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Renormalized Perturbation Theory: Vlasov Poisson System, Weak Turbulence Limit and Gyrokinetics

Y.Z. Zhang and S.M. Mahajan

Department of Physics
and Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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Abstract

The Self-consistency of the renormalized perturbation theory of Ref.1 is demonstrated by applying it to the Vlasov-Poisson System and showing that the theory has the correct weak turbulence limit. Energy conservation is proved to arbitrary high order for the electrostatic drift waves. The theory is applied to derive renormalized equations for a low-β gyrokinetic system. Comparison of our theory with other current theories is presented.

Introduction

This paper is devoted to an investigation of several generic problems in plasma turbulence using the techniques of renormalized perturbation theory presented in Ref.1. Our aim is two fold; 1) To apply our methods to problems which have already been studied so that we can compare and contrast our formalism with the existing theories. A detailed study of the Vlasov-Poisson system will be undertaken towards this end, 2) Use our formalism to solve new problems; the development and applications of renormalized gyro-kinetic theory (including the transport theory) is a case in point.

We begin by giving a short history of electrostatic turbulence in plasmas. There are two different popular approaches to deal with strong turbulence in a Vlasov-Poisson system; the perturbative and the non-perturbative, or functional approach. The former was initiated by Dupree's pioneering work[2], commonly known as "resonance broadening theory," in which the bare propagator $(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^{-1}$ in the Vlasov equation is intuitively replaced by a renormalized propagator $(\omega - \boldsymbol{k} \cdot \boldsymbol{v} + i\Gamma_k)^{-1}$, where $i\Gamma_k$ stands for the stochastic diffusion arising from the turbulent fields acting on particle. Further progress in perturbation theories consists mainly of the following two developments:

(i) The "single renormalization theory" (named by Horton and Choi[3]) is replaced by the "fully renormalized theory," which leads to an equation for $i\Gamma_k$ in terms of the spectrum; it is no longer determined through the bare propagator alone[3,4,5].

(ii) Dupree and Tetrault added the famous β-term to the nonlinear coherent dielectric function (the terminology of coherent dielectric function will be explained in Sec.II) in order to conserve energy in electrostatic drift waves, a property of the exact system which was being violated in earlier versions of perturbation theory.

The incorporation of these developments has led to the now widely used modern perturbation theories [6–10]. These theories are, so far, limited to second order (of coherent part) in expansion, although the concept of 'order' is not unambiguous. An important shortcoming of these conventional perturbation theories consists in their failure to reduce (in the corresponding limit) to the commonly accepted weak turbulence theory. For example the well-known Kadomtsev spectrum equation [Eq.(11·50) in Ref.11] does not follow from the weak trubulence limit of Dupree's equations.

The non-perturbative approach[12] for the Vlasov-Poisson system can be constructed on the basis of the rigorous MSR (Martin-Siggia-Rose) systematology[13]. It is basically a generating functional method for constructing the Green function. However, the system of equations in the MSR systemotology contains a functional differential equation for the renormalized vertex Γ. Its rigorous solution seems illusive with present mathematical techniques. In practice, progress is achieved by introducing a rather ad-hoc closure technique of replacing the renormalized vertex Γ by its lowest order approximation – the bare vertex γ, which is a known quantity. This is the DIA (Direct Interaction Approximation)[12,14–16]. Obviously, the DIA, although an approximate theory, has the advantage of keeping the form of the exact equations in the MSR systematology. To find a solution of the DIA is not easy. Krommes constructed the DIA equations for the Vlasov-Poisson system, and proposed the DIAC (Direct Interaction Approximation Coherent) to find a solution in its diffusion approximation and proved the energy conservation of electrostatic drift waves[17].

The approximations used in this solution as well as the proof of energy conservation are equivalent to the work of Dupree and Tetrault[6]. A significant sucess of the DIA lies in its ability to reproduce (in the appropriate limit) Kadomtsev's weak turbulence equation. Particularly, the dielectric function defined in statistical mechanics[18] is derived in the limit of weak turbulence, and found to be different from the coherent dielectric function used by Dupree[6]. Besides the diffusive part in the renormalized theory, the polarization part also contributes to the dielectric function. In a sense, this is equivalent to the correlation between the background waves and the induced waves. Therefore, the coherent dielectric function in Dupree's sense is not the dielectric function defined in statistical mechanics, because the contribution from the incoherent part, arising through the correlations mentioned above, is ignored.

Although the DIA has the correct weak turbulence limit, it has gone little beyond the perturbative results in physical application to strong turbulence problems. For example, the renormalized version of Kadomtsev's equation (including Compton scattering, nonlinear scattering, and three-wave interactions) and the dielectric function have still not been derived from this approach[12].

The failure of Dupree's theory to yield the correct weak turbulence limit has been attributed to its lack of proper, self-consistent treatment of the Vlasov-Poisson system. The difference between Dupree's approach, and the DIA emerges when we examine the renormalized propagator used in these two approaches. The propagator used in Dupree's approach contains only the diffusive part (which is related to the self-energy effects), while the propagator used in the MSR systematology (and the DIA) includes a polarization part also; this part is related to the polarization cloud around a test particle and to the statistical fluctuations in the dielectric response. We shall see in Sec.II that the effects like $\delta f_k(\varphi^{(e)}/\delta\varphi^{(e)}_{k'}\mid_{\varphi^{(e)}=0}$ for $k \neq k'$ are intrinsically related to the incoherent waves, or more precisely, to the correlation between the background waves and the induced (by the external source) waves. These physical effects do not need to be put in the propagator. The secularity in the bare Green function can be eliminated by self-energy renormalization alone, something like the mass renormalization for Dirac equation. Therefore, if the incoherent part is manipulated correctly in a properly matched perturbation theory, the physical effects represented by the polarization part will be automatically included in the new description.

The perturbative approach developed in Ref.1 fulfills this demand. In fact, the renormalization procedure of Ref.1, when applied to the Vlasov-Poisson system, is a generalization of Rudakov-Tsytovich's method[5] to arbitrary higher order. Renormalization, here, means that the counter (or compensating) term $i\Gamma_k$ (corresponding to frequency broadening and shift in the propagator), formally added to the equations, has to be eliminated in each perturbative order. The proof of renormalizability thus leads to a self-consistent determination of $i\Gamma_k$ [Eq.(11), Ref.1], and in addition provides a unique definition of 'order'.

The proof of renormalizability , however, is not enough to guarantee that the perturbative approach of Ref.1 is, at least, equaivalent to the DIA. To do this we must demonstrate that our approach can match the most important success of DIA in the weak turbulence limit, i.e. the calculation of the dielectric function and the derivation of Kadomtsev's spectrum equations. Therefore, we present in Sec.II a detailed derivation of the renormalized dielectric function (starting from its statistical mechanics definition), and then obtain its explicit form to second order and show that it reduces to the correct weak turbulence limit. In Sec.III, we derive the renormalized version of Kadomtsev's spectrum equation. Therefore, we present in Sec.II a detailed derivation of the renormalized dielectic function (starting from its statistical mechanics definition), and then obtain its explicit form to second order and show that it reduces to the correct weak turbulence limit. In Sec.III, we derive the renormalized version of Kadomtsev spectrum equation and discuss its weak turbulence limit. Since the diagrammatic technique (used in Ref.1) is not essential for these calculations, we adopt the usual conventional approach.

In Sec.IV we deal with the problem of proving energy conservation for the electrostatic drift waves to all orders in perturbation theory. We remind the reader that this was a crucial test for the validity of several theories and the perturbation approach passed this test to second order only after Dupree and Tetrault introduced their famous β-term. We present explicit calculations to order three (although we have proved energy conservation to arbitrary high orders). Notice that for orders greater than two, there are no existing proofs. We also show that, in these orders, DIAC (without diffusive approximation) is not expected to conserve energy. The energy conservation follows from some symmetry properties of the wave-particle interaction vertex. We show that these crucial properties are not mentioned in some presentations using ballooning representation[19]. Clearly energy conservation is violated in these theories.

In Sec. V, we turn to the second part of our program, i.e., use our procedure to elucidate and solve new problem. Since finite Larmor radius effects are considered quite important in plasma microturbulence we choose a low-β gyrokinetic system as our object of study. We first show that our method is ideally suited to develop 'finite-Larmor radius' renormalized theory [Sec.V.A], Then apply it to illustrate the nonlinear behavior of the kinetic η_i mode (Sec.V.B);i.e., we derive the coupled nonlinear wave equations. Sec.V.C is is devoted to the development of a transport theory with analytic general expressions for the transport coefficients. General properties of the energy transport coefficient ($D_\perp$) are discussed with particular emphasis on the effects of finite Larmor radius on $D_\perp$.

