\bar{\omega}_{di}|)-1\right], \qquad (41)$$

where

$$\aleph(k) = \omega_{*i}\int dy_- \ \exp(-iky_-)\int d\bar{E}_- \ (\tau_{cl} - \tau_c).$$

Note that unlike linear theory, the growth rate here is amplitude-dependent. The threshold condition (i.e., $\gamma_{nl} = 0$), however, is *independent* of amplitudes and depends only on global profiles (i.e., through η_i). To see this, first note that since the electrons are laminar, ${\epsilon_{Im}}^e$ cannot be amplitude-dependent. The only other place where amplitude dependence can enter is in $\aleph(k)$. The energy integral,

$$\aleph(y_-) = \int d\bar{E}_- \ (\tau_{cl} - \tau_c) = -\tau_c\int d\bar{E}_- \ \ln\left[\left(\sqrt{\frac{2}{e}}k_0\bar{v}_d\tau_c\bar{E}_- + l(r_-,y_-)\right)^2 + l^2(y_-,r_-)\right],$$

where $l(y_-, r_-) = (k_0 y_- + r_-/\Delta_0)/\sqrt{2e}$, and the limits of integration are $\sqrt{e}\,[\pm(1-l^2)^{1/2} - l]/\sqrt{2}k_o\bar{v}_d\tau_c$, can be easily carried out, yielding

$$\aleph(y_-) = \frac{4\sqrt{e/2}}{k_0\bar{v}_d}\left[(1-l^2)^{1/2} - |l|\cos^{-1}|l|\right].$$

Note that amplitude dependence has cancelled out. This is due to the fact that the clump velocity scale is determined by the decorrelation time [i.e., $E_- \propto {\tau_c}^{-1}$, cf. Eq. (22)]. This suggests that the only mechanism by which this nonlinear instability may be quenched is by profile modification through transport processes.

III.E Transport

The nonlinear instability we have just described gives rise to transport. In this section, we provide estimates of the transport fluxes that may be expected to ensue. To obtain these, we must take appropriate velocity moments of the nonlinear evolution equation for the equilibrium distribution function. The latter may be derived quite simply by noting that conservation of the phase space distribution function along particle orbits, constrains the evolution of the equilibrium distribution function to exactly balance that of the fluctuating component, i.e.,

$$\langle f_i\rangle \frac{\partial}{\partial t}\langle f_i\rangle = -\frac{1}{2}\frac{\partial}{\partial t}\langle \delta f_i^2\rangle.$$

Using the results of the preceding subsections, for small relative separation we then have

$$\frac{\partial\langle f_i\rangle}{\partial t} \simeq -\pi\, n_0 \sum_{k'} \omega_{*i}{}' \left(\frac{\eta_i}{\eta_{i,cr}(\bar{E}_+)} - 1\right) \frac{\operatorname{Im}\epsilon^e_{k',\bar{\omega}'_{di}\bar{E}_+}}{|\epsilon_{k',\bar{\omega}'_{di}\bar{E}_+}|^2} \langle \frac{\delta\tilde{n}}{n_0}\frac{\delta\tilde{h}^*}{n_0}\rangle_{k'}. \tag{42}$$

The particle and thermal fluxes are thus given by

$$\begin{pmatrix}\Gamma_i\\ Q_i\end{pmatrix} \simeq (2\epsilon_0)^{3/2}\pi^{-1/2}\,T_i^{3/2}\left(-\frac{dn_i}{dr}\right)\sum_{k'}\frac{\omega'^2_{*i}}{\nu_e}\int d\bar{E}_+\,\bar{E}_+^{1/2}$$
$$\times\begin{pmatrix}1\\ T_i\bar{E}_+\end{pmatrix}\frac{\left[(\epsilon_0/\tau)\,\bar{E}_+ + 1 + (3/2)\,\eta_e\right]\left[\eta_i/\eta_{i,cr}(\bar{E}_+) - 1\right]}{|\epsilon_{k',\bar{\omega}'_{di}\bar{E}_+}|^2}\langle\frac{\delta\tilde{n}}{n_0}\frac{\delta\tilde{h}^*}{n_0}\rangle_{k'}.$$

