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On Tokamak Equilibrium

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Abstract

Large aspect-ratio tokamak equilibrium is studied, including effects, such as ellipticity of the flux surfaces, of second order in the inverse aspect ratio. To facilitate the interpretation of experimental data, the analysis uses simple coordinates that are readily evaluated, rather than coordinates based on the exact flux surfaces. An explicit formula for the safety factor, including second order terms, is presented, along with a brief discussion of rotational effects. The analysis assumes near circularity of the outermost closed surface and low plasma pressure.

I. Introduction

The Grad-Shafranov equation for tokamak equilibrium,

$$\Delta^*\chi \equiv R^2\nabla\cdot\left(R^{-2}\nabla\chi\right) = -I\,dI(\chi)/d\chi - 4\pi R^2 dP(\chi)/d\chi \tag{1}$$

has been solved approximately by Shafranov[1] in the limit

$$\varepsilon \ll 1, \tag{2}$$

where $\varepsilon \equiv a/R_0$ is the inverse aspect ratio, with a the plasma radius and R_0 the major radius of the magnetic axis. [Here χ is the poloidal flux, $I(\chi)$ measures the toroidal field $B_\mathrm{T} = I/R$, P is the pressure and R the major radius.] The Shafranov solution is characterized by nested but nonconcentric circular flux surfaces. The poloidal flux is found only through first order in ε; indeed in second order one sees that the surfaces are no longer circular. Similarly the coordinates used—"Shafranov coordinates"—are based on the displaced circle geometry and are flux coordinates only through order ε. The exact flux surfaces have, in particular, a slight ellipticity.

The second order terms were studied by Greene, Johnson and Weimar[2] (GJW). Flux coordinates were used, so that, for example, ellipticity of the surfaces is implicit in the coordinate definitions and not otherwise visible. While this coordinate choice is analytically convenient (for stability studies), it complicates the interpretation of experimental data. For example the GJW radius is no simple distance from any circular center. Furthermore a closed-form expression for the second-order equilibrium configuration is not provided in GJW.

The present study is intended to simplify experimental interpretation by expressing equilibrium quantities in terms of very simple coordinates. [Thus our goal is sharply different from that of the "inverse moment" method.[3]] It nonetheless includes the higher order effects of GJW, which are here made explicit in certain cases of interest. Modifications due to plasma rotation are also briefly considered.

In view of recent experimental interest, we devote special attention to the safety factor.

II. Flux Coordinates and Shafranov Coordinates

By "flux coordinates" we mean the system (χ, θ_f, ζ) chosen to allow a certain representation of the magnetic field,

$$\mathbf{B} = q\nabla\chi\times\nabla\theta_f + \nabla\zeta\times\nabla\chi, \tag{3}$$

usually called the "flux representation." [Here θ_f is a poloidal angle and ζ is the toroidal (symmetry) angle.] In this system the field lines are straight in the sense that $\mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\theta_f$ is constant on each surface; in fact this ratio is precisely the safety factor:

$$q(\chi) = \mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\theta_f. \tag{4}$$

Equation (4) shows why flux coordinates simplify analytic studies of tokamak equilibrium and stability. In fact we shall use this system below to derive a convenient expression for q. However, for purposes of experimental interpretation, θ_f is less convenient than other measures of poloidal angle, such as the Shafranov angle coordinate that we consider next.

Shafranov coordinates are denoted here by $(r,\, \theta,\, \zeta)$ and defined by

$$R = R_0 + r\cos\theta - D(r)$$

$$z = r\sin\theta, \qquad \zeta = -\zeta_c, \tag{5}$$

where $(R,\, \zeta_c,\, z)$ are cylindrical coordinates about the tokamak symmetry axis and $D = O\left(\varepsilon^2\right)$ is the Shafranov displacement. Thus the poloidal cross-section of the surface labelled by r is a circle centered at $R_0 - D(r)$; since the innermost surface is centered by definition at R_0 we have

$$D(O) = 0.$$

The radius of the outermost surface is denoted by a; hence the geometric center of the plasma cross-section has major radius $R_0 - D(a)$. At this point we erect a Cartesian coordinate system (x, z). Evidently

$$R \;=\; R_0 + x - D(a),$$

so the Cartesian system is related to Shafranov coordinates by

$$\begin{aligned} r\cos\theta &= x - d(r), \\ r\sin\theta &= z, \end{aligned}$$

where

$$d(r) \;\equiv\; D(a) - D(r) \tag{6}$$

is the displacement with respect to the geometric center.