Section VI gives a short summary and discussion of the main points of this paper.

II Renormalized Dielectric Function

We now use the methodology developed in Ref.1 to calculate the renormalized dielectric function for the Vlasov-Poisson system. We use the standard statistical physics definition of the inverse dielectric function (ϵ_k^{-1}) pertaining to the electrostatic waves. Defined through the Poisson equation[18], ϵ_k^{-1} is given as a functional derivative

$$\epsilon_k^{-1} = \frac{\boldsymbol{k}^2}{4\pi} \cdot \frac{\delta \left\langle \varphi_k(\varphi^{(e)}) \right\rangle}{\delta \rho_k^{(e)}} \tag{1}$$

where φ_k is the potential induced in the plasma in response to the disturbance $\varphi^{(e)}$ generated by an infinitesimal external (unrandom) source $\rho^{(e)}$. We remind the reader that the bare source quantities $\varphi_k^{(e)}$ and $\rho_k^{(e)}$ are related through

$$\varphi_k^{(e)} = \frac{4\pi}{\boldsymbol{k}^2} \rho_k^{(e)} \tag{2}$$

and the potential φ_k is determined by the Poisson equation

$$\varphi_k = \varphi_k^{(e)} + (-4\pi q/\boldsymbol{k}^2) \int d\boldsymbol{v} f_k(\varphi^{(e)}) \equiv \varphi_k^{(e)} + O_k f_k(\varphi^{(e)}), \tag{3}$$

where f is the plasma distribution function which obeys the Vlasov equation ($\boldsymbol{\partial} \equiv \partial/\partial \boldsymbol{v}$)

$$[\partial_t + \boldsymbol{v} \cdot \nabla - (q/m) \nabla \varphi(\varphi^{(e)}) \cdot \boldsymbol{\partial}] f(\varphi^{(e)}) = 0. \tag{4}$$

Making use of Eqs.(2) and (3), Eq.(1) takes the form

$$\epsilon_k^{-1} = 1 + O_k \left\langle \frac{\delta f_k(\varphi^{(e)})}{\delta \varphi_k^{(e)}} \right\rangle_{\varphi^{(e)}=0} . \tag{5}$$

Thus to calculate ϵ_k^{-1}, we must solve f_k in terms of $\varphi_k^{(e)}$, and obtain the required functional derivative. Notice that the problem is not forbiddingly difficult because we need to calculate the response only linearly in the infinitesimal source $\varphi^{(e)}$. Before manipulating the Vlasov equation, we notice from Eq.(3) that the variation in the induced potential due to an infinitesimal source $\varphi^{(e)}$ is

$$\delta\varphi_k = \varphi_k(\varphi^{(e)}) - \varphi_k(0) = \left[\delta_{k,k'} + O_k \frac{\delta f_k(\varphi^{(e)})}{\delta \varphi_{k'}^{(e)}}\right] \varphi_{k'}^{(e)} \tag{6}$$

where $\varphi_k(0) = O_k f_k(\varphi^{(e)}) \mid_{\varphi^{(e)}=0}$, and $\delta\varphi_k$ is linear in $\varphi_{k'}^{(e)}$.

With the definitions $\left\langle f(\varphi^{(e)} = 0)\right\rangle = f_o$, and $f(\varphi^{(e)}) = f_o + \delta f(\varphi^{(e)})$, the Fourier transformed Vlasov equation becomes

$$\begin{aligned} \delta f_k &= G_k \hat{L}(k) f_o [\varphi_k(0) + \delta\varphi_k] \\ &+ \sum_{k_1} G_k \hat{L}(k_1)[\varphi_{k_1}(0) + \delta\varphi_{k_1}]\delta f_{k-k_1} + i G_k \Gamma_k \delta f_k, \end{aligned} \tag{7}$$

where $G_k = (\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\Gamma_k)^{-1}$ is the renormalized propagator, $i\Gamma_k$ is the compensating term to be determined, and $\hat{L}(k) \equiv -(q/m)\boldsymbol{k}\cdot\boldsymbol{\partial}$ is the wave-particle interaction vertex. In Eq.(7), it is understood that all the fluctuating quantities $(\delta f_k, \delta\varphi_k)$ are functionals of $\varphi^{(e)}$.

Eqation (7) is precisely of the form for which the renormalization procedure of Ref.1 was developed. The procedure will yield us an expression for $i\Gamma_k$ in terms of the operator G_k and $\varphi_k(0)$, and also a formal expression for δf_k (the solution of the Vlasov equation) in terms of $G_k, \varphi_k(0)$ and $\varphi^{(e)}$. From the latter, we can determine the functional derivative $\delta f_k/\delta\varphi_k^{(e)}$ to complete the evaluation of ϵ_k^{-1}. The frequency broadening, $i\Gamma_k$, is determined entirely by the fluctuating fields $\varphi_k(0)$ rather than $\varphi_k^{(e)}$ (which is an unrandom external source). Keeping this fact in mind, and following the procedure of Ref.1, we obtain a perturbation expansion of $i\Gamma_k$ by demanding that $i\Gamma_k$ cancel, order by order, all the self-energy terms (terms associated with the contraction of the fluctuating fields) produced in the iteration of Eq.7). The first four terms are

$$
\begin{aligned}
-i\Gamma_k &= \widehat{L}(k_1)G_{k-k_1}\widehat{L}(-k_1)\left\langle\varphi_k(0)\varphi_k^*(0)\right\rangle \\
&+ \widehat{L}(k_1)G_{k-k_1}\widehat{L}(k_2)G_{k-k_1-k_2}\widehat{L}(-k_1)G_{k-k_2}\widehat{L}(-k_2)\left\langle\varphi_{k_1}(0)\varphi_{k_1}^*(0)\right\rangle \\
&\cdot\left\langle\varphi_{k_2}(0)\varphi_{k_2}^*(0)\right\rangle \\
&+ \widehat{L}(k_1)G_{k-k_1}\widehat{L}(k_2)G_{k-k_1-k_2}\widehat{L}(k_3)G_{k-k_1-k_2-k_3} \\
&\cdot\widehat{L}(-k_1-k_2-k_3)\left\langle\varphi_{k_1}(0)\varphi_{k_2}(0)\varphi_{k_3}(0)\varphi_{k_1+k_2+k_3}^*(0)\right\rangle + \cdots .
\end{aligned}
\tag{8}
$$

After the aforementioned cancellation, we obtain the expansion of δf_k in terms of $\varphi_{k_1}^{(e)}$,

$$
\delta f_k = \mathcal{A}_k\delta\varphi_k + \tilde{f}_k(\varphi^{(e)}),
\tag{9}
$$

with

$$
\begin{aligned}
\mathcal{A}_k &\equiv G_k\widehat{L}(k)f_o + G_k\widehat{L}(k_1)G_{k-k_1}\widehat{L}(k)G_{-k_1}\widehat{L}(-k_1)\left\langle\varphi_k(0)\varphi_k^*(0)\right\rangle \\
&+ G_k\widehat{L}(k_1)G_{k-k_1}\widehat{L}(k)G_{-k_1}\widehat{L}(k_2)G_{-k_1-k_2} \\
&\cdot\widehat{L}(-k_1-k_2)f_o\left\langle\varphi_{k_1}(0)\varphi_{k_2}(0)\varphi_{k_1+k_2}^*(0)\right\rangle \\
&+ G_k\widehat{L}(k_1)G_{k-k_1}\widehat{L}(k_2)G_{k-k_1-k_2}\widehat{L}(k)G_{-k_1-k_2}\widehat{L}(-k_1-k_2)f_o \\
&\cdot\left\langle\varphi_{k_1}(0)\varphi_{k_2}(0)\varphi_{k_1+k_2}^*(0)\right\rangle + \cdots ,
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\tilde{f}_k(\varphi^{(e)}) &= G_k\widehat{L}(k_1)G_{k-k_1}\widehat{L}(_{k-k_1})f_o\left(\varphi_{k_1}(0)\delta\varphi_{k-k_1} + \delta\varphi_{k_1}\varphi_{k-k_1}(0)\right) \\
&+ G_k\widehat{L}(k_1)G_{k-k_1}\widehat{L}(k_2)G_{k-k_1-k_2}\widehat{L}(k-k_1-k_2)f_o(\varphi_{k_1}(0)\varphi_{k_2}(0)\delta\varphi_{k-k_1-k_2} \\
&+ \varphi_{k_1}(0)\delta\varphi_{k_2}\varphi_{k-k_1-k_2}(0) + \delta\varphi_{k_1}\varphi_{k_2}(0)\varphi_{k-k_1-k_2}(0)) + \cdots .
\end{aligned}
\tag{11}
$$