Needless to say, the gargantuan integrals appearing above are analytically intractable. However, we may make an order of magnitude estimate by invoking a simple mixing-length argument for the two-point correlation function. We note that the macroscopic length scale is that associated with the gradient of the average distribution function. Since the nonlinear instability is driven by the *temperature* gradient, the relevant scale is then $L_T = -(d \ln T_i/dr)^{-1}$. Thus,

$$\frac{\delta \tilde{n}}{n_0} \sim \frac{\Delta x_-}{L_T} \sim (k\hat{s}L_T)^{-1},$$

and

$$\frac{\delta \tilde{h}}{n_0} \sim \frac{\delta \tilde{n}}{n_0 T_i \, \Delta v_-} \sim \frac{\delta \tilde{n}}{n_0 T_i^{3/2} \sqrt{\bar{E}_-}} \sim \frac{\sqrt{\bar{\omega}_{di} \tau_c}}{T_i^{3/2} k\hat{s}L_T},$$

where $(\Delta x_-, \Delta v_-)$ are the clump scales in phase space. As an *upper-bound* estimate for the decorrelation time, we use the inverse of the precessional drift frequency. Finally, we note that clump production peaks about the resonance, i.e., $\bar{E}_+ = |\omega/\bar{\omega}_{di}|$. Then, our mixing-length estimate for the two-point correlation function can be written as

$$T_i^{3/2} \langle \frac{\delta \tilde{n}}{n_0} \frac{\delta \tilde{h}}{n_0} \rangle \sim (k\hat{s}L_T)^{-2} \, \delta(\bar{E}_+ - |\omega/\bar{\omega}_{di}|).$$

With this estimate, an order-of-magnitude expression for the fluxes can be given as

$$\begin{pmatrix} \Gamma_i \\ Q_i \end{pmatrix} \sim - \begin{pmatrix} 1 \\ |\omega/\bar{\omega}_{di}| T_i \end{pmatrix} \epsilon^{3/2} \frac{\omega_{*i}^2}{\nu_e \, (k\hat{s}L_T)^2} \left[\frac{\eta_i}{\eta_{i,cr}(|\omega/\bar{\omega}_{di}|)} - 1 \right] \frac{dn_i}{dr}. \tag{43}$$

Two points emerge from the above expressions. First, we conclude that the thermal flux per unit temperature is larger than the particle flux if $|\omega/\bar{\omega}_{di}| > 1$. When juxtaposed with the condition for nonlinear instability, the physically relevant frequency regime becomes $3/2 > |\omega/\bar{\omega}_{di}| > 1$. Above 3/2, there is no nonlinear instability; below 1, there is no possibility of nonlinear saturation. The second point to note is that the magnitudes of the fluxes are very large. Taken together, these two points suggest that with the onset of this nonlinear instability, transport processes will set in and act to rapidly reconfigure the equilibrium temperature and density profiles in such a way as to return η_i to marginality. Very large heating would be necessary to push η_i above $\eta_{i,cr}$, so that we may expect η_i to remain very close to its threshold value.

IV CONCLUSION

We have sought in this work, to investigate the nonlinear evolution of the trapped-ion temperature-gradient-driven mode. In contrast to the usual dissipative trapped-ion mode, the present mode propagates in the *ion* rather than the electron direction, thus allowing the possibility of resonant interaction between wave and particle ($\omega \sim \bar{\omega}_{di}$). Unlike an earlier attempt which considered the same problem from the point of view of *coherent* trapping of resonant ions in the electrostatic potential of the wave, we have been motivated by the fact that the strongly-overlapping nature of the modes is more likely to result in stochastic rather than spatially-ordered motion of the particle trajectories. Such a situation is then more appropriately analyzed within the framework of statistical theory. By following the self-consistent nonlinear evolution of the two-point correlation function for the trapped-ion distribution function in phase space, we have derived conditions under which a nonlinear (i.e., amplitude-dependent) instability, with a threshold on η_i, may occur. Any departure from this nonlinearly marginal state can be expected to lead to large levels of anomalous heat and particle transport, which by modifying the equilibrium temperature and density profiles, respectively, act as a negative feedback mechanism, and return η_i back to its threshold value. In some sense, then, the nonlinear instability is quite robust and accentuates the need for pellet injection in CIT and other reactor-like environments. We remark that the experimental signature of this resonant instability would probably show a great resemblance to the observations of "fishbone" activity.[16] As with the latter,[17–21] the present theory further underscores the undesirable consequences of confining a population of unfavorably-precessing trapped species, and suggests that attention needs to be focused on possible means by which the size of this population can be minimized. At least for the case of energetic particles produced by radio frequency auxiliary heating, one possibility that has been suggested has been to move the resonance layer to the inboard side of the tokamak.