It is assumed that experimental data are spatially resolved in terms of the Cartesian system; one might know for example the pressure at a point (x, z). To compare such data to various theoretical predictions one needs first of all the Shafranov coordinates of the point. These are easily found after it is recognized that the Shafranov picture demands large aspect ratio, as in (2), and that the shift D is only known through ε^2. (It will be seen that unknown corrections to the shift are of order ε^4.) Hence for consistency one retains only ε^2 terms, allowing a simple iterative solution to Eq. (5). We thus find

$$\begin{aligned} r(x,z) &= r_c - xd\left(r_c\right)/r_c, \\ \sin\theta(x,z) &= z/r(x,z), \end{aligned} \tag{7}$$

where

$$r_c = \left(x^2+z^2\right)^{1/2}$$

is the radius measured from the geometric center.

Having established the relation between Shafranov coordinates and the Cartesian system, we henceforth ignore the latter, and re-express the former in a more convenient form. Essentially we replace r and D by dimensionless versions, $\rho = r/a$ and

$$\Delta = -\varepsilon^{-2}D/R_0,$$

and define the Shafranov system, $(\rho,\ \theta,\ \zeta)$, by

$$\begin{aligned} R &= R_0\left[1+\varepsilon\rho\cos\theta+\varepsilon^2\Delta(\rho)\right], \\ \zeta_c &= -\zeta, \quad z = a\rho\sin\theta. \end{aligned}$$

We also presume a flux, χ, of the Shafranov form, depending only on ρ through first order:

$$\chi(\rho,\theta) = \psi(\rho)+\varepsilon^2\varphi(\rho,\theta). \tag{8}$$

Before proceeding further it is convenient to consider consequences of this *ansatz*.

III. Surface Functions

By a "surface function" or "flux function" we mean a quantity depending on position only through χ. The pressure and safety factor are examples. We note here that the θ-dependence at fixed ρ of any flux function, F, is determined by

$$\frac{dF}{d\chi} = \frac{(\partial F/\partial\rho)}{(\partial\chi/\partial\rho)} = \frac{(\partial F/\partial\theta)}{(\partial\chi/\partial\theta)},$$

whence

$$\frac{\partial F}{\partial \theta} = \varepsilon^2(\partial\varphi/\partial\theta)(\partial F/\partial\rho)/(\partial\chi/\partial\rho),$$

which can be integrated to give

$$F(\chi) = \bar{F}(\rho) + \varepsilon^2\varphi(\rho,\theta)(d\bar{F}/d\rho)/(d\psi/d\rho) + O\left(\varepsilon^3\right). \tag{9}$$

We also note that the coordinate ρ is not a flux function, but satisfies

$$\rho(\chi,\theta) = \bar{\rho}(\chi) + \varepsilon^2\tilde{\rho}(\bar{\rho},\theta), \tag{10}$$

where $\bar{\rho}$ is defined by

$$\psi(\bar{\rho}) = \chi. \tag{11}$$

Taylor expansion and Eq. (8) give

$$\tilde{\rho}(\bar{\rho},\theta) = -\varphi(\bar{\rho},\theta)/\psi'(\bar{\rho}) = -\left[\psi'(\rho)\right]^{-1}\varphi(\rho,\theta) + O\left(\varepsilon^2\right), \tag{12}$$

where the primes indicate derivatives with respect to argument.