where only the terms linear in $\delta\varphi_k \sim \varphi^{(e)}$ are retained (we are interested in determining the plasma response to an infinitesimal external perturbation). The breakdown implied in Eq.(9) is the familiar breakdown into the coherent ($\mathcal{A}_k\delta\varphi_k$) and the intrinsically incoherent part ($\tilde{f}_k$); the latter cannot have the fluctuating quantities with the momentum index k. Making use of Eq.(6) to evaluate $(\delta\varphi_k/\delta\varphi_{k'}^{(e)})$, we obtain from Eq.(9)

$$\frac{\delta f_k}{\delta\varphi_k^{(e)}} = \mathcal{A}_k \left[1 + O_k \frac{\delta f_k}{\delta\varphi_k^{(e)}}\right] + \frac{\delta\tilde{f}_k}{\delta\varphi_k^{(e)}} \tag{12}$$

leading to

$$\epsilon_k^{-1} = 1 + O_k \frac{\delta f_k}{\delta\varphi_k^{(e)}} = 1 + O_k \epsilon_k^{(c)^{-1}} \mathcal{A}_k + O_k \epsilon_k^{(c)^{-1}} \frac{\delta\tilde{f}_k}{\delta\varphi_k^{(e)}} = \epsilon_k^{(c)^{-1}} + \epsilon_k^{(c)^{-1}} O_k \frac{\delta\tilde{f}_k}{\delta\varphi_k^{(e)}}, \tag{13}$$

where

$$\epsilon_k^{(c)} \equiv 1 - O_k \mathcal{A}_k \tag{14}$$

is the coherent dielectric function defined by Dupree[6], and the second term on the right-hand side of Eq.(12) originates from the incoherent part of δf_k.

To complete the evaluation of ϵ_k^{-1}, we must calculate the right-hand side of Eq.(13) by using the expressions given by Eqs. (8)-(13). A general calculation, although straightforward, is obviously very complicated. But to check our theory against the results of earlier theories (the DIA, for example), we need to calculate the response correct to second order only. The final step in our calculation yields (in terms of the renormalized propagator G_k)

$$\epsilon_k^{-1} = \epsilon_k^{(c)^{-1}} + 4 \sum_{k_1} \epsilon_k^{(c)^{-1}} \tilde{\epsilon}^{(2)}_{k,k-k_1} \epsilon_{k-k_1}^{(c)^{-1}} \tilde{\epsilon}^{(2)}_{-k_1,k} \epsilon_k^{(c)^{-1}} I_{k_1} \tag{15}$$

where

$$\tilde{\epsilon}^{(2)}_{k_1,k_2} \equiv -\frac{1}{2} \cdot \frac{4\pi q}{|\boldsymbol{k}_1 + \boldsymbol{k}_2|^2} \int d\boldsymbol{v} G_{k_1+k_2} \left[\hat{L}(k_1) G_{k_2} \hat{L}(k_2) + \hat{L}(k_2) G_{k_1} \hat{L}(k_1)\right] f_o, \tag{16}$$

the coherent dielectric function is

$$\epsilon_k^{(c)} \equiv 1 + \frac{4\pi q}{\boldsymbol{k}^2} \int d\boldsymbol{v} G_k \left[\hat{L}(k) + \sum_{k_1} \hat{L}(k_1) G_{k-k_1} \hat{L}(k) G_{-k_1} \hat{L}(-k_1) I_{k_1}\right] f_o, \tag{17}$$

the spectral intensity is

$$I_k = \left\langle \varphi_k(0) \varphi_k^*(0) \right\rangle \tag{18}$$

and where we have spelled out the operator O_k[Eq.(3)]. We now invert Eq.(15) to obtain an expression for the dielectic function

$$\epsilon_k = \epsilon_k^{(c)} - 4 \sum_{k_1} \tilde{\epsilon}^{(2)}_{k_1,k-k_1} \epsilon_{k-k_1}^{(c)^{-1}} \tilde{\epsilon}^{(2)}_{k-k_1} I_{k_1} \tag{19}$$

where the quantity multiplying I_k is taken to be small. To obtain the results correct to second order, we substitute all the quatities calculated to the appropriate order (including expanding G_k to second order) and find the weak turbulence limit of our theory to be

$$\epsilon_k[\text{weak}] = \epsilon^{(c)}_k[\text{weak}] - 4\sum_{k_1} \frac{\epsilon^{(2)}_{k_1,k-k_1}\epsilon^{(2)}_{k,-k_1}}{\epsilon^{(\ell)}_{k-k_1}} I_{k_1}, \tag{20}$$

$$\epsilon^{(c)}_k[\text{weak}] = \epsilon^{(\ell)}_k + 2\sum_{k_1} \epsilon^{(3)}_{k_1,-k_1,k} I_{k_1}, \tag{21}$$

where $\epsilon^{(2)}_{k_1,k_2}$ is just $\tilde{\epsilon}^{(2)}_{k_1,k_2}$ with G_k replaced by $G^{(0)}_k$, and

$$\epsilon^{(3)}_{k_1,k_2,k_3} = \frac{4\pi q}{|\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3|^2}\int dv G^{(0)}_{k_1+k_2+k_3}\hat{L}(k_1)G^{(0)}_{k_2+k_3} \cdot \left[\hat{L}(k_2)G^{(0)}_{k_3}\hat{L}(k_2)G^{(0)}_{k_2}\hat{L}(k_2)\right] f_0. \tag{22}$$

The linear operator

$$\epsilon^{(\ell)} = 1 - O_k G^{(0)}_k \hat{L}(k) f_0, \tag{23}$$

and $G^{(0)}_k = (\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1}$ is the bare propagator.

Equation (20) with Eq. (21) is just the dielectric function (in the weak turbulence limit) obtained by Krommes and Kleva using DIA[17]. Thus the perturbation theory of Ref.1 has the correct weak turbulence limit for the dielectric function.

III Spectrum Equation

Since the concept of 'order' is clearly delineated in our perturbation theory (Ref.1), it is obvious that the contribution of the lowest order incoherent source terms (to the equation describing the fluctuation spectrum) should be more important than the higher order corrections due to the renormalization of the propagator. When the phase-space available for the three wave interaction with frequency broadening becomes significant, the incoherent source is expected to play a crucial role in the determination of the saturated level of turbulence. For electrostatic turbulence (Vlasov-Poisson system), a closed set of spectrum equations can be constructed from the renormalized perturbation theory of Ref.1 provided $e\varphi/T$ (the ratio of potential to the kinetic energy) is small enough to be a perturbation parameter.

From the formalism presented in Sec.II, it is straightforward to derive the nonlinear Poisson equation (to second order),

$$\epsilon_k^{(c)}\varphi_k = \sum_{k_1+k_2=k} \tilde{\epsilon}^{(2)}_{k_1,k_2}\varphi_{k_1}\varphi_{k_2} + \sum_{k_1+k_2+k_3=k} \tilde{\epsilon}^{(3)}_{k_1,k_2,k_3}\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}, \tag{24}$$

where the coherent dielectric function $\epsilon_k^{(c)}$ is given by Eq.(17), $\tilde{\epsilon}^{(2)}_{k_1,k_2}$ is given by Eq.(16), and

$$\tilde{\epsilon}^{(3)}_{k_1,k_2,k_3} \equiv -\frac{4\pi q}{|\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3|^2}\int d\boldsymbol{v} G_{k_1+k_2+k_3}\hat{L}(k_1)G_{k_2+k_3}\hat{L}(k_2)G_{k_3}\hat{L}(k_3)f_0. \tag{25}$$

Multiplying both sides of Eq.(25) by φ_k^* and ensemble averaging yields $[I_k = \left\langle\varphi_k\varphi_k^*\right\rangle]$

$$\epsilon_k^{(c)} I_k = \sum_{k_1+k_2=k} \tilde{\epsilon}^{(2)}_{k_1,k_2}\left\langle\varphi_{k_1}\varphi_{k_2}\varphi_k^*\right\rangle + \sum_{k_1+k_2+k_3=k} \tilde{\epsilon}^{(3)}_{k_1,k_2,k_3}\left\langle\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_k^*\right\rangle \tag{26}$$