ACKNOWLEDGMENTS

This work was supported by the U. S. Department of Energy Contract #DE-FG05-80ET-53088.

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FIGURE CAPTIONS

Fig. 1 Schematic illustration for the generation of granularity due to the conservation of the phase space distribution function along particle trajectories.

Fig. 2 Schematic of the nonlinear analysis.

Fig. 3 Relative diffusion as a function of relative separation.

Fig. 4 Nonlinear η_i threshold as a function of energy.

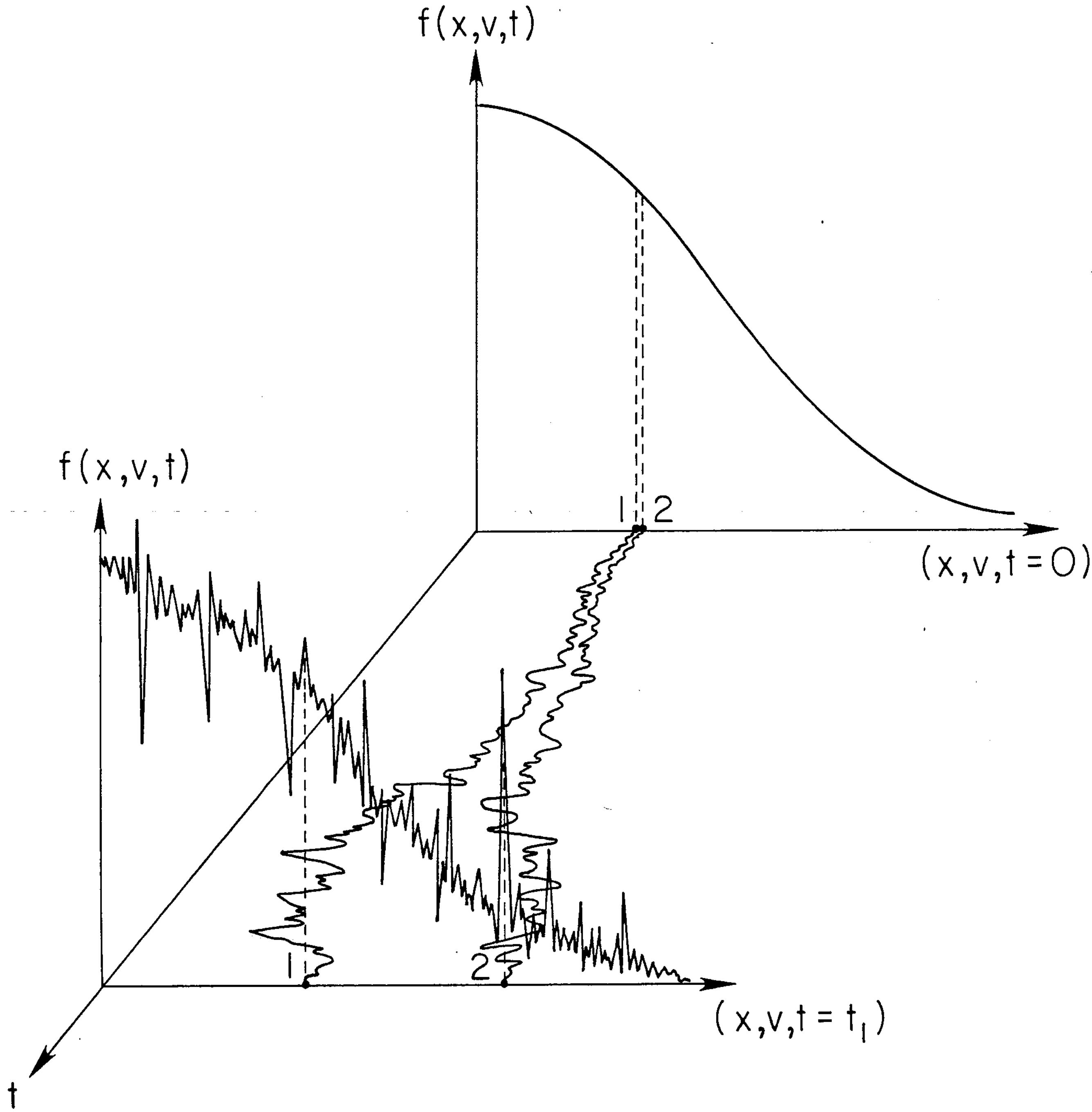

Fig. 1

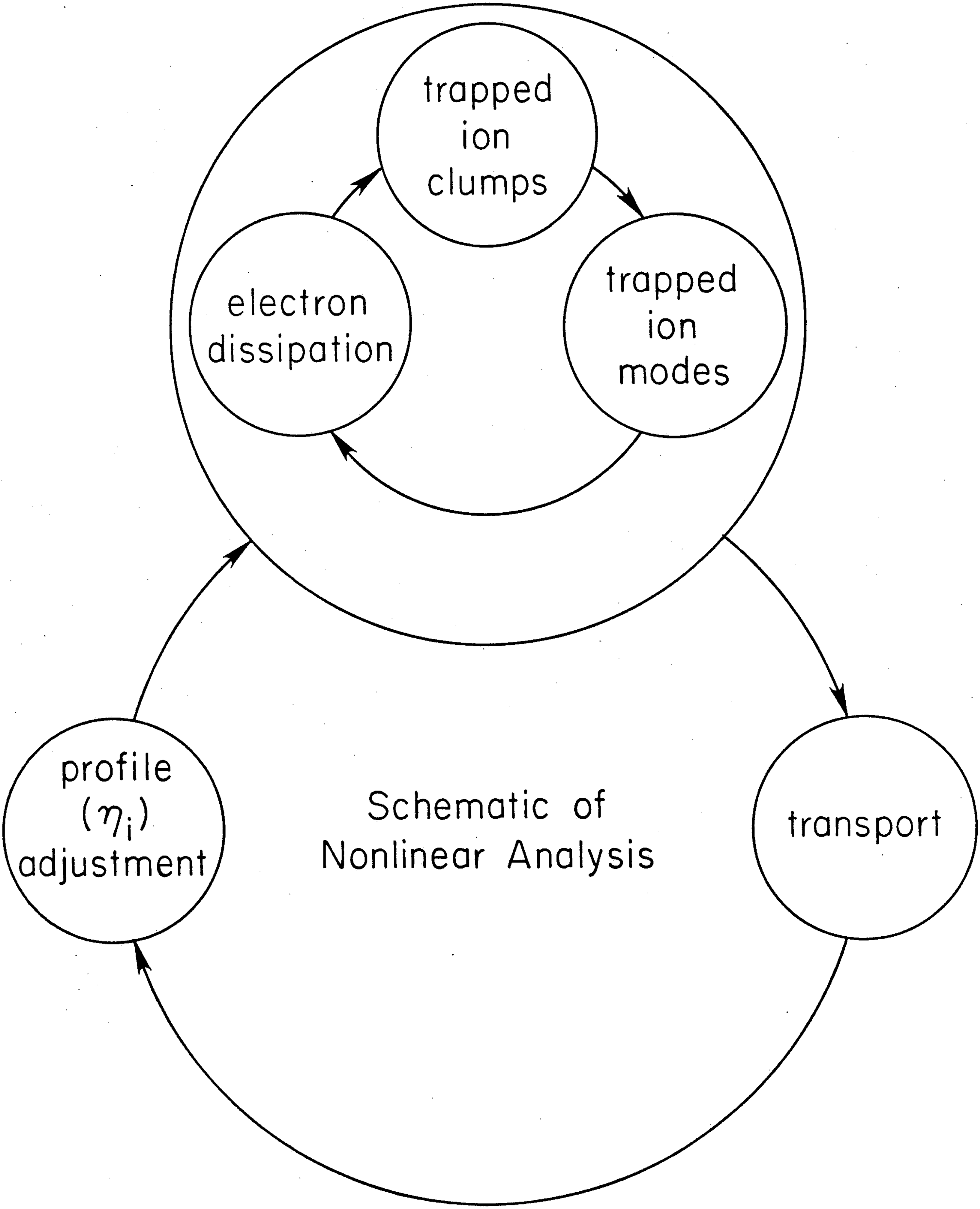
trapped
ion
clumps
electron
dissipation
trapped
ion
modes
profile
(η_i)
adjustment
Schematic of
Nonlinear Analysis
transport