IV. Expansion of the Grad-Shafranov Equation

In order to express Eq. (1) in terms of Shafranov coordinates we need the metric coefficients g_{ij}. From $ds^2 = dR^2 + R^2 d\varphi^2 + dz^2 = g_{11}d\rho^2 + g_{12}d\rho\, d\theta + g_{22}d\theta^2 + g_{33}d\zeta^2$, we calculate

$$\frac{g_{11}}{a^2} = w, \quad \frac{g_{22}}{a^2} = \rho^2, \quad \frac{g_{33}}{a^2} = \varepsilon^{-2}u^2, \quad \frac{g_{12}}{a^2} = -\varepsilon\rho\Delta'\sin\theta,$$

and

$$a^2 g^{11} = v^{-2}, \quad a^2 g^{22} = \frac{w}{(\rho^2 v^2)}, \quad a^2 g^{33} = \frac{\varepsilon^2}{u^2}, \quad a^2 g^{12} = \rho^{-1}\left(\frac{\partial}{\partial\theta}\right)\left(\frac{1}{v}\right).$$

Here we have introduced

$$u \equiv 1 + \varepsilon\rho\cos\theta + \varepsilon^2\Delta = \frac{R}{R_0}, \quad v \equiv 1 + \varepsilon\Delta'\cos\theta,$$

$$w \equiv 1 + 2\varepsilon\Delta'\cos\theta + \varepsilon^2\Delta'^2.$$

The metric determinant is

$$\sqrt{g} = a^2 R_0 \rho u v. \tag{13}$$

Now we can express the left-hand side of Eq. (1) as

$$\Delta^*\chi = \left(\frac{u^2}{\sqrt{g}}\right)\left(\frac{\partial}{\partial\xi^i}\right)\left(\frac{\sqrt{g}}{u^2}\right)g^{ij}\left(\frac{\partial\chi}{\partial\xi^j}\right),$$

or

$$
\begin{aligned}
a^2\Delta^*\chi &= \left(\frac{u}{\rho v}\right)\left\{\left(\frac{\partial}{\partial\rho}\right)\left[\left(\frac{\rho}{uv}\right)\frac{\partial\chi}{\partial\rho}\right]+\left(\frac{\partial}{\partial\theta}\right)\left[(w/\rho uv)\frac{\partial\chi}{\partial\theta}\right]\right.\\
&+ \left.\left(\frac{\partial}{\partial\rho}\right)\left[\left(\frac{v}{u}\right)\left(\frac{1}{v}\right)_\theta\frac{\partial\chi}{\partial\theta}\right]+\left(\frac{\partial}{\partial\theta}\right)\left[\left(\frac{v}{u}\right)\left(\frac{1}{v}\right)_\theta\frac{\partial\chi}{\partial\rho})\right]\right\},
\end{aligned}
$$

where $(1/v)_\theta \equiv (\partial/\partial\theta)(1/v)$.

The results above are exact for any ε. We now expand for small ε using the notation

$$
a^2\Delta^* = L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots
$$

and find

$$
\begin{aligned}
L_0\chi &= \left(\frac{1}{\rho}\right)\left(\frac{d}{d\rho}\right)\rho\left(\frac{d\psi}{d\rho}\right),\\
L_1\chi &= \left(\frac{\cos\theta}{\rho}\right)\left\{-\left(\frac{d}{d\rho}\right)\rho(\rho+\Delta')\left(\frac{d\psi}{d\rho}\right)+\Delta'\left(\frac{d\psi}{d\rho}\right)+(\rho-\Delta')\left(\frac{d}{d\rho}\right)\rho\left(\frac{d\psi}{d\rho}\right)\right\},\\
L_2\chi &= \left(\frac{1}{\rho}\right)\left(\frac{\partial}{\partial\rho}\right)\rho\left(\frac{\partial\varphi}{\partial\rho}\right)+\rho^{-2}\partial^2\varphi/\partial\theta^2\\
&+ \left(\frac{1}{\rho}\right)\left(\frac{\partial}{\partial\rho}\right)\left\{\rho\left[\left(\rho^2+\Delta'^2+\rho\Delta'\right)\cos^2\theta-\Delta\right]\frac{d\psi}{d\rho}\right\}\\
&- \Delta'(\rho+\Delta')(\cos 2\theta/\rho)\frac{d\psi}{d\rho}+\left[\Delta-\Delta'(\rho-\Delta')\cos^2\theta\right]\left(\frac{1}{\rho}\right)\left(\frac{d}{d\rho}\right)\rho\left(\frac{d\psi}{d\rho}\right)\\
&- (\rho-\Delta')\cos^2\theta\left\{\rho^{-1}\left(\frac{d}{d\rho}\right)\left[\rho(\rho+\Delta')\left(\frac{d\psi}{d\rho}\right)\right]-\left(\frac{\Delta'}{\rho}\right)\left(\frac{d\psi}{d\rho}\right)\right\}.
\end{aligned}
\tag{14}
$$