Notice that the last term of Eq.(26) does not contribute to the lowest order (quadratic in I_k), while the triplet is approximated by the quasi-Gaussian assumption

$$\left\langle\varphi_{k_1}\varphi_{k_2}\varphi_k^*\right\rangle = 2\frac{\tilde{\epsilon}^{(2)*}_{k_1,k_2}}{\epsilon_k^{(c)*}}I_{k_1}I_{k_2} + 2\frac{\tilde{\epsilon}^{(2)*}_{-k_1,k}}{\epsilon_{k_2}^{(c)}}I_{k_1}I_k + 2\frac{\tilde{\epsilon}^{(2)*}_{k_2,k}}{\epsilon_{k_1}^{(c)}}I_{k_2}I_k \tag{27}$$

converting Eq.(26) to

$$\left[\epsilon_k^{(c)} - 4\sum_{k_1}\frac{\tilde{\epsilon}^{(2)}_{k_1,k-k_1}\tilde{\epsilon}^{(2)}_{-k_1,k}}{\epsilon_{k-k_1}^{(c)}}I_{k_1}\right]I_k = 2\sum_{k_1+k_2=k}\frac{|\tilde{\epsilon}^{(2)}_{k_1,k_2}|^2}{\epsilon_k^{(c)*}}I_{k_1}, I_{k_2}, \tag{28}$$

which along with the other independent equation (correct to second order)

$$-i\Gamma_k = G_k^{(0)^{-1}} - G_k^{-1} = \sum_{k_1}\hat{L}(k_1)G_{k-k_1}\hat{L}(-k_1)I_{k_1} \tag{29}$$

where $G_k^{(0)^{-1}} = G_k^{-1} - i\Gamma_k$ is the inverse bare propagator, provides a closed set of equations for I_k and G_k (all $\tilde{\epsilon}_k$'s are functionals of G_k). Eqs. (28), (29) are quite similar to the equations proposed by Biskamp[20] (using diagramatical techniques), and by Orzag and Kraichnan[15].

If the turbulent level is so small that the renormalized propagator G_k^{-1} can be replaced by the bare propagator $G_k^{(0)^{-1}}$ [making Eq.(29) trivial], Eq.(28) with $\epsilon_k^{(c)}$ given by Eq.(21) reduces to the standard spectrum equation in weak turbulence theory [cf. Appendix of Ref.1]. In a sense the set of Eqs.(28),(29) form the renormalized version of weak turbulence theory, in which the resonance broadening (and shift), the three wave interaction, and the nonlinear scattering are all included in a consistent way.

For small, but finite I_k Eq.(27) is approximated by the single renormalization

$$G_k^{-1} \sim G_k^{(0)^{-1}} - \sum_{k_1} \hat{L}(k_1) G_{k-k_1}^{(0)} \hat{L}(-k_1) I_{k_1}. \tag{30}$$

Substituting Eq.(30) into Eq.(28) yields a spectrum equation for I_k only. The back-reaction of particle on waves still exists through the spectrum equation.

By doing explicit calculation to second order, we have demonstrated in Sec.II and III that our perturbation approach is, indeed, naturally reducible to the weak turbulence theory in the appropriate limit; this constitutes a definite advantage over the conventional perturbation approaches[4-10] which do not.

IV Energy Conservation in Electrostatic Drift Waves

It is well-known that the energy conservation is an essential test for the validity of a nonlinear wave-particle interaction theory. For the electrostatic drift waves the perturbed power term $\langle \delta \boldsymbol{j}_\perp \cdot \delta \boldsymbol{E}_\perp \rangle$ is rigorously equal to zero. This property should remain valid for any approximate theory with the implication that any plausible and acceptable perturbation theory must conserve energy in each order of the perturbation.

We now show that the perturbation theory proposed in Ref.1 does meet the required demands of energy conservation. In particular, we prove the energy conservation in the perturbation order higher than the second to which most analyses (including DIAC with diffusion approximation) have been limited[6,9,17].

For simplicity, only the shearless slab model is considered. The generalization to a sheared slab model is straightforward.

The starting point here is the Fourier transformed drift kinetic equation [Vlasov equation for the guiding center in a uniformly magnetized plasma],

$$f_k = G_k \hat{L}_0(k) f_0 \varphi_k + G_k \sum_{k_1} \hat{L}(k, k_1) \varphi_{k_1} f_{k-k_1} + G_k i \Gamma_k f_k \tag{31}$$

where

$$\hat{L}_0(k) \equiv -(e/T_e)(k_\| v_\| - \omega_{en}^*)$$

$$\widehat{L}(k,k_1) \equiv (e/m)k_{\|,1}\partial_{\|} - i(c/B)(\boldsymbol{k}\times\boldsymbol{k}_1)\cdot\boldsymbol{b}$$
$$G_k \equiv (\omega - k_{\|}v_{\|} + i\Gamma_k)^{-1} \quad \partial \equiv \partial/\partial v_{\|}$$

with the ambient magnetic field $\boldsymbol{B} = B\boldsymbol{b}$, the electron diamagnetic drift frequency $\omega_{en}^{*} = k_y(cT_e/eB)|d\ln n/dr|$ and $v_{\|} = \boldsymbol{v}\cdot\boldsymbol{b}$. The rate of work done by the drift current is expressed as

$$\langle \delta\boldsymbol{j}_{\perp}\cdot\delta\boldsymbol{E}_{\perp}\rangle = ie\int dv_{\|}dkG_k^{(0)^{-1}}\left\langle f_k\varphi_k^{*}\right\rangle, \tag{32}$$

where

$$G_k^{(0)^{-1}} \equiv \omega - k_{\|}v_{\|}.$$

Notice that Eq. (31) is formally equivalent to Eq. (7) analyzed in the last two sections.

Substituting $f_k = \mathcal{A}\varphi_k + \tilde{f}_k$ and $G_k^{(0)^{-1}} \equiv G_k^{-1} - i\Gamma_k$ into Eq. (32), we have

$$\begin{aligned}\langle \delta\boldsymbol{j}_{\perp}\cdot\delta\boldsymbol{E}_{\perp}\rangle &= ie\int dv_{\|}dk\{\left\langle\varphi_k^{*}G_k^{-1}\mathcal{A}\varphi_k\right\rangle \\ &+\left\langle\varphi_k^{*}\left(-i\Gamma_k\right)\mathcal{A}\varphi_k\right\rangle + \left\langle\varphi_k^{*}G_k^{-1}\tilde{f}_k\right\rangle + \left\langle\varphi_k^{*}\left(-i\Gamma_k\right)\tilde{f}_k\right\rangle\}.\end{aligned} \tag{33}$$

Energy conservation holds if we can show that the right-hand side of Eq. (33) is zero in every order. For convenience, we first prove the following two useful lemmata:

Lemma 1 *All terms of the form*

$$\sum_{k,k_1}\int dv_{\|}\widehat{L}\left(k,k_1\right)P\left(k-k_1\right)\varphi_k\varphi_{k_1}^{*} = 0,$$

where $P\left(k-k_1\right)$ is any functional of $k-k_1$.

Proof: Since

$$\widehat{L}\left(k,k_1\right) \equiv \widehat{L}_1\left(k,k_1\right) + \widehat{L}_2\left(k,k_1\right) = \left(\frac{e}{m}\right)k_{\|,1}\partial_{\|} - i\left(\frac{c}{B}\right)\left(\boldsymbol{k}\times\boldsymbol{k}_1\right)\cdot\boldsymbol{b},$$

it is obvious that the first term equals zero being purely a surface term in $v_{\|}$. The second term vanishes because $\widehat{L}_2\left(k,k_1\right)$ is odd under the transformation $(k \leftrightarrow -k_1)$ while the rest of the term are even.

It is helpful to define the adjoint of a term $\tilde{Q}$, i.e., if

$$Q = \sum_{k,k_1} \int dv_{\|} \widehat{L}(k,k_1)\, T(k,k_1),$$

then

$$\tilde{Q} = \sum_{k,k_1} \int dv_{\|} \widehat{L}(k,k_1)\, T(-k_1,-k),$$

where T is any functional of k and k_1. This leads us to the second lemma.

Lemma 2 *The sum of two adjoint terms is zero, i.e.,*

$$Q + \tilde{Q} = 0.$$

Proof: The contribution from $\widehat{L}_1$ is equal to zero after integration over $v_{\|}$. The contribution from $\widehat{L}_2(k,k_1)$ terms are cancelled because $T(k,k_1) + T(-k_1,-k)$ is invariant under the transformation $(k \leftrightarrow -k_1)$, while $\widehat{L}_2(k,k_1)$ changes sign.

We shall show the energy conservation in the first few orders explicitly by using the above two lemmata.

To zero order, only the linear term $\widehat{L}_0 f_0$ from $\left\langle \varphi_k^* G_k^{-1} \mathcal{A}_k \varphi_k \right\rangle$ contributes which changes sign under k-inversion and thus integration over k yields zero.