Fig. 2

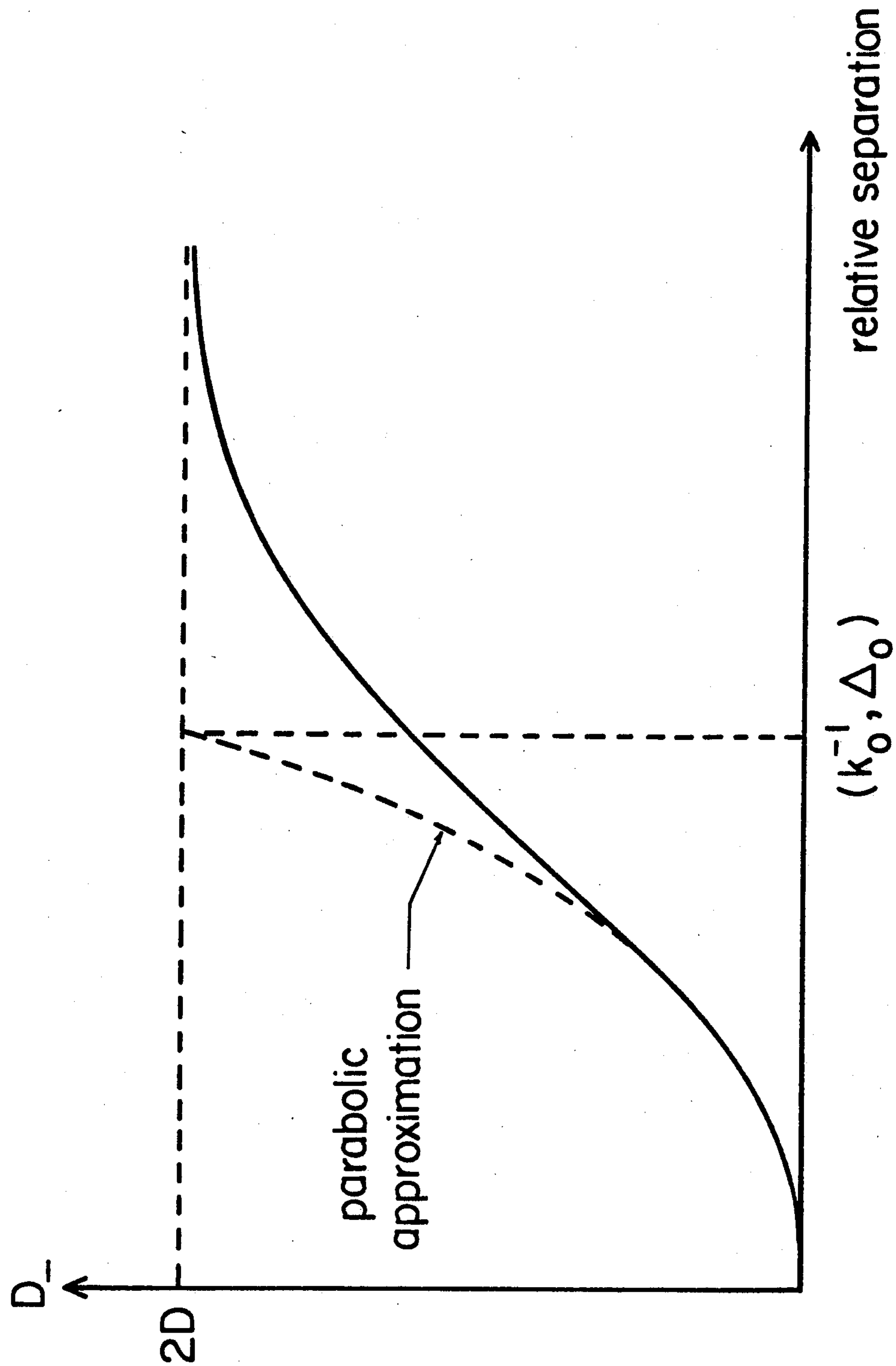
D_
2D
parabolic approximation
(k_o^{-1}, Δ_o)
relative separation