Next consider the expansion of the right-hand side,

$$
a^2Q \equiv a^2\left[I\,dI(\chi)/d\chi + 4\pi R^2 dP(\chi)/d\chi\right].
$$

Since the Shafranov picture requires $\beta = O\left(\varepsilon^2\right)$, we introduce a normalized pressure through

$$
a^4P = \varepsilon^2 p,
$$

and then use Eq. (9) to write

$$
a^4P(\rho,\theta) = \varepsilon^2\left\{\bar{p}(\rho)+\varepsilon^2\left(\frac{p'}{\psi'}\right)\varphi(\rho,\theta)\right\}.
$$

To simplify notation, the overbar on $\bar{p}$ is suppressed: $\bar{p} \to p$. Thus

$$\begin{aligned} 4\pi R^2 a^2 dP/d\chi &= 4\pi u^2 \left(\psi' + \varepsilon^2 \partial\varphi/\partial\rho\right)^{-1} \left(\frac{\partial}{\partial\rho}\right) \left[p + \varepsilon^2 \left(\frac{p'}{\psi'}\right)\varphi\right] \\ &= \left(4\pi u^2\right)\left(\frac{p'}{\psi'}\right)\left[1 + \varepsilon^2 (p')^{-1}\left(\frac{p'}{\psi'}\right)'\varphi\right]. \end{aligned}$$

With regard to the $I\,dI/d\chi$ term, we define the quantity $B_0(\chi)$ through

$$I = B_{\mathrm{T}} R \equiv B_0 R_0$$

and treat B_0 very similarly to P to obtain

$$B_0(\rho,\theta) = \bar{B}_0(\rho) + \varepsilon^2 \left(d\bar{B}_0/d\rho\right)\varphi(\rho,\theta)/\psi'.$$

However, since radial gradients of the toroidal field are of order β, we now assume $\bar{B}_0(\rho) = a^{-2} b\left(\varepsilon^2\rho\right)$ and then find

$$a^2 B_0(\rho,\theta) = b + \varepsilon^4 b'\varphi/\psi',$$

where we consistently use primes to denote differentiation with respect to argument. Hence we have the following expansion of the right-hand side:

$$\begin{aligned} a^2 Q &= bb'/\psi' + 4\pi p'/\psi' + \varepsilon\cos\theta\,[8\pi\rho p'/\psi'] \\ &+ \varepsilon^2 \left\{(4\pi p'/\psi')\left[\left(\frac{p'}{\psi'}\right)'\left(\frac{\varphi}{p'}\right) + 2\Delta + \rho^2\cos^2\theta\right] - (bb'/\psi')\left(\frac{\psi''}{\psi'^2}\right)\varphi\right\}. \qquad (15) \end{aligned}$$

We combine the foregoing results to write the first two orders of Eq. (1) as

$$\left(\frac{1}{\rho}\right)\left(\frac{d}{d\rho}\right)\rho\, d\psi/d\rho = -\left[b_0 b_0' + 4\pi p'\right]/\psi',$$

and

$$-\left(\frac{1}{\rho}\right)\left(\frac{d}{d\rho}\right)\rho(\rho+\Delta')\frac{d\psi}{d\rho} + \left(\frac{\Delta'}{\rho}\right)\frac{d\psi}{d\rho} + (\rho-\Delta')\left(\frac{1}{\rho}\right)\left(\frac{d}{d\rho}\right)\rho\, d\psi/d\rho = -\left(8\pi\rho p'/\psi'\right).$$

After some rearrangement these results can be expressed as

$$\left(\frac{d}{d\rho}\right)(\rho\psi')^2 = -\rho^2\left(\frac{d}{d\rho}\right)\left(b^2 + 8\pi p\right) \qquad (16)$$

a well-known equation for $\psi'(\rho)$; and

$$\left(\frac{d}{d\rho}\right)\left(\Delta'\rho\psi'^2\right) + \rho\psi'^2 = 8\pi\rho^2 p', \qquad (17)$$

a well-known equation for $\Delta'(\rho)$.