To first order, only the incoherent source $\left\langle \varphi_k^* G_k^{-1} \tilde{f}_k \right\rangle$ contributes a term

$$C_1^{(1)} = \sum_{k,k_1} \int dv_{\|} \widehat{L}(k,k_1)\, G_{k-k_1} \widehat{L}_0(k-k_1)\, f_0 \varphi_k^* \varphi_{k_1} \varphi_{k-k_1} \tag{34}$$

which is zero because of Lemma 1.

In second order, we have three nontrivial terms making up the energy:

$$\begin{aligned} C_2^{(2)} &= \sum_{k,k_1,k_2} \int dv_{\|} \widehat{L}(k,k_1)\, G_{k-k_1} \widehat{L}(k-k_1,k_2) \\ &\quad \cdot G_{k-k_1-k_2} \widehat{L}_0(k-k_1-k_2)\, f_0 \varphi_k^* \varphi_{k_1} \varphi_{k_2} \varphi_{k-k_1-k_2} \end{aligned} \tag{35}$$

from the incoherent term $\left\langle \varphi_k^* G_k^{-1} \tilde{f}_k \right\rangle$,

$$C_2^{(2)} = \sum_{k,k_1} \int dv_{\|} \widehat{L}(k,k_1)\, G_{k-k_1} \widehat{L}(k-k_1,k)\, G_{-k_1} \widehat{L}_0(-k_1) \left\langle \varphi_{k_1}^* \varphi_{k_1} \right\rangle \left\langle \varphi_k^* \varphi_k \right\rangle \tag{36}$$

from $\left\langle \varphi_k^* G_k^{-1} \mathcal{A}_k \varphi_k \right\rangle$, and

$$C_3^{(2)} = \sum_{k,k_1} \int dv_{||} \widehat{L}(k,k_1) G_{k-k_1} \widehat{L}(k-k_1,-k_1) G_k \widehat{L}_0(k) f_0 \left\langle \varphi_{k_1}^* \varphi_{k_1} \right\rangle \left\langle \varphi_k^* \varphi_k \right\rangle \tag{37}$$

from $\left\langle \varphi_k^* (-i\Gamma_k) \mathcal{A}_k \varphi_k \right\rangle$. It is obvious that energy conservation is valid in this order because $C_1^{(2)} = 0$ by Lemma 1 and $C_2^{(2)} + C_3^{(2)} = 0$ by Lemma 2. Notice that violation of energy conservation would occur if we had kept only the term proportional to the resonance broadening $i\Gamma_k \left[\left\langle \varphi_k^* (-i\Gamma_k) \mathcal{A}_k \varphi_k \right\rangle\right]$ and neglected the second order contribution from the term $\left\langle \varphi_k^* G_k^{-1} \mathcal{A}_k \varphi_k \right\rangle$ which represents the renormalized averaged distribution function. This latter is just the famous β term introduced by Dupree and Tetrault to insure energy conservation.

The energy conservation in second order is essentially a consequence of the cancellation of the two coherent contributions because the incoherent contribution $C_1^{(2)}$ was identically zero due to Lemma 1. Thus up to second order, a theory would pass the energy conservation test even if the incoherent part is neglected altogether. The situation changes entirely in the third and higher orders where incoherent terms are not individually (and hence trivially) equal to zero. Thus the proof of energy conservation in third (and higher) order becomes a crucial test to any progress beyond Dupree and Tetrault. In third order, $\left\langle \varphi_k^* (-i\Gamma_k) \tilde{f}_k \right\rangle$ gives a term

$$\begin{aligned} C_1^{(3)} &= \sum_{k,k_1,k_2} \int dv_{||} \widehat{L}(k,k_1) G_{k-k_1} \widehat{L}(k-k_1,-k_1) G_k \widehat{L}(k,k_2) G_{k-k_2} \widehat{L}_0(k-k_2) f_0 \\ &\quad \cdot \left\langle \varphi_{k_1}^* \varphi_{k_1} \right\rangle \left\langle \varphi_{k_2} \varphi_{k-k_2} \varphi_k^* \right\rangle \end{aligned} \tag{38}$$

cancelled by its adjoint

$$C_2^{(3)} = \tilde{C}_1^{(3)} \Rightarrow C_2^{(3)} + C_1^{(3)} = 0 \tag{39}$$

contributed by $\left\langle \varphi_k^* G_k^{-1} \mathcal{A}_k \varphi_k \right\rangle$, which also gives

$$\begin{aligned} C_3^{(3)} &= \sum_{k,k_1,k_2} \int dv_{||} \widehat{L}(k,k_1) G_{k-k_1} \widehat{L}(k-k_1,k_2) G_{k-k_1-k_2} (k-k_1-k_2,-k) G_{-k_1-k_2} \\ &\quad \cdot \widehat{L}_0(-k_1,-k_2) f_0 \left\langle \varphi_k^* \varphi_k \right\rangle \left\langle \varphi_{k_1} \varphi_{k_2} \varphi_{k_1+k_2}^* \right\rangle \end{aligned} \tag{40}$$

cancelling its own adjoint

$$C_4^{(3)} = \tilde{C}_3^{(3)} \Rightarrow C_4^{(3)} + C_3^{(3)} = 0 \tag{41}$$

originating from $\left\langle \varphi_k^* G_k^{-1} \tilde{f}_k \right\rangle$. The non-Gaussian frequency broadening term

$$C_5^{(3)} = \sum_{k,k_1,k_2} \int dv_{\|} \widehat{L}(k,k_1)\, G_{k-k_1} \widehat{L}(k-k_1,k_2)\, G_{k-k_1-k_2} \widehat{L}(k-k_1-k_2,-k_1,-k_2)$$

$$\cdot\, G_k \widehat{L}_0(k) f_0 \left\langle \varphi_k^* \varphi_k \right\rangle \left\langle \varphi_{k_1} \varphi_{k_2} \varphi_{k_1+k_2}^* \right\rangle, \qquad (42)$$

added to its adjoint $C_6^{(3)} = \tilde{C}_5^{(3)}$ coming from $\left\langle \varphi_k^* G_k^{-1} \tilde{f}_k \right\rangle$, also gives zero contribution to the right-hand side of Eq. (31); the conservation of energy in third order, thereby, is proved.

At this point we would like to stress that the energy conservation in third order resulted from exact cancellation between the non-Gaussian contributions from the coherent part, and similar contributions from the intrinsically incoherent source. Clearly a theory like DIAC (without diffusive approximation) will violate energy conservation in this order because it neglects the incoherent part altogether. To the best of our knowledge, energy conservation has not yet been proved for DIAC.

We have also succeeded in proving energy conservation up to arbitrary high orders by using diagrammatic approach. The results, not relevant to the main thrust of this paper, will be presented elsewhere.

We wish to emphasize that energy conservation in electrostatic drift waves is a consequence of the antisymmetry (change of sign for $k \leftrightarrow k_1$) of the wave-particle interaction vertex. Explicit antisymmetry must be maintained in any treatment, otherwise energy conservation will be violated. To illustrate our point, we make a small digression to show that there exist examples in published literature[19] (using ballooning representation) where antisymmetry and hence energy conservation is not preserved.

Manipulation of the gyrokinetic equation in the ballooning representation (in the electrostatic, and small gyro radius approximation) yields the explicitly antisymmetric nonlinear term

$$\pi(c/B)\tilde{F} \sum_{n_1+n_2=n} n_1 n_2 \sum_{\ell} \ell \{ \exp(-2\pi i \bar{q} n_1 \ell) \delta\hat{\varphi}_{n_1}(\psi,\theta+2\pi\ell) \delta\hat{h}_{n_2}(\psi,\theta)$$
$$- \quad \exp(-2\pi i \bar{q} n_2 \ell) \delta\hat{\varphi}_{n_1}(\psi,\theta) \delta\hat{h}_{n_2}(\psi,\theta+2\pi\ell) \} \qquad (43)$$

where $\delta\hat{h}_n(\psi,\theta)$ is the non-adiabatic part of ballooning amplitude $\delta\hat{\varphi}_n$ defined by

$$\delta\varphi(\psi,\theta,\zeta) = \sum_n \exp\left[in \left(\zeta - \int^{\theta} q(\psi,\theta')\, d\theta' \right) \right] \sum_{\ell} \delta\hat{\varphi}_n(\psi,\theta+2\pi\ell) \exp(-2\pi i n \bar{q} \ell) \qquad (44)$$

with

$$\bar{q} = \oint q(\psi, \theta')d\theta',$$

and

$$\tilde{F} = \nabla\left(\zeta - \int^{\theta} q(\psi, \theta')d\theta'\right) \times \nabla q \cdot \boldsymbol{b} \tag{45}$$

It is obvious that it is only the combination of the two terms on the right-hand side of Eq.(43) that guarantees the antisymmetry which makes Eq.(43) correspond to the slab model nonlinear term

$$(c/B)(\nabla\delta\varphi \times \boldsymbol{b}) \cdot \nabla\delta\varphi = 0.$$

Unfortunately, the second term in Eq.(43) is missing in the aforementioned treatment, and in that treatment there is no way to assure energy conservation in each order of the perturbation theory.