Fig. 3

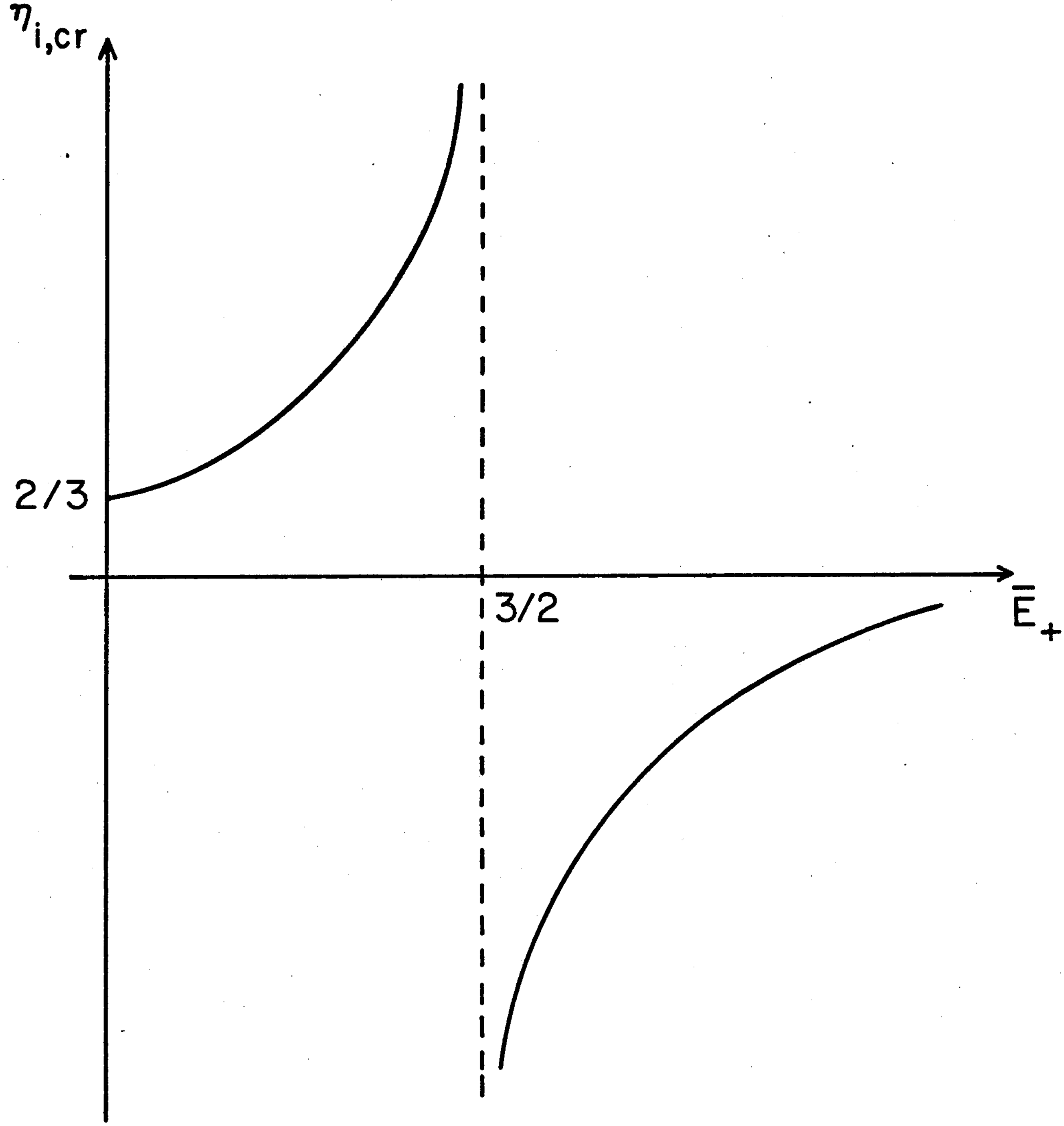

Fig. 4