The second order equation is only slightly more complicated. We first simplify $L_2\chi$, which can be expressed as

$$L_2\chi = \nabla^2\varphi + X_0(\rho) + X_2(\rho)\cos 2\theta,$$

where

$$X_0 = \left(3\Delta'^2/2\right)\nabla^2\psi + \left(\frac{\psi'}{2}\right)\left[\Delta'\left(3\Delta'' + 1 - \Delta'/\rho\right) + \rho\right],$$

$$X_2 = \left(3\Delta'^2/2\right)\nabla^2\psi + \left(\frac{\psi'}{2}\right)\left[\Delta'\left(3\Delta'' + 1 - 3\Delta'/\rho\right) + \rho\right].$$

Next, considering the right-hand side, we express the second order terms in (15) as

$$a^2 Q_2 = \left[4\pi\left(\frac{p'}{\psi'}\right)' + \left(bb'/\psi'\right)'\right]\left(\frac{\varphi}{\psi'}\right) + \left(4\pi p'/\psi'\right)\left(2\Delta + \rho^2\cos^2\theta\right),$$

since $db/d\rho = \varepsilon^2 b'$. By inspecting the lower order results we see that

$$a^2 Q_2 = -\left(\nabla^2\psi\right)'\left(\frac{\varphi}{\psi'}\right) + \left(4\pi p'/\psi'\right)\left(2\Delta + \rho^2\cos^2\theta\right).$$

Thus the second-order terms in the Grad-Shafranov equation are

$$\nabla^2\varphi - (\psi')^{-1}\left(\nabla^2\psi\right)'\varphi = -\left(4\pi p'/\psi'\right)\left(2\Delta + \rho^2\cos^2\theta\right) - X_0(\rho) + X_2(\rho)\cos 2\theta,$$

an equation for $\varphi(\rho,\theta)$. We see this takes the form of a driven Schroedinger equation, in which the gradient of the lowest order current, $(\nabla^2\psi)'$, plays the role of the potential. It is evident that φ has only $m = 0$ and $m = 2$ components:

$$\varphi(\rho,\theta) = \varphi_0(\rho) + \varphi_2(\rho)\cos 2\theta;$$

these satisfy

$$\nabla^2\varphi_0 - (\psi')^{-1}J'\varphi_0 = -\left(2\pi p'/\psi'\right)\left(\rho^2 + 4\Delta\right) - \left(3\Delta'^2/2\right)J + \left(\frac{\psi'}{2}\right)\left[\Delta'\left(3\Delta'' + 1 - \Delta'/\rho\right) + \rho\right], \tag{18}$$

$$\nabla^2\varphi_2 - 4\varphi_2/\rho^2 - (\psi')^{-1}J'\varphi_2 = -\left(2\pi p'/\psi'\right)\rho^2 - \left(3\Delta'^2/2\right)J + \left(\frac{\psi'}{2}\right)\left[\Delta'\left(3\Delta'' + 1 - 3\Delta'/\rho\right) + \rho\right], \tag{19}$$

where $\nabla^2\varphi_m \equiv (1/\rho)(d/d\rho)\rho(d\varphi_m/d\rho)$ and

$$J \equiv \nabla^2\psi$$

measures the lowest order toroidal current.

Thus the second order equilibrium problem is reduced to the solution of four ordinary differential equations, (16)–(19). The first two of these are well known and straightforwardly solved in terms of the plasma beta and internal inductance. Special solutions to (18) and (19), which appear to be new, are discussed after we have commented upon their application.