V Renormalized Gyro-Kinetics:

The methodology developed in Ref.1 and presented (in some detail) in earlier sections of the present paper can be readily generalized to deal with more complicated and realistic problems in plasma physics.

A: Collisionless Plasma with Finite Larmor Radius Effects

An obvious extension of the previous work is to develop a renormalized theory of collisionless plasmas including finite Larmor radius effects. Clearly one must begin with the nonlinear gyrokinetic version of the Vlasov equation. To make the problem more realistic and interesting, we allow the equilibrium to have temperature and density gradients also.

Notice that in a gyrokinetic theory, the perpendicular (to the magnetic field) thermal motion of the particles comes into its own implying that self energy terms, etc. will become function of the perpendicular velocity $\boldsymbol{v}_\perp$. Thus the evaluation of all the renormalized (linear as well as nonlinear) terms will involve cumbersome $\boldsymbol{v}_\perp$ integrations[17] It turns out however, that in slab model to second order in perturbations, the $\boldsymbol{v}_\perp$-dependent self energy terms can be replaced by $\boldsymbol{v}_\perp$-averaged terms (independent of $\boldsymbol{v}_\perp$) provided drift frequencies are ignored. As a result, only the familiar $v_{\parallel}$ integrations (just like the drift-kinetic treatment) will be necessary to derive the nonlinear wave equation.

In a slab model, the non-adiabatic part of the perturbed distribution function obeys (in Fourier space with $\delta \boldsymbol{A}_{\perp} = 0$)

$$(\omega - k_{\parallel} v_{\parallel})\delta h_k = \widehat{L}_0(k) J_0(v_{\perp} k_{\perp}/\Omega) F_0 \delta\psi_k + \sum_{k_1+k_2=k} \widehat{L}_1(k_1, k_2) J_0(v_{\perp} k_{1\perp}/\Omega)\delta\psi_{k_1}\delta h_{k_2}, \quad (46)$$

with the definitions

$$\widehat{L}_0(k) \equiv (q/T)\left\{\omega - \omega_n^*[1 + \eta(H/T - 3/2)]\right\} \quad (47)$$

$$\widehat{L}_1(k_1, k_2) \equiv i(c/B)(\boldsymbol{k}_1 \times \boldsymbol{k}_2) \cdot \boldsymbol{b} \quad (48)$$

and

$$\delta\psi_k = \delta\varphi_k - (v_{\parallel}/c)\delta A_k \quad (49)$$

where $\omega_n^* = (cT/eB)(k_y/L_n)$ is the density gradient induced diamagnetic frequency, L_n is the density scale length, $\eta = L_n/L_T, L_T$ is the temperature scale length, $H = (m/2)(v_{\parallel}^2 + v_{\perp}^2)$ is the particle energy, Ω is the gyrofrequency, B is the ambient magifield and J_0 representing finite gyroradius effects, is the Bessel function of order zero. Direct renormalization of Eq.(46) would yield the $\boldsymbol{v}_{\perp}$ dependent self energy term

$$-i\Gamma_k = \sum_{k_1}(c/B)^2[(\boldsymbol{k}_1 \times \boldsymbol{k}) \cdot \boldsymbol{b}]^2 J_0^2(v_{\perp} k_{1\perp}/\Omega) G_{k-k_1} \left|\delta\psi_{k_1}\right|^2, \quad (50)$$

and an appropriate expression for the distribution function $\delta h_k(\boldsymbol{v})$. As pointed out earlier, this would lead to complicated $\boldsymbol{v}_{\perp}$ integration. Indeed, we prefer to deal with the $\boldsymbol{v}_{\perp}$ averaged distribution function

$$\delta\bar{h}_k(v_{\parallel}) \equiv I_{v_{\perp}} \delta h_k(\boldsymbol{v}) \equiv \int_0^{\infty} v_{\perp} dv_{\perp} J_0(v_{\perp} k_{\perp}/\Omega)\delta h_k(\boldsymbol{v}) \quad (51)$$

whose equation of motion is obtained by applying $I_{v_{\perp}}$ to both sides of Eq.(46). Standard manipulation of the resulting equation with the assumption $F_0(\boldsymbol{v}) = F_0(v_{\parallel}{}^2) \cdot F_0(v_{\perp}{}^2)$ yields the renormalized second order expression,

$$\begin{aligned} \delta\bar{h}_k(v_{\parallel}) = {} & G_k \widehat{L}_0^{(0)}(k) F_0(v_{\parallel}^2)\delta\psi_k + \sum_{k_1+k_2=k} G_k \widehat{L}(k_1, k_2) G_{k_2} \widehat{L}_0^{(1)}(k_1, k_2) F_0(v_{\parallel}^2)\delta\psi_{k_1}\delta\psi_{k_2} \\ & + \sum_{\substack{k_1+k_2+k_3=k \\ k_3 \neq k}} G_k \widehat{L}(k_1, k - k_1) G_{k-k_1} \widehat{L}(k_2, k_3) G_{k_3} \widehat{L}_0^{(2)}(k_1, k_2, k_3) F_0(v_{\parallel}^2) \\ & \cdot \delta\psi_{k_1}\delta\psi_{k_2}\delta\psi_{k_3} \end{aligned} \quad (52)$$

with

$$\widehat{L}_0^{(0)}(k) \equiv (q/T)\left\{(\omega-\omega_n^*[1+\eta(v_\parallel^2/v_0^2-1/2-b)])\Gamma_0(b)-\omega_n^*\eta b\Gamma_1(b)\right\} \quad (53)$$

$$\widehat{L}_0^{(1)}(k_1,k_2) \equiv \int_0^\infty dv_\perp J_0(v_\perp|\boldsymbol{k}_{1\perp}+\boldsymbol{k}_{2\perp}|/\Omega) \cdot J_0(v_\perp k_{1\perp}/\Omega)J_0(v_\perp k_{2\perp}/\Omega)\widehat{L}_0(k_2)F_0(v_\perp{}^2) \quad (54)$$

$$\widehat{L}_0^{(2)}(k_1,k_2,k_3) \equiv \int_0^\infty dv_\perp v_\perp J_0(v_\perp|\boldsymbol{k}_{1\perp}+\boldsymbol{k}_{2\perp}+\boldsymbol{k}_{3\perp}|/\Omega) \cdot J_0(v_\perp k_{1\perp}/\Omega)J_0(v_\perp k_{3\perp}/\Omega)J_0(v_\perp k_{3\perp}/\Omega)\widehat{L}_0(k_3)F_0(v_\perp{}^2) \quad (55)$$

$$G_k = (\omega-k_\parallel v_\parallel+i\Gamma_k)^{-1} \quad (56)$$

$$-i\Gamma_k = \sum_{k_1} J(k_\perp,k_{1\perp})\widehat{L}(k_1,k)G_{k-k_1}\widehat{L}(-k_1,k-k_1)\,|\delta\psi_{k_1}|^2 \quad (57)$$

$$\text{with } J(k_\perp,k_{1\perp}) \equiv \frac{\int_0^\infty dv_\perp v_\perp J_0^2(v_\perp k_1/\Omega)J_0^2(v_\perp k_{1\perp}/\Omega)\widehat{L}_0(k)F_0(v_\perp{}^2)}{\int_0^\infty dv_\perp v_\perp J_0^2(v_\perp k_\perp/\Omega)\widehat{L}_0(k)F_0(v_\perp{}^2)} \quad (58)$$

Substituting $\delta\bar{h}_k(v_\parallel)$ into Poisson equation and Ampere's law will yield the set of nonlinear coupled wave equations in φ_k and A_k.

B: Nonlinear Kinetic η_i mode:

The renormalized gyrokinetic formalism presented in V.A can be used to delineate some general features of the behavior of the η_i mode which is thought to be a possible candidate responsible for anomalous particle transport in Tokamaks[21]. Since the η_i mode is essentially electrostatic, the basic wave equation is the equation of quasineutrality. Perturbed density can be readily calculated from Eq.(52). We begin with the first term on the r.h.s which leads to the renormalized linear ion response (G_k is the renormalized propagator)

$$\delta n_k^{(r\cdot\ell)} = -(en/T_i)\delta\varphi_k\left[1-(1-\omega_{in}^*/\omega)\Gamma_0(b_i)-(\omega_{iT}^*/\omega)b_i[\Gamma_i(b_i)-\Gamma_0(b_i)]-\tilde{\sigma}(k)\right] \quad (59)$$

where

$$\tilde{\sigma}_{(k)} \equiv (\omega_{in}^*/\omega-1)\Gamma_0(b_i)[1+\zeta Z(\zeta)] + (\omega_{iT}^*/\omega)\left\{(\zeta+\zeta^2 Z(\zeta)-Z(\zeta)/2)\tilde{\zeta}\Gamma_0(b_i)+[1+\tilde{\zeta}Z(\zeta)]b_i[\Gamma_1(b_i)-\Gamma_0(b_i)]\right\} \quad (60)$$

Z is the plasma dispersion functions, $\Gamma_n(b) = \exp(-b)\Gamma_n(b), \Gamma_n(b)$ is the modified Bessel function, and $b_i = {k_\perp}^2\rho_i^2/2$. Notice that the argument of the Z-function

$$\zeta = (\omega + i\Gamma_k)/k_{||}v_i = \tilde{\zeta} + i\Gamma_k/k_{||}v_i$$

contains the effects of renormalization (frequency broadening and shift). Since the electron response for the η_i mode is adiabatic $\left(\delta n_k^{(e)} = e\varphi_k/T_e\right)$ a nonlinear dispersion relation is obtained by equating $\delta n_k^{(i)} = e\varphi_k/T_e$. The solution (of the nonlinear dispresion relation) at marginal stability implies a saturation frequency broadening which can be associated with the transport coefficients. At this level, the result has no contribution from the incoherent part, particularly, the three wave interaction. Since longer perpendicular wavelength ($k_\perp\rho_i \ll 1$) modes cause a larger transport, while the shorter wavelengths ($k_\perp\rho_i \simeq 1$) modes are more easily destabilized (due to finite ion Larmor radius effects), it is important to examine the three wave interaction processes which can carry energy from the shorter to the longer wavelength part of the spectrum.

Let $k_{1\perp}$ and $k_{2\perp}$ be the two short wavelength modes which produce a long wave length (small k) beat wave $k_\perp$. This process will be mediated by the term $\widehat{L}_0(k_1, k_2)$ given by Eq.(54). Straight-forward algebra leads to the nonlinear perturbed density (the density characterized by $k_\perp$ driven by the interaction of waves $k_{1\perp}$ and $k_{2\perp}$) corresponding to the second term on the right-hand side of Eq.(50),

$$\begin{aligned} \delta n_k^{(n\cdot\ell)'} &= i(c/Bv_i)(en/T_i) \sum_{k_1+k_2=k} (\boldsymbol{k}_1 \times \boldsymbol{k}_2)\cdot\boldsymbol{b} \\ &\quad \cdot \{[\widehat{\Omega}(2)\left\{\mathrm{sgn}(k_{||})Z(\zeta) - \mathrm{sgn}(k_{2,||})Z(\zeta_2)\right\} - \omega_{iT}^*(2)\Gamma_0(b_i)\{\mathrm{sgn}(k_{||})\widehat{W}(\zeta) \\ &\quad - \mathrm{sgn}(k_{2,||})\widehat{W}(\zeta_2)\}]/[k_{2,||}(\omega + i\Gamma_{k_2}) - k_{||}(\omega_2 + i\Gamma_{k_2})] \\ &\quad -(1 \rightleftharpoons 2)\}\delta\varphi_{k_1}\delta\varphi_{k_2}, \end{aligned} \tag{61}$$

where

$$\widehat{\Omega}(2) \equiv [\omega_2 - \omega_{in}^*(2)]\Gamma_0(b_i) + \omega_{iT}^*(2)b_i[\Gamma_0(b_i) - \Gamma_1(b_i)] \tag{62}$$

$$\widehat{W}(\zeta) \equiv \zeta[1 + \zeta Z(\zeta)] - Z(\zeta)/2 \tag{63}$$

which, incidentally, is just the term $\sum_{k_1+k_2=k} \tilde{\epsilon}^{(2)}(k_1, k_2)\delta\varphi_{k_1}\delta\varphi_{k_2}$ in the nonlinear wave equation Eq.(24). In a similar manner, we can evaluate $\delta n_k^{(n\cdot\ell)''}$ which corresponds to the third term on the right-hand side of Eq.(52), and is naturally divided into the coherent and the incoherent part. As expected these terms are nothing but the appropriate explicit expressions for the terms in the nonlinear wave Eq.(24); the coherent part is determined by $\epsilon^{(3)}(k_1, k, -k_1)$, and the incoherent part by $\epsilon^{(3)}(k_1, k_2, k_3)$. Further manipulation of these equations will depend upon the detailed physics of the η_i mode, and are not relevant for this paper.

C. Transport Equations in Gyrokinetic Theory:

Several of the recent candidates for causing turbulent transport in Tokamaks are modes which depend on finite larmor radius effects. In this section, we derive the transport theory relevant for these modes. We begin with the (for simplicity, the drift frequency ω_d has been neglected) collisonless gyro-kinetic equation[22]

$$[\partial/\partial t + v_{\|}\boldsymbol{b}\cdot\widehat{\nabla} - \Omega\partial/\partial\alpha]f = -q(\delta R f) \tag{64}$$

with

$$\delta R \equiv -\nabla\delta\psi\cdot[\boldsymbol{v}\partial/\partial H + (\boldsymbol{v}_\perp/B)\partial/\partial\mu - (\boldsymbol{v}\times\boldsymbol{b}/m v_\perp{}^2)\partial/\partial\alpha] - (\nabla\delta\psi\times\boldsymbol{b}/m\Omega)\cdot\widehat{\nabla}, \tag{65}$$

and where $\widehat{\nabla}$ acts on the guiding center motion, and α is the gyroangle. Ensemble averaging Eq.(64), we obtain

$$\begin{aligned}(\partial/\partial t + v_{\|}\boldsymbol{b}\cdot\nabla - \Omega\partial/\partial\alpha)F &= q\langle\nabla\psi\cdot[\boldsymbol{v}\partial/\partial H + (v_\perp/B)\partial/\partial\mu \\ &\quad - (\boldsymbol{v}\times\boldsymbol{b}/m v_\perp{}^2)\partial/\partial\alpha + (m\Omega)^{-1}\boldsymbol{b}\times\widehat{\nabla}]\delta f\rangle\end{aligned} \tag{66}$$

where $F = \langle f\rangle$ is the ensemble averaged distribution function; $\delta f[\langle\delta f\rangle = 0]$ is the perturbed or the fluctuating part. From the point of view of transport, the quantity of basic physical interest is the gyroaveraged distribution function $\langle F\rangle_\alpha = \langle\langle f\rangle\rangle_\alpha$ which obeys

$$\begin{aligned}(\partial/\partial t + v_{\|}\boldsymbol{b}\cdot\nabla)\langle F\rangle_\alpha &= q\left\langle v_{\|}\boldsymbol{b}\cdot\nabla\langle\delta\psi\rangle_\alpha(\partial/\partial H)\delta f_0\right\rangle \\ &\quad + (q/m\Omega)\left\langle\langle\nabla\delta\psi\rangle_\alpha\cdot\boldsymbol{b}\times\widehat{\nabla}\delta h_0\right\rangle\end{aligned} \tag{67}$$

which is obtained by gyroaveraging Eq.(66), and where δh_0 satisfies the nonlinear equation

$$\begin{aligned}(\partial/\partial t + v_{\|}\boldsymbol{b}\cdot\nabla)\delta h_0 &= q[(\partial F_0/\partial H)i\omega + (m\Omega)^{-1}\boldsymbol{b}\times\widehat{\nabla}F_0\cdot\widehat{\nabla}] \\ &\quad \cdot[\langle\delta\psi\rangle_\alpha + (c/B)\boldsymbol{b}\times\widehat{\nabla}\delta h_0\cdot\widehat{\nabla}\langle\delta\psi\rangle_\alpha].\end{aligned} \tag{68}$$

Notice that we have not yet specified the nature of ensemble averaging; a convenient ensemble will be defined later. We shall also show that the second term on the right-hand side of Eq.(68) is the diffusion term (the first term is clearly the power term). The analysis becomes much more perspicuous in the Fourier representation in which the aforementioned term takes the form

$$\langle\nabla\delta\psi\rangle_\alpha\cdot\boldsymbol{b}\times\widehat{\nabla}\delta h_0=\widehat{\nabla}\cdot[\langle\nabla\delta\psi\rangle_\alpha\times\boldsymbol{b}\delta h_0]=\widehat{\nabla}\cdot\sum_{k_1,k_2}i(\boldsymbol{k}_1\times\boldsymbol{b})\delta\psi_{k_1}J_0(k_{1\perp}v_\perp/\Omega)\delta h_{k_2}\quad(69)$$