V. The Safety Factor

The significance of φ_0 and φ_2 is seen in particular when we evaluate the safety factor, q. It is convenient to begin with Eq. (3), which implies

$$\mathbf{B}\cdot\nabla\zeta = q\nabla\zeta\cdot\nabla\chi\times\nabla\theta_f.$$

Alternatively, $\mathbf{B}\cdot\nabla\zeta = I/R^2$, so

$$\nabla\zeta\cdot\nabla\chi\times\nabla\theta_f = \frac{I}{(qR^2)}.$$

After expressing the gradients in terms of Shafranov coordinates we have

$$\left\{\left(\frac{\partial\chi}{\partial\rho}\right)\left(\frac{\partial\theta_f}{\partial\theta}\right) - \frac{\partial\chi}{\partial\theta}\left(\frac{\partial\theta_f}{\partial\rho}\right)\right\} = \sqrt{g}I/\left(qR^2\right),$$

which reduces easily to

$$\frac{\partial\theta_f}{\partial\theta} + \left(\frac{\varepsilon^2}{\psi'}\right)\left[\left(\frac{\partial\varphi}{\partial\rho}\right)\left(\frac{\partial\theta_f}{\partial\theta}\right) - \frac{\partial\varphi}{\partial\theta}\left(\frac{\partial\theta_f}{\partial\rho}\right)\right] = \sqrt{g}I/\left(qR^2\psi'\right). \tag{20}$$

We show next that an expression for the safety factor $q(\rho,\theta)$ can be derived from Eq. (20), together with the constraint

$$q = q(\chi). \tag{21}$$

First expand

$$\begin{aligned}\theta_f(\rho,\theta) &= \theta + \varepsilon p_1(\rho,\theta) + \varepsilon^2 p_2(\rho,\theta) + \cdots,\\ q(\rho,\theta) &= q_0(\rho) + \varepsilon q_1(\rho,\theta) + \varepsilon^2 q_2(\rho,\theta) + \cdots,\end{aligned}$$

where the p_n and q_n must be periodic functions of θ. After substitution into Eq. (20) and straightforward expansion, we find

$$
\begin{aligned}
1+\varepsilon\partial p_1/\partial\theta \quad &+ \quad \varepsilon^2\left[\frac{\partial p_2}{\partial\theta}+(\psi')^{-1}\frac{\partial\varphi}{\partial\rho}\right] = \left[a^2\rho\bar{B}_0(\rho)/\psi' q_0\right]\left\{1-\varepsilon\left[\frac{q_1}{q_0}+(\rho-\Delta')\cos\theta\right]\right. \\
&+ \quad \varepsilon^2\left[\left(\frac{q_1}{q_0}\right)^2+2\left(\frac{q_1}{q_0}\right)(\rho-\Delta')\cos\theta+(\rho-\Delta')^2\cos^2\theta\right. \\
&\left.\left.\qquad -\left(\frac{q_2}{q_0}\right)-\Delta-\Delta'(\Delta'-\rho)\cos^2\theta\right]\right\}. \qquad (22)
\end{aligned}
$$

Here we recalled that $d\bar{B}_0/d\rho = O\left(\varepsilon^2\right)$ and neglected terms of higher than second order as usual. From the zeroth- and first-order terms we find respectively that

$$
q_0(\rho) \quad = \quad a^2\rho\bar{B}_0/\psi'(\rho), \qquad (23)
$$

and

$$
\frac{\partial p_1}{\partial\theta} \quad = \quad -\left[\frac{q_1}{q_0}+(\rho-\Delta')\cos\theta\right]. \qquad (24)
$$

But Eqs. (9) and (21) require

$$
\frac{\partial q}{\partial\theta} \quad = \quad \varepsilon^2\left(\frac{\partial\varphi}{\partial\theta}\right)\frac{q_0'}{\psi'}, \qquad (25)
$$

so q_1 must be independent of θ; the θ-average of (24) then shows that

$$
q_1 \quad = \quad 0.
$$

The second-order terms in (22) are therefore

$$
\frac{\partial p_2}{\partial\theta}+(\psi')^{-1}\frac{\partial\varphi}{\partial\rho} = (\rho-\Delta')^2\cos^2\theta-\Delta-\Delta'(\Delta'-\rho)\cos^2\theta-\left(\frac{q_2}{q_0}\right),
$$

and, after averaging over θ, we may conclude that

$$
\frac{\bar{q}_2}{q_0} = \left(\frac{1}{2}\right)\rho(\rho-\Delta')-\Delta-\frac{\varphi_0'}{\psi'}.
$$