The ensemble is now so chosen that the ensemble averaging implies $k_1+k_2=0$, i.e., one chooses only those modes which have equal and opposite four momenta. The operator $\widehat{\nabla}$ outside the summation over Fourier component acts only on the macroscopic quantities contained in the summation (these quantities are *D.C.*, and hence not Fourier transformed). Equation (69) becomes

$$\left\langle\langle\nabla\delta\psi\rangle_\alpha\cdot\boldsymbol{b}\times\widehat{\nabla}\delta h_0\right\rangle=\widehat{\nabla}\cdot\sum_{k_1+k_2=0}i(\boldsymbol{k}_1\times\boldsymbol{b})\delta\psi_{k_1}J_0(k_{1\perp}v_\perp/\Omega)\delta h_{k_2}\quad(70)$$

At this stage, we go back and apply the standard renormalizing procedure to Eq.(68) and obtain the linear response

$$\delta h_k=-G_kq\{(\partial F_0/\partial H)\omega+(m\Omega)^{-1}\boldsymbol{b}\times\widehat{\nabla}F_0\cdot\boldsymbol{k}_1\}J_0(k_\perp v_\perp/\Omega)\delta\psi_k\quad(71)$$

where G_k is the renormalized propagator [Eq.(56)]. The nonlinear power term (the first term on the right hand side of Eq.(68)) is associated with transport in velocity space and will not be investigated in this paper. Substituting Eqs.(70), (71) into Eq.(68), and integrating over $\boldsymbol{v}_\perp$, we obtain

$$\begin{aligned}(\partial/\partial t+v_\parallel\boldsymbol{b}\cdot\widehat{\nabla})F_0(v_\parallel)&=\widehat{\nabla}\cdot(c/B)^2Im\sum_{k_1+k_2=0}(\boldsymbol{k}_1\times\boldsymbol{b})G_{k_2}(\boldsymbol{k}_2\times\boldsymbol{b})\\&\cdot[\widehat{\nabla}F_0(v_\parallel)]\Gamma_0(\rho^2k_{1\perp}^2/2)\left\langle\delta\psi_{k_1}\delta\psi_{k_1}^*\right\rangle+....\quad(72)\end{aligned}$$

The transport equations can be built up by taking various $v_\parallel$ moments of Eq.(72); the associated transport coefficient can also be readily identified. In particular, the energy transport coefficient is associated with the term (from the second moment)

$$\widehat{\nabla}\cdot(c/B)^2Im\sum_k(\boldsymbol{k}\times\boldsymbol{b})\Gamma_0(\rho^2{k_\perp}^2/2)(\boldsymbol{k}\times\boldsymbol{b})\cdot m\int_{-\infty}^{+\infty}dv_\parallel{v_\parallel}^2G_k\widehat{\nabla}F_0(v_\parallel)\left\langle\delta\psi_k\delta\psi_k^*\right\rangle\quad(73)$$

where $F_0(v_{||}) \equiv (\sqrt{\pi}v_0)^{-1}\exp(-v_{||}{}^2/v_0^2)$ is the one dimensional Maxwellian. From Eq.(72) we see that the turbulent transport coefficient can be written in the familiar form $D_\perp^{(T)} \sim (\delta v_\perp)^2\tau_c$, where $\delta v_\perp$ is the perturbed perpendicular velocity induced by the fluctuations, and τ_c is the shortest correlation times. Several general properties of the diffusion coefficient are easily gleaned from Eq.(72):

1) Larmor radius effects cause the diffusion coefficient to be suppressed by the factor $\Gamma_0(b = k_\perp^2\rho^2/2) = I_0(b)\exp(-b) \sim b^{-1/2}(b \to \infty)$,

2) In the expression for the diffusion coefficient there are two competing time scales [provided ω is small so that $\omega^{-1} \gg \tau_c$] $(k_{||}v_{||})^{-1}$ and $\Gamma_k^{-1} = (k_\perp^2 D_\perp)^{-1}$; their origin is in the renormalized propagator $G_k \sim (\omega - k_{||}v_{||} + i\Gamma_k)^{-1}$. Thus τ_c will equal the shorter of these times, $(k_{||}v_0)^{-1}$ if $k_{||}v_0 > k_\perp{}^2 D_\perp$, and $k_\perp^2 D_\perp$ if $k_\perp{}^2 D_\perp > k_{||}v_0$,

3) It is possible to obtain the ratio of electron to ion turbulent transport coefficient ($R = \Gamma_k^{(e)}/\Gamma_k^{(i)}$ although neither of these can be evaluated without knowing the saturation fluctuation level which, of course, requires the complete solution of the nonlinear problem. As an example, we work out the ratio R for the energy transport caused by kinetic electrostatic drift waves which are characterized by the inequality $k_{||}v_i < \omega < k_{||}v_e$. For the regime $k_{||}v_e > \Gamma_k^{(e)}$, $k_{||}v_i < \Gamma_k^{(i)}$, it is straightforward to evaluate $\tau_c^{(e)} \sim (\pi^{1/2}/k_{||}v_e)(\omega/k_{||}v_e)^2$, and $\tau_c^{(i)} = \Gamma_0^{-1}\Gamma_k^{(i)}/(\omega^2 + (\Gamma_k^{(i)})^2)$ leading to the ratio between the transport coefficients

$$R = \chi_e/\chi_i = \tau_c^{(e)}/\tau_c^{(i)} = \frac{\sqrt{\pi}}{\Gamma_0(k_\perp{}^2\rho_i^2/2)}\left(\frac{\omega}{k_{||}v_e}\right)^3\frac{\omega}{\Gamma_k^{(i)}}. \tag{74}$$

Notice that $R < 1$ because for the drift waves $k_\perp\rho_s \sim 1$ and $\omega < k_{||}v_e$. The conclusion $R < 1$ does not change even if the other limit $(k_{||}v_{||} < \Gamma_k^{(e)})$ were examined. This conclusion seems to be in direct conflict with the tokamak experiments where $R = \chi_e/\chi_i > 1$. It is thus difficult to believe that the electrostatic drift wave could be responsible for the observed anomalous electron transport in tokamaks.

VI Summary and Discussion

In this paper, we have made use of the general methodology of renormalized perturbation theory (Ref.1) to deal with a variety of problems of interest to plasma physics.

We first investigated the much studied Vlasov-Poisson system to compare and contrast our approach with DIA (Direct Interaction Approximation) and with the theory of Dupree and Tetrault. It was remarked earlier that DIAC with diffusive approximation was equivalent to Dupree-Tetrault. Both of these theories are equivalent to our perturbation theory to second order provided the incoherent terms, (even in the first order) present in our theory, are neglected. The reason for equivalence is that the incoherent terms do not contribute to energy up to this order in perturbation.

Since DIAC with diffusive approximation neglects incoherent terms in DIA (though it keeps the polarization terms), it cannot have the correct weak turbulence limit; the three wave interaction and nonlinear scattering is neglected both in DIAC and in Dupree-Tetrault. In fact DIA has not yielded any significant practical results beyond DIAC with diffusive approximation; it has not been possible, for example, to prove energy conservation in electrostatic drift waves beyond second order, or to derive correct weak turbulence equations.

The incoherent term was studied non-perturbatively by Dupree in his clump theory[7]; and if the incoherent term is properly treated in Dupree's perturbation theory, the correct weak turbulence limit should follow. Thus the failure of Dupree's theory to give the correct weak turbulence limit must be attributed to a mismatched perturbation theory and not due to a faulty iteration procedure[12].

Our perturbation theory, on the other hand, is a properly matched perturbation theory with an unambiguous difinition of 'order' which allows us to retain all the relevant terms (including the incoherent terms) to a particular order. As a result, we can routinely reproduce all the appropriate weak turbulence expressions as limiting cases of our theory (Secs. II and III). The perturbative nature of the theory allows us to do a whole variety of calculations (like proving the energy conservation in electrostatic drift waves to all orders in Sec.IV) not accessible to non-perturbative schemes.

It should be pointed out that our perturbation theory is quite different from DIA; the propagators in the two theories are different even in the lowest non-trivial order (the polarization term is included in the DIA propagator).

In addition to showing the usefulness of our approach to standard problems like Vlasov-Poisson turbulence, we have also applied it successfully to derive the relevant equations for a gyro-kinetic system, i.e., the nonlinear coupled wave equations and the equations of transport of particle, energy etc. By comparing expressions for the ion and electron fluctuation induced transport, we show that it is difficult to believe that electrostatic drift waves could be responsible for anomalous electron transport in low β tokamaks.

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