Combining this result with (25) we can express the safety factor as

$$
q(\rho,\theta)/q_0 = 1+\varepsilon^2\left[\left(\frac{1}{2}\right)\rho(\rho-\Delta')-\Delta-\frac{\varphi_0'}{\psi'}+(\ln q_0)'\left(\frac{\varphi_2}{\psi'}\right)\cos 2\theta\right]. \qquad (26)
$$

Conventional measures of poloidal and toroidal fields in Shafranov geometry are B_{PO} and B_{TO}, defined by

$$B_{\mathrm{P}} \equiv |\nabla\zeta \times \nabla\chi| = B_{\mathrm{PO}}\left[1 - \varepsilon(\rho + \Delta')\cos\theta\right] + O\left(\varepsilon^2\right),$$
$$B_{\mathrm{T}} \equiv |I\nabla\zeta| = B_{\mathrm{TO}}(1 - \varepsilon\rho\cos\theta) + O\left(\varepsilon^2\right).$$

We note that

$$\psi' = aR_0 B_{\mathrm{PO}}, \qquad \bar{B}_0 = B_{\mathrm{TO}},$$

and thus obtain from (23) the conventional lowest order formula,

$$q_0 = rB_{\mathrm{TO}}/R_0 B_{\mathrm{PO}}. \tag{27}$$

VI. Second-Order Flux

Equations (18) and (19) become invariant with respect to changes of scale in the case

$$\psi = \alpha r^k + \beta r^{2-k}, \qquad k = 0, 1, 2, \ldots \tag{28}$$

which is consistent with (16) if the pressure is also polynomial:

$$\psi \propto r^k \Rightarrow P \propto r^{2k-2}. \tag{29}$$

It is then possible to calculate a particular solution for φ. The result gives useful information about the general solution insofar as (28) is locally valid.

For each solution one can straightforwardly compute $q(r)$; the result will be usefully accurate if (8) is *locally* valid. For simplicity we display the result when only one of the two terms is present:

$$\psi = \alpha r^k.$$

We restore dimensional variables to (26), write it as

$$\frac{q}{q_0} - 1 = -\left(\frac{r^2}{R_0^2}\right)(Y_0 + Y_2\cos 2\theta), \tag{30}$$

and find

$$Y_0 = -\left(\frac{1}{2}\right)\left\{\left(\frac{A}{k}\right) + 1 + [(k+2)/4(3k+2)]\left[\left(\frac{A^2}{k^2}\right)(5k/2 - 1) - A - 1\right]\right\}, \tag{31}$$

$$Y_2 = \left[(k-2)/16k^2\right]\left[A^2 - \left(\frac{2}{3}\right)(k-1)A - \left(\frac{2}{3}\right)k\right], \tag{32}$$

where

$$A = 1+\beta_*,$$
$$\beta_* \equiv \left|8\pi r P'/B_{\mathrm{PO}}^2\right|.$$

Evidently β_* measures the poloidal beta; the conditions (23) and (24) make β_* and therefore A consistently constant.

For example when $k=2$ (locally constant lowest-order current) we find $Y_2 = 0$ and

$$Y_0 = -\left(\frac{1}{16}\right)\left(A^2+3A+7\right).$$

It is interesting to note that neglect of φ in this case would have led to

$$Y_0(\varphi=0) = -\left(\frac{1}{4}\right)(A+2).$$

When $A \approx 1$ these two Y_0's are nearly equal.

VII. Rotation

A toroidally confined plasma always rotates slowly, without significantly affecting the equilibrium, essentially because of diamagnetic drifts and their associated return flows. Here we consider the more rapid rotation, zeroth-order in the Larmor radius, that is externally driven (by beam injection, for example). Such rapid plasma motion must be in the toroidal direction (because variation of B damps any rapid poloidal motion)[4]; it is related to the lowest order electrostatic potential, $\phi(\chi)$, according to

$$\mathbf{V}\cdot\nabla\zeta = -c\phi'(\chi) \equiv \omega(\chi), \tag{33}$$

the prime denoting a χ-derivative as usual. Since ω is the angular frequency of rotation, we see that each flux surface rotates rigidly.

Rotation enters equilibrium force balance though the convective inertial terms:

$$\nabla p + mn\mathbf{V}\cdot\nabla\mathbf{V} = \left(\frac{1}{c}\right)\mathbf{J}\times\mathbf{B}.$$

In combination with Ampere's law, the toroidal component implies

$$\frac{\partial I}{\partial \theta} = 0,$$

as before; the radial and poloidal components respectively yield

$$\frac{\partial p}{\partial \chi} - \left(\frac{1}{2}\right) mn\omega^2 \partial R^2/\partial \chi \;=\; -\left(1/4\pi R^2\right)\left(II' + \Delta^* \chi\right), \tag{34}$$

$$\frac{\partial p}{\partial \theta} - \left(\frac{1}{2}\right) mn\omega^2 \partial R^2/\partial \theta \;=\; 0. \tag{35}$$

To proceed further we use a well-known[4,5] result of kinetic theory,

$$\frac{\partial T}{\partial \theta} \approx 0,$$

where T is temperature of either species; isothermality within a given flux surface results from the very rapid conduction of heat along field lines. In combination with Eq. (35), it implies that the quantity

$$p^*(\chi) \;\equiv\; p(\chi, \theta) \exp\left(-\alpha u^2\right) \tag{36}$$

is constant on flux surfaces. Here $u \equiv R/R_0$, as before, and

$$\alpha(\chi) \;\equiv\; \left(\frac{1}{2}\right) m_i \omega^2 R_0^2 / \left(T_i + T_e\right),$$

also constant on each surface, is a normalized rotation energy:

$$\alpha \;\approx\; \frac{V^2}{v_{\text{th}}^2}.$$

By "rapid rotation" we mean specifically that α can be of order unity. The θ-variation of pressure predicted by Eq. (35) or (36) is estimated by

$$\Delta p \equiv |p(\theta = 0) - p(\theta = \pi)| \approx 2\alpha\varepsilon, \tag{37}$$

and is not inconsistent with some experimental observations.

The rotating version of the Grad-Shafranov equation is obtained straightforwardly from the combination of Eqs. (34) and (36). Of interest here is the small-ε version—the rotational generalizations of Eqs. (16) and (17). These take the form

$$\left(\frac{d}{d\rho}\right)(\rho\psi')^2 = -\rho^2 \left(\frac{d}{d\rho}\right)\left(b^2 + 8\pi p_0\right),$$

and

$$\left(\frac{d}{d\rho}\right)\left(\Delta' \rho \psi'^2\right) + \rho\psi'^2 = 8\pi\rho^2 \left[(1+\alpha)p_0\right]', \tag{38}$$

where

$$p_0(\chi) \equiv \exp(\alpha) p^*$$

is, in view of Eq. (36), simply the lowest-order (in ε) pressure. The only significant change is the appearance of α in Eq. (38); its affect is sensitive to the temperature and rotation profiles, but will typically enhance the outward shift. We note that Eq. (38) is closely equivalent to previous descriptions[4,5] of Tokamak equilibrium rotation.

To see the effects of rotation more explicitly, we integrate Eq. (38) in terms of the radial or volume average[5]:

$$\bar{f}(\rho) \equiv \left(\frac{2}{\rho^2}\right) \int_0^\rho d\rho' \rho' f(\rho').$$

Thus

$$\Delta' \rho = -\left(\frac{\rho^2}{2}\right) \bar{\psi}'^2 / \psi'^2 + 8\pi \left[(1+\alpha)p_0 - \overline{(1+\alpha)p_0}\right] / \psi'^2. \tag{39}$$

Note that the hoop force or internal inductance—the first term on the right-hand side of Eq. (39)—is unaffected by rotation, which therefore modifies the shift only when the plasma beta is substantial.

VIII. Conclusion

This work formulates the second-order tokamak equilibrium problem in terms of Shafranov coordinates. Its main new result is Eq. (26), expressing the spatial dependence of the safety factor through second order in the inverse aspect ratio. For a family of physically reasonable, model profiles, the formula is made explicit in Eqs. (30)–(33).

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